Algebraic Geometry

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Notation 0.0.1. *In this course:*

- *Rings and fields are associative, commutative, with a unit.*
- *The zero ring is not an integral domain (and therefore not a field).*

1 Towards schemes

1.1 Motivation

Notation 1.1.1. *Let k be a ring. We define:*

(i) *The* affine n-space $\mathbb{A}_k^n = k^n$.

(ii) *The* projective n-space $\mathbb{P}_k^n = \left(k^{n+1}\backslash\{0\}\right)/k^\times$.

The class of $(x_1, \ldots, x_{n+1}) \in k^{n+1}\backslash\{0\}$ in $\mathbb{P}_k^n$ will be denoted by $[x_1, \ldots, x_{n+1}]$.

Remark 1.1.2. *The aim of classical algebraic geometry is to study* affine varieties *(resp.* projective varieties*), i.e. subsets of $\mathbb{A}_k^n$ (resp. $\mathbb{P}_k^n$) defined by some* polynomial equations *(resp.* homogeneous polynomial equations*). A variety defined by only one polynomial equation is called a* hypersurface*.*

Example 1.1.3. *Let k be a field.*

(i) Polynomials with one variable*. If $P \in k[X]$, then the affine variety $V = \{x \in k,\ P(x) = 0\}$ is the set of roots of P.*

(ii) Linear algebra*. If $P_1, \ldots, P_m \in k[X_1, \ldots, X_n]$ are homogeneous polynomials of degree 1, then the affine variety $\{x \in \mathbb{A}_k^n,\ P_1(x) = \cdots = P_m(x) = 0\}$ is the kernel of a linear map f; it has dimension $n - \operatorname{rk} f$.*

(iii) Riemann surfaces*. Cartan's Theorem states that any compact Riemann surface is a projective variety in $\mathbb{P}_{\mathbb{C}}^3$.*

(iv) Some real affine varieties*. Circles, ovals, parabolas, spheres, etc. are affine varieties in $\mathbb{A}_{\mathbb{R}}^3$ or $\mathbb{A}_{\mathbb{R}}^2$.*

(v) Number theory*. Fermat's Last Theorem is a statement about a variety in $\mathbb{A}_{\mathbb{Q}}^3$.*

Remark 1.1.4. *Why schemes?*

(i) *If k is not algebraically closed, then very different equations can define the same variety. For instance, $\{x \in \mathbb{R},\ x^2 + 1 = 0\} = \varnothing = \{x \in \mathbb{R},\ 1 = 0\}$.*

(ii) *Roots can have multiplicities. For instance, $\{(x, y) \in \mathbb{R}^2,\ y = x^{2n} \text{ and } y = 0\} = \{(0, 0)\}$ does not depend on n.*

(iii) *One would like to talk about "a point at general position".*

1.2 Towards the notion of schemes

Remark 1.2.1. *We have the following analogy:*

- Topological spaces *are the geometric objects of* topology*.*
- Manifolds *are the geometric objects of* differential geometry*.*
- Complex manifolds *are the geometric objects of* complex geometry*.*
- Schemes *are the geometric objects of* algebraic geometry*.*

The key point will be the correspondence between algebra and geometry : to understand a geometric space is "equivalent" to understand the algebra of functions on this space, as illustrated by the following theorem.

Theorem 1.2.2 (Gelfand–Naimark, 1943)**.** *Let M be a compact Hausdorff topological space. Let $\mathcal{C}(M)$ be the $\mathbb{C}$-algebra of continuous complex functions on M. Then there is a natural bijection between M and the set of maximal ideals of $\mathcal{C}(M)$, given by $x \mapsto \{f \in \mathcal{C}(M), f(x) = 0\}$. Moreover, this bijection defines an equivalence between the category of compact Hausdorff topological spaces and the opposite category of commutative $\mathbb{C}^{\times}$-algebras with a unit.*

Remark 1.2.3. *The principle given by the Gelfand-Naimark Theorem works fine for topological spaces (resp. smooth manifolds) because there are a lot of continuous (resp. smooth) functions. But for compact complex manifolds, every holomorphic function is constant, so the study of $\mathcal{O}(M)$ is not enough to recover M. For algebraic geometry, the problem is the same: there are not enough polynomial functions to recover the variety or scheme. There are two solutions to this problem:*

- *We shall not only look at globally defined functions, but also at locally defined functions. This will lead to the notion of* sheaves*.*
- *We shall also include functions that are not everywhere defined (similar to meromorphic functions in complex geometry, or rational functions in classical algebraic geometry). This will lead to the notion of* coherent sheaves*.*

Remark 1.2.4. *To define a scheme, we shall define the following elements:*

(i) *The* set*, i.e. the points of the scheme.*

(ii) *The* topology*, i.e. the closed or open subsets of the scheme.*

(iii) *The* sheaf of functions*, i.e. the local algebraic functions on the scheme.*

(iv) *The* local structure*, i.e. an open covering of the scheme by* affine schemes*.*

1.3 Affine schemes

Definition 1.3.1 (Spectrum of a ring)**.** *If A is a ring, then the* spectrum *of A, denoted by* $\operatorname{Spec}(A)$*, is the set of prime ideals of A.*

Example 1.3.2.

(i) $\operatorname{Spec}(0) = \varnothing$.

(ii) *If k is a field,* $\operatorname{Spec}(k) = \{(0)\}$.

(iii) $\operatorname{Spec}(\mathbb{Z}) = \{(p),\ p \text{ prime}\} \cup \{(0)\}$.

(iv) *If k is an algebraically closed field, then we have* $\operatorname{Spec}(k[T]) = \{(T-a),\ a \in k\} \cup \{(0)\}$*. This scheme will be called the* scheme-theoretic affine line*.*

(v) *If k is a field, then* $\operatorname{Spec}\left(k[T]/\left(T^2\right)\right) = \{(T)\}$

(vi) *If k is an algebraically closed field, then we have* $\operatorname{Spec}(k[X,Y]) = \{(X-a, Y-b),\ a, b \in k\} \cup \{(P),\ P \in k[X,Y] \text{ irreducible}\} \cup \{(0)\}$*. This scheme will be called the* scheme-theoretic affine plane*.*

Definition 1.3.3 (Ring of functions on an affine scheme)**.** *Let A be a ring and let $X = \operatorname{Spec}(A)$. Then A will be called the* ring of functions *on X. If $f \in A$ is a function on X and $\mathfrak{p} \in X$ is a point of X, then the value $f(\mathfrak{p})$ of f at $\mathfrak{p}$ is defined as the image of f under the following map:*

$$\begin{tikzcd}
 & A/\mathfrak{p} \arrow[r] & \operatorname{Frac}(A/\mathfrak{p}) \arrow[d, leftrightarrow, "="'] \\
A \arrow[ur] \arrow[dr] & & \\
 & A_{\mathfrak{p}} \arrow[r] & A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}}
\end{tikzcd}$$

Hence, $f(\mathfrak{p})$ *is an element of the* residue field $\kappa(\mathfrak{p}) = \mathrm{Frac}\,(A/\mathfrak{p}) = A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}}$.

Remark 1.3.4. *Note that the values of a function on a scheme live in different fields.*

Example 1.3.5.

(i) *Let* A *be an integral domain. Consider* $X = \mathrm{Spec}(A)$*. If* $\eta = (0) \in X$*, then for* $f \in A$*,* $f(\eta) = f \in \mathrm{Frac}(A)$.

(ii) *Let* k *be an algebraically closed field. Consider* $X = \mathrm{Spec}\,(k[T])$*. For* $a \in k$*, let* $\mathfrak{m}_a = (T - a) \in X$*. Then for* $f \in k[T]$*,* $f(\mathfrak{m}_a) = f(a) \in k$.

Notation 1.3.6. *Let* A *be a ring.*

- *For* $I \trianglelefteq A$ *(i.e.* I *ideal of* A*), we write* $V(I) = \{\mathfrak{p} \in \mathrm{Spec}(A),\ \mathfrak{p} \supseteq I\}$.
- *For* $f \in A$*, we write* $D_f = \{\mathfrak{p} \in \mathrm{Spec}(A),\ f \notin \mathfrak{p}\} = \{\mathfrak{p} \in \mathrm{Spec}(A),\ f(\mathfrak{p}) \neq 0)$.

Remark 1.3.7. *Let* A *be a ring. If* $I \subseteq J$*, then* $V(I) \supseteq V(J)$.

Definition 1.3.8 (Zariski topology)**.** *If* A *is a ring, then the collection* $\{V(I),\ I \trianglelefteq A\} \subseteq \mathcal{P}\,(\mathrm{Spec}(A))$ *satisfies the axioms of closed sets of a topology. The topology defined by this collection will be called the* Zariski topology *on* $\mathrm{Spec}(A)$.

Proof. Note that $\varnothing = V(A)$ and $\mathrm{Spec}(A) = V\,((0))$ are closed sets. Moreover, if $(V\,(I_\lambda))_{\lambda\in\Lambda}$ is a collection of closed sets, then $\bigcap_{\lambda\in\Lambda} V\,(I_\lambda) = V\,((\bigcup_{\lambda\in\Lambda} I_\lambda))$ is also a closed set. Finally, if $V(I)$ and $V(J)$ are two closed sets, then $V\,(I) \cup V\,(J) = V\,(I \cap J)$ (this is due to properties of prime ideals), so $V\,(I) \cup V\,(J)$ is a closed set. $\square$

Proposition 1.3.9. *Let* A *be a ring. Then* $\{D_f,\ f \in A\}$ *forms a basis of the Zariski topology on* $\mathrm{Spec}(A)$*. The sets in this basis are called* principal open subsets.

Remark 1.3.10. *Let* A *be a ring.*

(i) *The Zariski topology is highly non-Hausdorff: for* $\mathfrak{p} \in \mathrm{Spec}(A)$*,* $\{\mathfrak{p}\}$ *is closed* iff $\mathfrak{p}$ *is a maximal ideal of* A.

(ii) *Closed subsets are stable by specialisation: if* Z *is a closed subset of* $\mathrm{Spec}(A)$*,* $\mathfrak{p} \in Z$*, then for every* $\mathfrak{q} \in \mathrm{Spec}(A)$ *s.t.* $\mathfrak{q} \supseteq \mathfrak{p}$*, we have* $\mathfrak{q} \in Z$.

(iii) *Open subsets are stable by generalisation: if* U *is an open subset of* $\mathrm{Spec}(A)$*,* $\mathfrak{p} \in U$*, then for every* $\mathfrak{q} \in \mathrm{Spec}(A)$ *s.t.* $\mathfrak{q} \subseteq \mathfrak{p}$*, we have* $\mathfrak{q} \in U$.

1.4 Relation between ideals and closed sets

Definition 1.4.1 (Radical of an ideal)**.** *Let* A *be a ring. For* $I \trianglelefteq A$*, we define the* radical *of* I *by:*

$$\sqrt{I} = \{a \in A,\ \exists n \in \mathbb{N},\ a^n \in I\}\,.$$

$\sqrt{I}$ *is an ideal of* A *s.t.* $A/\sqrt{I}$ *does not have any nilpotent element. Moreover, we have* $\sqrt{\sqrt{I}} = \sqrt{I}$.

Lemma 1.4.2. *Let* A *be a ring and* $I \trianglelefteq A$.

(i) $\sqrt{I} = \bigcap_{\mathfrak{p}\in V(I)} \mathfrak{p}$.

(ii) $V(I) = V\left(\sqrt{I}\right)$.

(iii) $\sqrt{I}$ *is maximal among all ideals* $J \trianglelefteq A$ *s.t.* $V(J) = V(I)$.

Proof. (i) Note that, if $x \in \sqrt{I}$, then there exists $n \in \mathbb{N}$ s.t. $x^n \in I$. Therefore, for all $\mathfrak{p} \in V(I)$, $x^n \in \mathfrak{p}$, and since $\mathfrak{p}$ is a prime ideal, $x \in \mathfrak{p}$. This proves that $\sqrt{I} \subseteq \bigcap_{\mathfrak{p} \in V(I)} \mathfrak{p}$. Conversely, we may assume that $I = (0)$ by replacing A by $A/\sqrt{I}$. If $x \notin \sqrt{(0)}$, then the set $S = \{x^n,\ n \in \mathbb{N}\}$ does not contain 0. Hence, the ring $A_x = S^{-1}A$ is nonzero, so $\operatorname{Spec}(A_x) \neq \varnothing$. But $\operatorname{Spec}(A_x)$ is in bijection with $\{\mathfrak{p} \in \operatorname{Spec}(A),\ S \cap \mathfrak{p} = \varnothing\}$, so there exists $\mathfrak{p} \in \operatorname{Spec}(A)$ s.t. $S \cap \mathfrak{p} = \varnothing$. In particular, $x \notin \mathfrak{p}$ so $x \notin \bigcap_{\mathfrak{p} \in \operatorname{Spec}(A)} \mathfrak{p}$. □

Corollary 1.4.3. *Let A be a ring. If $\mathcal{F}_{\operatorname{Spec}(A)}$ is the set of closed subsets of* $\operatorname{Spec}(A)$ *and $\mathcal{R}_A$ is the set of ideals $I \trianglelefteq A$ s.t. $I = \sqrt{I}$, then there is a bijection $\mathcal{F}_{\operatorname{Spec}(A)} \to \mathcal{R}_A$ given by:*

$$\left|\begin{array}{rcl} \mathcal{F}_{\operatorname{Spec}(A)} & \longrightarrow & \mathcal{R}_A \\ Z & \longmapsto & \bigcap_{\mathfrak{p} \in Z} \mathfrak{p} \end{array}\right. \quad \text{and} \quad \left|\begin{array}{rcl} \mathcal{R}_A & \longrightarrow & \mathcal{F}_{\operatorname{Spec}(A)} \\ I & \longmapsto & V(I) \end{array}\right. .$$

1.5 Functoriality

Proposition 1.5.1. *Let $\varphi : A \to B$ be a ring homomorphism. Then the map:*

$$\varphi^* : \mathfrak{p} \in \operatorname{Spec}(B) \longmapsto \varphi^{-1}(\mathfrak{p}) \in \operatorname{Spec}(A),$$

is a well-defined continuous map.

Proposition 1.5.2. *Let $\varphi : A \to B$ and $\psi : B \to C$ be two ring homomorphisms. Then $(\psi \circ \varphi)^* = \varphi^* \circ \psi^*$.*

Example 1.5.3. *Let A be a ring.*

(i) *If $I \trianglelefteq A$ is an ideal, then the canonical projection $\pi : A \to A/I$ induces a homeomorphism $\pi^* : \operatorname{Spec}(A/I) \to V(I) \subseteq \operatorname{Spec}(A)$.*

(ii) *If $f \in A \backslash \sqrt{(0)}$, then the natural localisation map $\phi : A \to A_f$ induces a homeomorphism $\phi^* : \operatorname{Spec}(A_f) \to D_f \subseteq \operatorname{Spec}(A)$.*

(iii) *If $\mathfrak{p} \in \operatorname{Spec}(A)$, then the natural localisation map $\phi : A \to A_{\mathfrak{p}}$ induces a homeomorphism $\phi^* : \operatorname{Spec}(A_{\mathfrak{p}}) \to \{\mathfrak{q} \in \operatorname{Spec}(A),\ \mathfrak{q} \subseteq \mathfrak{p}\} \subseteq \operatorname{Spec}(A)$.*

1.6 Quasi-compactness of $\operatorname{Spec}(A)$

Definition 1.6.1 (Quasi-compactness)**.** *A topological space X is said to be* quasi-compact *if one of the following two equivalent conditions is satisfied:*

(i) *If $(U_i)_{i \in I}$ is a collection of open subsets of X s.t. $\bigcup_{i \in I} U_i = X$, then there exists a finite set $J \subseteq I$ s.t. $\bigcup_{i \in J} U_i = X$.*

(ii) *If $(F_i)_{i \in I}$ is a collection of closed subsets of X s.t. $\bigcap_{i \in I} F_i = \varnothing$, then there exists a finite set $J \subseteq I$ s.t. $\bigcap_{i \in J} F_i = \varnothing$.*

Note that, in English, such a topological space is usually called compact.

Proposition 1.6.2. *If A is a ring, then* $\operatorname{Spec}(A)$ *is quasi-compact.*

Proof. Let $(F_\lambda)_{\lambda \in \Lambda} = (V(I_\lambda))_{\lambda \in \Lambda}$ be a collection of closed subsets of $\operatorname{Spec}(A)$. Then we have:

$$\bigcap_{\lambda \in \Lambda} V(I_\lambda) = \varnothing \iff V\left(\left(\bigcup_{\lambda \in \Lambda} I_\lambda\right)\right) = \varnothing \iff \left(\bigcup_{\lambda \in \Lambda} I_\lambda\right) = A \iff 1 \in \left(\bigcup_{\lambda \in \Lambda} I_\lambda\right).$$

But it is clear that if $1 \in (\bigcup_{\lambda \in \Lambda} I_\lambda)$, then there exists a finite set $M \subseteq \Lambda$ s.t. $1 \in (\bigcup_{\lambda \in M} I_\lambda)$. This proves the result. □

1.7 Noether Normalisation and Nullstellensatz

Definition 1.7.1 (Finite algebras, algebras of finite type)**.** *Let B be an A-algebra (i.e. a ring equipped with a homomorphism $f : A \to B$).*

(i) *B is said to be* of finite type *over A if there exists a surjective map of algebras $A[X_1, \dots, X_r] \to B$, with $r \in \mathbb{N}$.*

(ii) *B is said to be* finite *over A if there exists a surjective map of modules $A^r \to B$, with $r \in \mathbb{N}$.*

Note that a finite A-algebra is also of finite type, but the converse is false.

Example 1.7.2. *If k is a field, then $k[X]$ is of finite type, but not finite, over k.*

Definition 1.7.3 (Integral elements)**.** *Let A be an A_0-algebra. An element $x \in A$ is said to be* integral *over A_0 if there exists a monic polynomial $P \in A_0[T]$ s.t. $P(x) = 0$. The algebra A is said to be* integral *over A_0 if every element of A is integral over A_0.*

Lemma 1.7.4. *Let A be an A_0-algebra. The following assertions are equivalent:*

(i) *A is finite over A_0.*

(ii) *A is of finite type and integral over A_0.*

Theorem 1.7.5 (Noether Normalisation Theorem)**.** *Let A be a k-algebra of finite type, where k is a field. Then there exists $d \in \mathbb{N}$ and an injective homomorphism of k-algebras $k[T_1, \dots, T_d] \to A$ s.t. A is finite over $k[T_1, \dots, T_d]$.*

Corollary 1.7.6. *Let F/k be a field extension. If F is a k-algebra of finite type, then the field extension is finite.*

Proof. By Theorem 1.7.5, there exists $d \in \mathbb{N}$ and an injective map $k[T_1, \dots, T_d] \to F$ s.t. F is finite over $k[T_1, \dots, T_d]$. Assume for contradiction that $d \geq 1$. Then the element $T_1^{-1} \in F$ is integral over $k[T_1, \dots, T_d]$, so there exists a monic polynomial $P \in k[T_1, \dots, T_d][X]$ s.t. $P\left(T_1^{-1}\right) = 0$. Using this equality, we show that $T_1^{-1} \in k[T_1, \dots, T_d]$, which is false. Therefore, $d = 0$ and F is finite over k. □

Corollary 1.7.7. *Let A be a k-algebra of finite type, where k is a field. Let $\mathfrak{m} \trianglelefteq A$ be a maximal ideal. Then $A/\mathfrak{m}$ is a finite extension of k.*

Proof. Note that $A/\mathfrak{m}$ is a field extension of k that is of finite type, and apply Corollary 1.7.6. □

Corollary 1.7.8 (Nullstellensatz)**.** *Let k be an algebraically closed field. Then the maximal ideals of $k[T_1, \dots, T_n]$ are the ideals of the form $(T_1 - a_1, \dots, T_n - a_n)$, with $a_1, \dots, a_n \in k$.*

Proof. Start by showing that ideals of the given form are maximal. Conversely, let $\mathfrak{m} \trianglelefteq k[T_1, \dots, T_n]$ be a maximal ideal. By Corollary 1.7.7, $k[T_1, \dots, T_n]/\mathfrak{m}$ is a finite extension of k, so we have a map $f : k \to k[T_1, \dots, T_n]/\mathfrak{m}$. This map is actually an isomorphism because k is algebraically closed. Hence, for $i \in \{1, \dots, n\}$, let $a_i = f^{-1}\left(\overline{T}_i\right)$. Now show that $\mathfrak{m} \supseteq (T_1 - a_1, \dots, T_n - a_n)$. This inclusion is actually an equality because the right-hand side is known to be a maximal ideal. □

Remark 1.7.9. *The Nullstellensatz shows that the set of maximal ideals of $k[T_1, \dots, T_n]$ is in bijection with k^n. More generally, if $f_1, \dots, f_r \in k[T_1, \dots, T_n]$, then the set of maximal ideals of $k[T_1, \dots, T_n]/(f_1, \dots, f_r)$ is in bijection with $\{x \in k^n,\ f_1(x) = \dots = f_r(x) = 0\}$. These are the fundamental objects of classical algebraic geometry; and going from the classical point of view to the modern one amounts to adding the prime ideals to obtain the notion of scheme.*

2 Sheaves

2.1 Sheaves

Definition 2.1.1 (Presheaf)**.** *Let X be a topological space. A* presheaf *of abelian groups (resp. rings, A-modules, A-algebras, etc.) over X is the data of an abelian group (resp. ring, A-module, A-algebra, etc.) $\mathcal{F}(\mathcal{U})$ for each open set $\mathcal{U} \subseteq X$, together with homomorphisms $\rho_{\mathcal{V}\mathcal{U}} : \mathcal{F}(\mathcal{U}) \to \mathcal{F}(\mathcal{V})$ for each pair of open subsets $\mathcal{V} \subseteq \mathcal{U} \subseteq X$, satisfying $\mathcal{F}(\varnothing) = 0$, $\rho_{\mathcal{U}\mathcal{U}} = \mathrm{id}_{\mathcal{F}(\mathcal{U})}$ and $\rho_{\mathcal{W}\mathcal{V}} \circ \rho_{\mathcal{V}\mathcal{U}} = \rho_{\mathcal{W}\mathcal{U}}$ for $\mathcal{W} \subseteq \mathcal{V} \subseteq \mathcal{U} \subseteq X$.*

Vocabulary 2.1.2. *Let $\mathcal{F}$ be a presheaf over a topological space X. If $\mathcal{U} \subseteq X$ is an open set, then elements of $\mathcal{F}(\mathcal{U})$ will be called* sections *of $\mathcal{F}$ over $\mathcal{U}$. Moreover, for $s \in \mathcal{F}(\mathcal{U})$, $\mathcal{V} \subseteq \mathcal{U}$ open, $\rho_{\mathcal{V}\mathcal{U}}(s) \in \mathcal{F}(\mathcal{V})$ will be called the* restriction *of s to $\mathcal{V}$, and denoted by $s_{|\mathcal{V}}$. The set $\mathcal{F}(\mathcal{U})$ will sometimes be denoted by $\Gamma(\mathcal{U}, \mathcal{F})$ or $H^0(\mathcal{U}, \mathcal{F})$.*

Remark 2.1.3. *A presheaf of abelian groups (resp. rings, A-modules, A-algebras, etc.) over a topological space X is a contravariant functor from the opposite category of open subsets of X to the category of abelian groups (resp. rings, A modules, A-algebras, etc.).*

Definition 2.1.4 (Sheaf)**.** *Let X be a topological space equipped with a presheaf $\mathcal{F}$. We say that $\mathcal{F}$ is a* sheaf *if it satisfies the following two conditions:*

(i) Uniqueness. *Let $\mathcal{U} \subseteq X$ be an open set and let $(\mathcal{U}_i)_{i\in I}$ be an open covering of $\mathcal{U}$. If $s \in \mathcal{F}(\mathcal{U})$ is s.t. $\forall i \in I$, $s_{|\mathcal{U}_i} = 0$, then $s = 0$.*

(ii) Existence *(or* gluing condition*). Let $\mathcal{U} \subseteq X$ be an open set and let $(\mathcal{U}_i)_{i\in I}$ be an open covering of $\mathcal{U}$. If $(s_i)_{i\in I} \in \prod_{i\in I} \mathcal{F}(\mathcal{U}_i)$ s.t. $s_{i|\mathcal{U}_i\cap\mathcal{U}_j} = s_{j|\mathcal{U}_i\cap\mathcal{U}_j}$ for all $i, j \in I$, then there exists $s \in \mathcal{F}(\mathcal{U})$ s.t. $\forall i \in I$, $s_{|\mathcal{U}_i} = s_i$.*

Remark 2.1.5. *A presheaf $\mathcal{F}$ over a topological space X is a sheaf if and only if for any open set $\mathcal{U}$ and for any open covering $(\mathcal{U}_i)_{i\in I}$ of $\mathcal{U}$, the following sequence is exact:*

$$0 \longrightarrow \mathcal{F}(\mathcal{U}) \longrightarrow \prod_{i\in I} \mathcal{F}(\mathcal{U}_i) \longrightarrow \prod_{i,j\in I} \mathcal{F}(\mathcal{U}_i \cap \mathcal{U}_j),$$

where the map $\prod_{i\in I} \mathcal{F}(\mathcal{U}_i) \to \prod_{i,j\in I} \mathcal{F}(\mathcal{U}_i \cap \mathcal{U}_j)$ is given by $(s_i)_{i\in I} \mapsto \left(s_{i|\mathcal{U}_i\cap\mathcal{U}_j} - s_{j|\mathcal{U}_i\cap\mathcal{U}_j}\right)_{i,j\in I}$.

Example 2.1.6.

(i) *If X is a topological space, we have a sheaf $\mathcal{C}_X$ of continuous functions on X.*

(ii) *If X is a differentiable manifold, we have a sheaf $\mathcal{C}_X^\infty$ of smooth functions on X and a sheaf $\mathcal{A}_X^k$ of differential k-forms on X.*

(iii) *If X is a differentiable manifold and E is a smooth vector bundle over X, we have a sheaf $\mathcal{E}$ of smooth sections of E.*

(iv) *If X is a complex manifold, we have a sheaf $\mathcal{O}_X$ of holomorphic functions on X and a sheaf Ω_X^k of holomorphic differential k-forms on X.*

(v) *Let X be a topological space and A be an abelian group. Then we have a sheaf $\mathcal{F}$ of locally constant functions on X with values in A. This sheaf will be denoted by $\underline{A}$.*

Remark 2.1.7. *If X is a compact and connected complex manifold, then $\mathcal{O}_X(X) = \mathbb{C}$. On the other hand, if X is a compact Hausdorff topological space, X is determined by $\mathcal{C}_X(X)$ (according to Theorem 1.2.2).*

Example 2.1.8.

(i) *Let X be a topological space and A be an abelian group. Then we have a presheaf defined by $\mathcal{F}(\mathcal{U}) = A$ for all $\mathcal{U} \subseteq X$, but this presheaf is not a sheaf.*

(ii) *Let X be a differentiable manifold, and let $i \in \mathbb{N}$. Then we have a presheaf on X defined by $\mathcal{F}(\mathcal{U}) = H^i_{\mathrm{dR}}(\mathcal{U}, \mathbb{R})$ (the de Rham cohomology of $\mathcal{U}$) for all $\mathcal{U} \subseteq \mathbb{R}$, where the restriction maps are given by $j^* : H^i_{\mathrm{dR}}(\mathcal{V}, \mathbb{R}) \to H^i_{\mathrm{dR}}(\mathcal{U}, \mathbb{R})$ when we have an inclusion $j : \mathcal{U} \to \mathcal{V}$. This defines a presheaf but not a sheaf (because the information given by de Rham cohomology is global, not local).*

2.2 Digression on inductive limits

Definition 2.2.1 (Inductive limit)**.** *Let $(A_i)_{i \in I}$ be a collection of abelian groups indexed by a partially ordered set I. Assume that, for $i > j$, we have a homomorphism $\rho_{ji} : A_i \to A_j$ s.t. $\rho_{kj} \circ \rho_{ji} = \rho_{ki}$ if $i > j > k$. In other words, $(A_i)_{i \in I}$ defines a contravariant functor from I to the category of abelian groups. Now, the* inductive limit *(or* direct limit*) $\varinjlim_{i \in I} A_i$ is the abelian group defined by:*

$$\varinjlim_{i \in I} A_i = \left(\bigoplus_{i \in I} A_i \right) / H,$$

where H is the subgroup of $\bigoplus_{i \in I} A_i$ generated by $\{\rho_{ji}(x) - x,\ i > j,\ x \in A_i\}$. The abelian group $\varinjlim_{i \in I} A_i$ is equipped with natural maps $f_j : A_j \to \varinjlim_{i \in I} A_i$ for all j.

Proposition 2.2.2 (Universal property of inductive limits)**.** *Let $(A_i)_{i \in I}$ be a collection of abelian groups indexed by a partially ordered set I and equipped with maps $\rho_{ji} : A_i \to A_j$ for $i > j$. Let B be an abelian group and let $(g_i : A_i \to B)_{i \in I}$ be a collection of homomorphisms s.t. $g_j \circ \rho_{ji} = g_i$ for $i > j$. Then there exists a unique homomorphism $h : \varinjlim_{i \in I} A_i \to B$ s.t. $g_i = h \circ f_i$ for all i.*

$$\begin{tikzcd}
A_i \arrow[d] \arrow[dr] \arrow[drr, bend left] & & \\
A_j \arrow[d] \arrow[r] \arrow[rr, bend left] & \varinjlim_{i \in I} A_i \arrow[r, dashed] & B \\
A_k \arrow[ur] \arrow[urr, bend right] & &
\end{tikzcd}$$

Definition 2.2.3 (Filtrant partially ordered set)**.** *A partially ordered set I is said to be* filtrant *(or* directed*) if $\forall i, j \in I$, $\exists k \in I$, $i \geq k$* and *$j \geq k$.*

Proposition 2.2.4. *Let $(A_i)_{i \in I}$ be a collection of abelian groups indexed by a partially ordered set I and equipped with maps $\rho_{ji} : A_i \to A_j$ for $i > j$. If I is filtrant, then any element of $\varinjlim_{i \in I} A_i$ can be represented by an element of $\bigsqcup_{i \in I} A_i$.*

Proof. Consider an element u of $\varinjlim_{i \in I} A_i$ represented by $x_{i_1} + \cdots + x_{i_r} \in \bigoplus_{i \in I} A_i$, with $i_1, \ldots, i_r \in I$. Since I is filtrant, there exists $k \in I$ s.t. $k \leq i_1, \ldots, i_r$. Now, by definition, u is represented by $\rho_{ki_1}(x_{i_1}) + \cdots + \rho_{ki_r}(x_{i_r}) \in A_k$. $\square$

2.3 Back to sheaves

Definition 2.3.1 (Stalk)**.** *Let X be a topological space and let $\mathcal{F}$ be a presheaf on X. The* stalk *of $\mathcal{F}$ at a point $x \in X$ is defined by:*

$$\mathcal{F}_x = \varinjlim_{\mathcal{U} \ni x} \mathcal{F}(\mathcal{U}).$$

Elements of $\mathcal{F}_x$ are called germs *of $\mathcal{F}$ at x.*

Example 2.3.2.

(i) *If X is a topological space, then $\mathcal{C}_{X,x}$ is the group of germs of continuous functions around x.*

(ii) *If X is a differentiable manifold, then $\mathcal{C}^{\infty}_{X,x}$ (resp. $\mathcal{A}^{k}_{X,x}$) is the group of germs of smooth functions (resp. of differential k-forms) around x.*

Remark 2.3.3. *Let X be a topological space equipped with a presheaf $\mathcal{F}$. Since the partially ordered set $\{\mathcal{U} \subseteq X \text{ open}, \mathcal{U} \ni x\}$ is filtrant, Proposition 2.2.4 implies that each germ of $\mathcal{F}$ at x is represented by a pair $(\mathcal{U}, s)$, with $\mathcal{U} \subseteq X$ open, $\mathcal{U} \ni x$, and $s \in \mathcal{F}(\mathcal{U})$.*

Definition 2.3.4 (Value of a section)**.** *Let X be a topological space equipped with a presheaf $\mathcal{F}$. If $\mathcal{U}$ is an open subset of X containing a point x, and $s \in \mathcal{F}(\mathcal{U})$, the* value *of s at x, denoted by s_x, is the class of $(\mathcal{U}, s)$ in $\mathcal{F}_x$.*

Proposition 2.3.5. *Let X be a topological space equipped with a sheaf $\mathcal{F}$. Let $\mathcal{U}$ be an open subset of X. For $s \in \mathcal{F}(\mathcal{U})$, we have $s = 0$ iff $\forall x \in \mathcal{U}$, $s_x = 0$.*

Remark 2.3.6. *In Example 2.3.2, the sections of the sheaf $\mathcal{F}$ on X are functions defined on open subsets of X. However, the value of an element $f \in \mathcal{F}(\mathcal{U})$ at a point $x \in \mathcal{U}$ is a germ; it carries more information than the real number $f(x)$.*

Definition 2.3.7 (Morphisms of presheaves)**.** *Let $\mathcal{F}, \mathcal{G}$ be two presheaves (resp. sheaves) on a topological space X. A* morphism of presheaves *(resp.* of sheaves*) $\varphi : \mathcal{F} \to \mathcal{G}$ is the data, for all open subset $\mathcal{U} \subseteq X$, of a homomorphism $\varphi_{\mathcal{U}} : \mathcal{F}(\mathcal{U}) \to \mathcal{G}(\mathcal{U})$ s.t. for all open subsets $\mathcal{U} \supseteq \mathcal{V}$, the following diagram commutes:*

$$\begin{tikzcd}
\mathcal{F}(\mathcal{U}) \arrow[r, "\varphi_{\mathcal{U}}"] \arrow[d, "\rho_{\mathcal{V}\mathcal{U}}"'] & \mathcal{G}(\mathcal{U}) \arrow[d, "\rho_{\mathcal{V}\mathcal{U}}"] \\
\mathcal{F}(\mathcal{V}) \arrow[r, "\varphi_{\mathcal{V}}"] & \mathcal{G}(\mathcal{V})
\end{tikzcd}$$

In other words, a morphism from $\mathcal{F}$ to $\mathcal{G}$, viewed as functors from the opposite category of open subsets of X to the category of abelian groups, is simply a natural transformation between them.

Example 2.3.8. *Let X be a differentiable manifold. For $k \in \mathbb{N}$, the differential $\mathrm{d} : \mathcal{A}^{k}_{X} \to \mathcal{A}^{k+1}_{X}$ is a morphism of sheaves.*

Remark 2.3.9. *Let $\varphi : \mathcal{F} \to \mathcal{G}$ be a morphism between two presheaves on a topological space X. Then, for all $x \in X$, φ induces a (unique) natural homomorphism $\varphi_x : \mathcal{F}_x \to \mathcal{G}_x$, defined using the universal proprety of inductive limits (Proposition 2.2.2).*

2.4 Sheafification

Theorem 2.4.1. *Let $\mathcal{F}$ be a presheaf on a topological space X. Then there exists a unique sheaf (up to unique isomorphism) $\mathcal{F}^{+}$ on X, equipped with a morphism of presheaves $a : \mathcal{F} \to \mathcal{F}^{+}$, s.t. for any sheaf $\mathcal{G}$ on X and for any morphism of presheaves $\varphi : \mathcal{F} \to \mathcal{G}$, there exists a unique morphism of sheaves $\varphi^{+} : \mathcal{F}^{+} \to \mathcal{G}$ s.t. $\varphi = \varphi^{+} \circ a$.*

$$\begin{tikzcd}
\mathcal{F} \arrow[r, "\varphi"] \arrow[d, "a"'] & \mathcal{G} \\
\mathcal{F}^{+} \arrow[ur, dashed, "\varphi^{+}"'] &
\end{tikzcd}$$

The data of $\mathcal{F}^{+}$ together with a is called the sheafification *of $\mathcal{F}$.*

Proof. Given an open subset $\mathcal{U} \subseteq X$, define $\mathcal{F}^+(\mathcal{U})$ to be the set of maps $s : \mathcal{U} \to \bigsqcup_{x\in\mathcal{U}} \mathcal{F}_x$ s.t. $\forall x \in \mathcal{U}$, $s(x) \in \mathcal{F}_x$ and for all $x \in \mathcal{U}$, there exists an open neighbourhood $\mathcal{V}$ of x in $\mathcal{U}$ s.t. $s_{|\mathcal{V}} \in \mathcal{F}(\mathcal{V})$ (i.e. $\exists t \in \mathcal{F}(\mathcal{V}), \forall y \in \mathcal{V},\ s(y) = t_y$). The morphism $a_{\mathcal{U}} : \mathcal{F}(\mathcal{U}) \to \mathcal{F}^+(\mathcal{U})$ is defined in the natural way: for $s \in \mathcal{F}(\mathcal{U})$, $a_{\mathcal{U}}(s)$ is the map $\hat{s} : x \in \mathcal{U} \longmapsto s_x \in \bigsqcup_{x\in\mathcal{U}} \mathcal{F}_x$. It is clear that this defines a sheaf $\mathcal{F}^+$ and a morphism of presheaves $a : \mathcal{F} \to \mathcal{F}^+$. Moreover, it is easy to check that the universal property of sheafification is satisfied. □

Remark 2.4.2. *The sheafification of a sheaf is itself.*

Remark 2.4.3. *Let $\mathcal{F}$ be a presheaf on a topological space X. If $\mathcal{F}^+$ is the sheafification of $\mathcal{F}$, then $\mathcal{F}_x = \mathcal{F}^+_x$ for all $x \in X$.*

Example 2.4.4.

(i) *Let X be a topological space and let A be an abelian group. Consider the constant presheaf $\mathcal{F}$ on X equal to A (as in Example 2.1.8). Then the sheafification of $\mathcal{F}$ is the sheaf $\underline{A}$ of locally constant functions on X with values in A (c.f. Example 2.1.6).*

(ii) *Let X be a differentiable manifold and consider the presheaf $\mathcal{F}$ on X given by de Rham cohomology of degree $i \in \mathbb{N}^*$ (c.f. Example 2.1.8). Then the sheafification $\mathcal{F}^+$ of $\mathcal{F}$ is given by $\mathcal{F}^+(\mathcal{U}) = 0$ for all $\mathcal{U}$.*

2.5 Operations on sheaves

Definition 2.5.1 (Kernel)**.** *Let $\varphi : \mathcal{F} \to \mathcal{G}$ be a morphism between two sheaves on a topological space X. Then we have a presheaf* $\operatorname{Ker}\varphi$ *on X given by:*

$$(\operatorname{Ker}\varphi)(\mathcal{U}) = \operatorname{Ker}(\varphi_{\mathcal{U}}) \subseteq \mathcal{F}(\mathcal{U}),$$

with the restriction maps induced by those of $\mathcal{F}$. The presheaf $\operatorname{Ker}\varphi$ *is actually a sheaf.*

Definition 2.5.2 (Image and cokernel)**.** *Let $\varphi : \mathcal{F} \to \mathcal{G}$ be a morphism between two sheaves on a topological space X.*

(i) $\operatorname{Im}\varphi$ *is the sheafification of the presheaf on X given by $\mathcal{U} \mapsto \operatorname{Im}(\varphi_{\mathcal{U}})$.*

(ii) $\operatorname{Coker}\varphi$ *is the sheafification of the presheaf on X given by $\mathcal{U} \mapsto \operatorname{Coker}(\varphi_{\mathcal{U}})$.*

Proposition 2.5.3. *Let $\varphi : \mathcal{F} \to \mathcal{G}$ be a morphism between two sheaves on a topological space X. For $x \in X$, we have:*

$$(\operatorname{Ker}\varphi)_x = \operatorname{Ker}(\varphi_x), \qquad (\operatorname{Im}\varphi)_x = \operatorname{Im}(\varphi_x), \qquad (\operatorname{Coker}\varphi)_x = \operatorname{Coker}(\varphi_x).$$

Definition 2.5.4 (Injectivity, surjectivity, bijectivity)**.** *A morphism of sheaves $\varphi : \mathcal{F} \to \mathcal{G}$ is called* injective *(resp.* surjective*,* bijective*) if for all $x \in X$, $\varphi_x : \mathcal{F}_x \to \mathcal{G}_x$ is injective (resp. surjective, bijective).*

Definition 2.5.5 (Exact sequence)**.** *A sequence $\mathcal{E} \xrightarrow{\varphi} \mathcal{F} \xrightarrow{\psi} \mathcal{G}$ of morphisms of sheaves on X is called* exact *if for all $x \in X$, the sequence $\mathcal{E}_x \xrightarrow{\varphi_x} \mathcal{F}_x \xrightarrow{\psi_x} \mathcal{G}_x$ is exact.*

Example 2.5.6. *Let X be a complex manifold. Consider the sheaf $\mathcal{O}_X$ of holomorphic functions on X and the sheaf $\mathcal{O}_X^\times$ of holomorphic functions on X with values in $\mathbb{C}^\times$. Then we have a morphism of sheaves* $\exp : \mathcal{O}_X \to \mathcal{O}_X^\times$ *given by $f \mapsto e^f$. This morphism is surjective (this is a consequence of the local existence of the logarithm). However,* $\exp_{\mathcal{U}} : \mathcal{O}_X(\mathcal{U}) \to \mathcal{O}_X^\times(\mathcal{U})$ *is often non-surjective; for instance, if $X = \mathcal{U} = \mathbb{C}^\times$, then $z \in \mathcal{O}_X^\times(\mathcal{U})$ has no preimage by* $\exp$.

Definition 2.5.7 (Direct image and inverse image)**.** *Let $f : X \to Y$ be a continuous map between topological spaces.*

(i) *Consider a sheaf $\mathcal{F}$ on X. We define a presheaf $f_*\mathcal{F}$ on Y, called the* pushforward, *or* direct image *of $\mathcal{F}$, by:*

$$f_*\mathcal{F}(\mathcal{V}) = \mathcal{F}\left(f^{-1}(\mathcal{V})\right),$$

with restriction maps defined in the natural way. Then $f_*\mathcal{F}$ is a sheaf on Y.

(ii) *Consider a sheaf $\mathcal{G}$ on Y. We define a presheaf $\mathcal{F}$ on X by $\mathcal{F}(\mathcal{U}) = \varinjlim_{\mathcal{V}\supseteq f(\mathcal{U})} \mathcal{G}(\mathcal{V})$, and we denote by $f^{-1}\mathcal{G}$ the sheafification of $\mathcal{F}$. Hence, $f^{-1}\mathcal{G}$ is a sheaf on X, called the* inverse image *of $\mathcal{G}$. For $x \in X$, we have:*

$$\left(f^{-1}\mathcal{G}\right)_x = \mathcal{G}_{f(x)}.$$

Example 2.5.8. *Let Y be a topological space and $X \subseteq Y$ be an open subset. Denote by $j : X \to Y$ the inclusion map. If $\mathcal{F}$ is a sheaf on Y, then the sheaf $j^{-1}\mathcal{F}$ on X will be denoted by $\mathcal{F}_{|X}$ and called the* restriction *of $\mathcal{F}$ to X; it satisfies:*

$$\mathcal{F}_{|X}(\mathcal{U}) = \mathcal{F}(\mathcal{U}).$$

3 Schemes

3.1 Structure sheaf on $\mathrm{Spec}(A)$

Remark 3.1.1. *Let A be a ring. Our goal is now to construct a sheaf $\mathcal{O}_X$ of rings on the topological space $X = \mathrm{Spec}(A)$ s.t.*

(i) $\forall \mathfrak{p} \in X$, $\mathcal{O}_{X,\mathfrak{p}} \simeq A_{\mathfrak{p}}$,

(ii) $\forall f \in A$, $\mathcal{O}_X(D_f) \simeq A_f$.

In particular, we will have $A = A_1 \simeq \mathcal{O}_X(D_1) = \mathcal{O}_X(X)$.

Definition 3.1.2 (Structure sheaf)**.** *Let A be a ring. Consider the topological space $X = \mathrm{Spec}(A)$, equipped with the Zariski topology. We define a presheaf $\mathcal{O}_X$ on X as follows. If $\mathcal{U} \subseteq X$ is an open subset, we define $\mathcal{O}_X(\mathcal{U})$ to be the set of maps $s : \mathcal{U} \to \bigsqcup_{\mathfrak{p}\in\mathcal{U}} A_{\mathfrak{p}}$ s.t. $\forall \mathfrak{p} \in \mathcal{U}$, $s(\mathfrak{p}) \in A_{\mathfrak{p}}$ and for all $\mathfrak{p} \in \mathcal{U}$, there exists an open neighbourhood $\mathcal{V}$ of $\mathfrak{p}$ in $\mathcal{U}$ and some elements $a, f \in A$ s.t. for all $\mathfrak{q} \in \mathcal{V}$, we have $f \notin \mathfrak{q}$ and $s(\mathfrak{q}) = \frac{a}{f}$. The restriction maps are the usual ones. Then $\mathcal{O}_X$ is a sheaf on X, called the* structure sheaf *of X. When there is no ambiguity, we shall write $\mathcal{O}$ instead of $\mathcal{O}_X$.*

Lemma 3.1.3. *Let A be a ring and $f, g \in A$. Then $D_g \subseteq D_f \iff V((g)) \supseteq V((f)) \iff g \in \sqrt{(f)}$. In this case, there is a natural homomorphism $\rho_{g,f} : A_f \to A_g$.*

Theorem 3.1.4. *Let A be a ring. For all $f \in A$, there exists an isomorphism $\psi_f : A_f \to \mathcal{O}(D_f)$ s.t. if $f, g \in A$ with $D_g \subseteq D_f$, then the following diagram commutes:*

$$\begin{tikzcd}
A_f \arrow[r, "\rho_{g,f}"] \arrow[d, "\psi_f"'] & A_g \arrow[d, "\psi_g"] \\
\mathcal{O}(D_f) \arrow[r, "\rho_{D_g D_f}"] & \mathcal{O}(D_g)
\end{tikzcd}$$

Proof. *Construction of ψ_f.* Set $\psi_f : \frac{a}{f^m} \in A_f \longmapsto \left(\mathfrak{p} \mapsto \frac{a}{f^m}\right) \in \mathcal{O}(D_f)$. The commutativity of the above diagram is clear. *Injectivity of ψ_f.* Let $\frac{a}{f^m} \in \mathrm{Ker}\,\psi_f$, i.e. $\forall \mathfrak{p} \in D_f$, $\frac{a}{f^m} = 0$ in $A_{\mathfrak{p}}$. Therefore, $\forall \mathfrak{p} \in D_f$, $\exists y \in A\backslash\mathfrak{p}$, $ya = 0$. In other words, $\mathrm{Ann}(a) \not\subseteq \mathfrak{p}$ for all $\mathfrak{p} \in D_f$; this means that $D_f \subseteq X\backslash V(\mathrm{Ann}(a))$, i.e. $V(\mathrm{Ann}(a)) \subseteq X\backslash D_f = V((f))$, so $f \in \sqrt{\mathrm{Ann}(a)}$ and $a = 0$ in A_f. *Surjectivity of ψ_f.* Let $s \in \mathcal{O}(D_f)$. By definition of the structure sheaf, there exists an open cover $(V_i)_{i\in I}$ of D_f, where we can assume that $V_i = D_{f_i}$ for some $f_i \in A$, and there exist $a_i, g_i \in A$ s.t. for all $\mathfrak{q} \in V_i$, $g_i \notin \mathfrak{q}$ and $s(\mathfrak{q}) = \frac{a_i}{g_i}$ in $A_\mathfrak{q}$. As D_f is homeomorphic to $\operatorname{Spec}(A_f)$ (by Example 1.5.3), which is quasi-compact (by Proposition 1.6.2), we may assume I to be finite. Now, for every $i \in I$, we see that $D_{f_i} \subseteq D_{g_i}$; by Lemma 3.1.3, we have $f_i \in \sqrt{(g_i)}$, so we may write $b_i g_i = f_i^{\ell_i}$. By setting $c_i = a_i b_i$, we obtain:

$$\forall \mathfrak{q} \in V_i,\ s(\mathfrak{q}) = \frac{c_i}{f_i^{\ell_i}} \quad \text{in } A_\mathfrak{q}.$$

Now, note that $D_{f_i} = D_{f_i^{\ell_i}}$, so we may replace f_i by $f_i^{\ell_i}$; in other words, we may assume that $\ell_i = 1$, which gives:

$$\forall \mathfrak{q} \in V_i,\ s(\mathfrak{q}) = \frac{c_i}{f_i} \quad \text{in } A_\mathfrak{q}.$$

Now, for $i, j \in I$, we have $D_{f_i} \cap D_{f_j} = D_{f_i f_j}$, and $\left(\frac{c_i}{f_i}\right)_{|D_{f_i f_j}} = \left(\frac{c_j}{f_j}\right)_{|D_{f_i f_j}}$, i.e. $\psi_{f_i f_j}\left(\frac{c_i}{f_i}\right) = \psi_{f_i f_j}\left(\frac{c_j}{f_j}\right)$ in $\mathcal{O}\left(D_{f_i f_j}\right)$. As $\psi_{f_i f_j}$ is injective, there exist $n_{ij} \in \mathbb{N}$ s.t. $(f_i f_j)^{n_{ij}}(c_i f_j - c_j f_i) = 0$. As I is finite, there exists $n \in \mathbb{N}$ s.t.

$$\forall i, j \in I,\ (f_i f_j)^n (c_i f_j - c_j f_i) = 0.$$

We now replace c_i by $c_i f_i^n$ and f_i by f_i^{n+1} (because $D_{f_i} = D_{f_i^{n+1}}$), which gives:

$$\forall i, j \in I,\ c_i f_j = c_j f_i.$$

But $D_f = \bigcup_{i \in I} V_i = \bigcup_{i \in I} D_{f_i}$, so $V((f)) = V((f_i,\ i \in I))$ and $f \in \sqrt{(f_i,\ i \in I)}$. Therefore, there exist $m \in \mathbb{N}$, $d_i \in A$ s.t.

$$f^m = \sum_{i \in I} f_i d_i.$$

Therefore, for all $i \in I$, $c_i f^m = \sum_{j \in I} c_i f_j d_j = \left(\sum_{j \in I} c_j d_j\right) f_i$, so $\frac{c_i}{f_i} = \frac{\sum_{j \in I} c_j d_j}{f^m}$ in A_{f_i}, which shows that:

$$s = \psi_f\left(\frac{\sum_{j \in I} c_j d_j}{f^m}\right).$$

□

Remark 3.1.5. *To prove the surjectivity of ψ_f, we used a technique that is analogous to partitions of unity in differential geometry.*

Corollary 3.1.6. *Let A be a ring. If $X = \operatorname{Spec}(A)$, then:*

$$\mathcal{O}_X(X) \simeq A.$$

Remark 3.1.7. *By Corollary 3.1.6, we can recover a ring A from* $\operatorname{Spec}(A)$ *equipped with its structure sheaf. For instance, if k is a field, then* $\operatorname{Spec}(k)$ *and* $\operatorname{Spec}\left(k[T]/(T^2)\right)$ *are homeomorphic (they are both singletons) but they have different structure sheaves.*

Lemma 3.1.8. *If A is a ring and $\mathfrak{p} \in \operatorname{Spec}(A)$, there is a natural isomorphism $\varphi : \varinjlim_{f \notin \mathfrak{p}} A_f \to A_\mathfrak{p}$.*

Proposition 3.1.9. *Let A be a ring, $X = \operatorname{Spec}(A)$. Then, for any $\mathfrak{p} \in X$:*

$$\mathcal{O}_{X,\mathfrak{p}} \simeq A_\mathfrak{p}.$$

3.2 Ringed spaces and affine schemes

Definition 3.2.1 (Ringed space)**.** *A* (locally) ringed space *is a topological space X, together with a sheaf of rings $\mathcal{O}_X$ s.t. the stalk $\mathcal{O}_{X,x}$ is a local ring for any $x \in X$. In this case:*

- *$\mathcal{O}_X$ is called the* structure sheaf *of X.*

- *For $x \in X$, $\mathcal{O}_{X,x}$ is called the* local ring *of X at x, its maximal ideal $\mathfrak{M}_x$ is called the* maximal ideal *at x and its residue field $\kappa(x) = \mathcal{O}_{X,x}/\mathfrak{M}_x$ is called the* residue field *at x.*

Example 3.2.2.

(i) *If A is a ring, then $\left(\operatorname{Spec}(A), \mathcal{O}_{\operatorname{Spec}(A)}\right)$ is a ringed space.*

(ii) *If X is a topological space, then $(X, \mathcal{C}_X)$ is a ringed space.*

(iii) *If X is a differentiable manifold, then $(X, \mathcal{C}_X^\infty)$ is a ringed space.*

Definition 3.2.3 (Local ring homomorphism)**.** *If A and B are two local rings with respective maximal ideals $\mathfrak{M}_A$ and $\mathfrak{M}_B$, then a ring homomorphism $f : A \to B$ is said to be* local *if $f^{-1}(\mathfrak{M}_B) = \mathfrak{M}_A$.*

Definition 3.2.4 (Morphism of ringed spaces)**.** *Let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be two ringed spaces. A* morphism of ringed spaces *between X and Y is the following data:*

- *A continuous map $f : X \to Y$,*
- *A morphism $f^\sharp : \mathcal{O}_Y \to f_*\mathcal{O}_X$ of sheaves of rings on Y s.t. for all $x \in X$, the natural homomorphism $\mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$ is local.*

Definition 3.2.5 (Affine scheme, affine variety)**.**

(i) *An* affine scheme *is a ringed space $(X, \mathcal{O}_X)$ that is isomorphic to $\left(\operatorname{Spec}(A), \mathcal{O}_{\operatorname{Spec}(A)}\right)$ for some ring A.*

(ii) *An* affine variety *is an affine scheme $(X, \mathcal{O}_X)$ that is isomorphic to $\left(\operatorname{Spec}(A), \mathcal{O}_{\operatorname{Spec}(A)}\right)$, where A is a k-algebra of finite type and k is a field (i.e. $A \simeq k[T_1, \ldots, T_n] / (f_1, \ldots, f_m)$).*

(iii) *A* morphism *between affine schemes is a morphism between the underlying ringed spaces.*

Example 3.2.6. *Let A be a ring.*

(i) $\operatorname{Spec}(A)$ *is an affine scheme by definition.*

(ii) *If $f \in A$, then D_f is an affine scheme that is isomorphic to* $\operatorname{Spec}(A_f)$.

(iii) *If $I \trianglelefteq A$ is an ideal, then $V(I)$ is an affine scheme that is isomorphic to* $\operatorname{Spec}(A/I)$.

Theorem 3.2.7. *Let $\varphi : A \to B$ be a homomorphism of rings. Then there is a naturally associated morphism of affine schemes $\varphi^* : \operatorname{Spec}(B) \to \operatorname{Spec}(A)$. Moreover, this defines an equivalence between the opposite category of rings and the category of affine schemes. More precisely, for all rings A and B, the map $\Phi : \operatorname{Hom}_{\text{rings}}(A, B) \longrightarrow \operatorname{Mor}(\operatorname{Spec}(B), \operatorname{Spec}(A))$ defined by $\varphi \mapsto \varphi^*$ is a bijection.*

Proof. Let $X = \operatorname{Spec}(A)$ and $Y = \operatorname{Spec}(B)$. By Proposition 1.5.1, $\varphi : A \to B$ induces a continuous map $f : Y \to X$. It remains to define a morphism $f^\sharp : \mathcal{O}_X \to f_*\mathcal{O}_Y$. To do this, it suffices to define $f^\sharp$ on the basis of the topology of X given by principal open subsets. But for $g \in A$, we have:

$$f_*\mathcal{O}_Y(D_g) = \mathcal{O}_Y\left(f^{-1}(D_g)\right) = \mathcal{O}_Y\left(D_{\varphi(g)}\right).$$

Therefore, by Theorem 3.1.4, we have $\mathcal{O}_X(D_g) \simeq A_g$ and $f_*\mathcal{O}_Y(D_g) \simeq B_{\varphi(g)}$. Now, we have a natural homomorphism $A_g \to B_{\varphi(g)}$ induced by φ, so we also have a homomorphism $f^\sharp_{D_g} : \mathcal{O}_X(D_g) \to f_*\mathcal{O}_Y(D_g)$. For the second part of the theorem, it suffices to construct an inverse for Φ. To do this, define:

$$\Psi : \left(f, f^\sharp\right) \in \operatorname{Mor}(\operatorname{Spec}(B), \operatorname{Spec}(A)) \longmapsto f^\sharp_{\operatorname{Spec}(A)} \in \operatorname{Hom}_{\text{rings}}(A, B).$$

Ψ is well-defined: as $\mathcal{O}_{\operatorname{Spec}(A)}(\operatorname{Spec}(A)) \simeq A$ and $\mathcal{O}_{\operatorname{Spec}(B)}(\operatorname{Spec}(B)) \simeq B$, the homomorphism $f^\sharp_{\operatorname{Spec}(A)} : \mathcal{O}_{\operatorname{Spec}(A)}(\operatorname{Spec}(A)) \to f_*\mathcal{O}_{\operatorname{Spec}(B)}(\operatorname{Spec}(A)) = \mathcal{O}_{\operatorname{Spec}(B)}(\operatorname{Spec}(B))$ can be identified with a map $A \to B$. It remains to check that Φ and Ψ are inverse maps. $\square$

Proposition 3.2.8. *If $\varphi : A \to B$ and $\psi : B \to C$ are ring homomorphisms, then $(\psi \circ \varphi)^* = \varphi^* \circ \psi^*$.*

3.3 Schemes

Definition 3.3.1 (Schemes)**.**

- *A* scheme *is a ringed space* $(X, \mathcal{O}_X)$ *admitting an open cover* $(\mathcal{U}_i)_{i\in I}$ *s.t. for all* $i \in I$, $\left(\mathcal{U}_i, \mathcal{O}_{X|\mathcal{U}_i}\right)$ *is an affine scheme.*

- *A* morphism of schemes *is simply a morphism between the underlying ringed spaces.*

Definition 3.3.2 (Regular functions)**.** *Let X be a scheme. For any open subset $\mathcal{U} \subseteq X$, elements of $\mathcal{O}_X(\mathcal{U})$ will be called* regular functions *on $\mathcal{U}$. For $x \in \mathcal{U}$ and $f \in \mathcal{O}_X(\mathcal{U})$, the element $f(x) \in \kappa(x) = \mathcal{O}_{X,x}/\mathfrak{M}_x$ will be called the* value *of f at x.*

Remark 3.3.3. *Let $\varphi = \left(f, f^\sharp\right) : X \to Y$ be a morphism of schemes.*

(i) *If $x \in X$ and $y = f(x) \in Y$, we have a local homomorphism $\mathcal{O}_{Y,y} \to \mathcal{O}_{X,x}$ inducing a field extension $\kappa(y) \to \kappa(x)$.*

(ii) *If $g \in \mathcal{O}_Y(Y)$, then the* pullback *of g is defined by $\varphi^* g = f_Y^\sharp(g) \in \mathcal{O}_X(X)$.*

If $g \in \mathcal{O}_Y(Y)$, $x \in X$ and $y = f(x)$, then $g(y) = (\varphi^* g)(x)$ (viewing $\kappa(y)$ as a subfield of $\kappa(x)$).

Proposition 3.3.4. *Let X be a scheme. For any open subset $\mathcal{U} \subseteq X$, the ringed space $\left(\mathcal{U}, \mathcal{O}_{X|\mathcal{U}}\right)$ is a scheme.*

Proof. Let $(\mathcal{U}_i)_{i\in I}$ be an open cover of X consisting of affine schemes. For $i \in I$, $\mathcal{U} \cap \mathcal{U}_i$ is an open subset of the affine scheme $\mathcal{U}_i$, so it admits an open cover $\left(V_j^i\right)_{j\in J_i}$ consisting of principal open subsets of $\mathcal{U}_i$ (which are in particular affine schemes). Now $\mathcal{U} = \bigcup_{i\in I} \bigcup_{j\in J_i} V_j^i$ is a scheme. □

Example 3.3.5. *Let $X = \mathbb{A}^2 = \operatorname{Spec}(k[t_1, t_2])$ be the affine plane, and let 0 be the (closed) point corresponding to the maximal ideal (t_1, t_2). Then $\mathcal{U} = \mathbb{A}^2 \backslash \{0\}$ is a scheme by Proposition 3.3.4 but it is not an affine scheme. To prove this fact, note that $\mathcal{U} = D_{t_1} \cup D_{t_2}$, so we have an exact sequence:*

$$0 \longrightarrow \mathcal{O}_X(\mathcal{U}) \longrightarrow \mathcal{O}_X(D_{t_1}) \oplus \mathcal{O}_X(D_{t_2}) \longrightarrow \mathcal{O}_X(D_{t_1} \cap D_{t_2}) = \mathcal{O}_X(D_{t_1 t_2}).$$

Using this exact sequence, we compute $\mathcal{O}_{\mathcal{U}}(\mathcal{U}) = \mathcal{O}_X(\mathcal{U}) = k[t_1, t_2]$. Thus, we have an isomorphism $\mathcal{O}_X(X) \to \mathcal{O}_{\mathcal{U}}(\mathcal{U})$ but the induced map is the inclusion $\mathcal{U} \to X$, which is not an isomorphism. By Theorem 3.2.7, X is not an affine scheme.

4 Morphisms

4.1 Points

Definition 4.1.1 (Relative schemes)**.** *Fix a base scheme S. An* S-scheme *(or a* scheme over S*) is a scheme X, together with a morphism to S, denoted by X/S. A* morphism of S-schemes *(or a* S-morphism*) $f : X/S \to Y/S$ is a commutative diagram:*

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr] & & Y \arrow[dl] \\ & S & \end{tikzcd}$$

We denote by $\operatorname{Mor}_S(X, Y)$ the set of all S-morphisms.

Remark 4.1.2. *Any scheme is naturally a* $\operatorname{Spec}(\mathbb{Z})$*-scheme. This is because in general, if X is a scheme and A is a ring,* $\operatorname{Mor}(X, \operatorname{Spec}(A))$ *is in bijection with* $\operatorname{Hom}(A, \mathcal{O}_X(X))$.

Definition 4.1.3 (Points)**.** *Let A be a ring and let B be an A-algebra. Then* $\mathrm{Spec}(B)$ *is naturally an A-scheme (i.e. a* $\mathrm{Spec}(A)$*-scheme). Let X be another A-scheme. A B-*point *of X is an A-morphism (i.e. a* $\mathrm{Spec}(A)$*-morphism) from* $\mathrm{Spec}(B)$ *to X, i.e. a commutative diagram:*

$$\begin{tikzcd} \mathrm{Spec}(B) \arrow[rr] \arrow[dr] & & X \arrow[dl] \\ & \mathrm{Spec}(A) & \end{tikzcd}$$

The set $\mathrm{Mor}_{\mathrm{Spec}(A)}(\mathrm{Spec}(B), X)$ *of B-points of X will be denoted by $X(B)$.*

Example 4.1.4.

(i) *If L/k is a field extension, and X is a k-scheme, then an L-point of X is the data of a point $x \in X$ together with a morphism $\kappa(x) \to L$ which fixes k.*

(ii) *If k is a field and X is a k-scheme, then a k-point of X is a point $x \in X$ s.t. $\kappa(x) = k$.*

(iii) *Let $X = \mathrm{Spec}\left(\mathbb{Z}\left[T_1, \ldots, T_n\right] / \left(f_1, \ldots, f_r\right)\right)$. If A is a ring, then:*

$$X(A) = \left\{(a_1, \ldots, a_n) \in A^n,\ \forall i \in \{1, \ldots, r\},\ f_i\left(a_1, \ldots, a_n\right) = 0\right\}.$$

(iv) *Let k be a field and let X be a k-scheme. If $A = k[T]/\left(T^2\right)$, then:*

$$X(A) = \left\{(x, v),\ x \in X,\ \kappa(x) = k,\ v \in \left(\mathfrak{M}_x/\mathfrak{M}_x^2\right)^{\wedge}\right\}.$$

For $x \in X$, the $\kappa(x)$-vector space $T_xX = \left(\mathfrak{M}_x/\mathfrak{M}_x^2\right)^{\wedge}$ will be called the tangent space *of X at x. In some sense, $X(A)$ corresponds to the* tangent bundle *of X.*

4.2 Immersions

Definition 4.2.1 (Open subschemes and open immersions)**.** *Let X be a scheme.*

- *An* open subscheme *of X is an open subset $\mathcal{U} \subseteq X$ equipped with the structure sheaf $\mathcal{O}_{X|\mathcal{U}}$.*
- *An* open immersion *is a morphism of schemes $f : X \to Y$ s.t. f induces an isomorphism of schemes between X and an open subscheme of Y.*

Example 4.2.2. *Let A be a ring. If $f \in A \backslash \sqrt{(0)}$, then the natural map $A \to A_f$ induces an open immersion* $\mathrm{Spec}\left(A_f\right) \to D_f \subseteq \mathrm{Spec}(A)$.

Definition 4.2.3 (Closed subschemes and closed immersions)**.** *Let Y be a scheme.*

- *A* closed immersion *is a morphism of schemes $f : X \to Y$ s.t.* (1) *f induces a homeomorphism from X to a closed subset of Y and* (2) *the morphism $f^\sharp : \mathcal{O}_Y \to f_*\mathcal{O}_X$ is surjective.*
- *A* closed subscheme *of Y is a closed subset X together with a closed immersion $f : X \to Y$ whose underlying set-theoretical map is the inclusion.*

Remark 4.2.4. *The notion of closed subscheme is much more subtle than that of open subscheme, because a closed subset of a scheme does not have a canonical scheme structure.*

Example 4.2.5. *Let A be a ring and let $I, J \trianglelefteq A$ be two ideals of A s.t. $\sqrt{I} = \sqrt{J}$. If $Z = V(I) = V(J)$, then we have two closed immersions* $\mathrm{Spec}\left(A/I\right) \to Z \subseteq \mathrm{Spec}(A)$ *and* $\mathrm{Spec}\left(A/J\right) \to Z \subseteq \mathrm{Spec}(A)$ *(given by the projection $A \to A/I$ and $A \to A/J$), and they induce different subscheme structures on Z.*

4.3 Closed immersions to an affine scheme

Lemma 4.3.1. *Let A be a ring and let $f \in A$. Then:*

$$f \in A^\times \iff \forall x \in \operatorname{Spec}(A),\ f(x) \neq 0.$$

Moreover, if $f \in A^\times$, then its inverse is unique.

Lemma 4.3.2 (Non-vanishing loci of functions)**.** *Let X be a scheme and let $f \in \mathcal{O}_X(X)$. Define:*

$$X_f = \{x \in X,\ f(x) \neq 0\}.$$

(i) *X_f is open, and hence has a canonical open subscheme structure.*

(ii) *Suppose that X satisfies the following condition: $(*)$ There exists a finite affine open cover $(\mathcal{U}_i)_{i\in I}$ of X s.t. for all $i, j \in I$, $\mathcal{U}_i \cap \mathcal{U}_j$ has a finite affine open cover. Then the restriction map $\mathcal{O}_X(X) \to \mathcal{O}_X(X_f)$ induces an isomorphism $\mathcal{O}_X(X)_f \to \mathcal{O}_X(X_f)$.*

Proof. Note that, in the case where X is affine, the lemma is a consequence of Theorem 3.1.4. (i) It suffices to show that for any affine open subscheme $\mathcal{U} \subseteq X$, $\mathcal{U} \cap X_f$ is open. But $\mathcal{U} \cap X_f = \{x \in \mathcal{U},\ f(x) \neq 0\} = \mathcal{U}_f$, so we may assume that X is affine, and we already know the result for affine schemes. (ii) We need to show firstly that f is invertible in $\mathcal{O}_X(X_f)$ (so that the map $\mathcal{O}_X(X)_f \to \mathcal{O}_X(X_f)$ is well-defined). Cover X_f with an affine open cover $(\mathcal{U}_i)_{i\in I}$. Using Lemma 4.3.1, we see that $f_{|\mathcal{U}_i}$ is invertible for all i, with inverse g_i. Using an affine open cover of $\mathcal{U}_i \cap \mathcal{U}_j$, and the uniqueness of the inverse, we show that $g_{i|\mathcal{U}_i\cap\mathcal{U}_j} = g_{j|\mathcal{U}_i\cap\mathcal{U}_j}$, so we may use the gluing axiom to construct $g \in \mathcal{O}_X(X_f)$ s.t. $g_i = g_{|\mathcal{U}_i}$. Hence, g is an inverse of f. Thus, the map $\mathcal{O}_X(X)_f \to \mathcal{O}_X(X_f)$ is well-defined. We now consider a finite affine open cover $(\mathcal{U}_i)_{1\leq i\leq n}$ of X s.t. $\mathcal{U}_i \cap \mathcal{U}_j$ has a finite affine open cover for all i, j. Using the sheaf axioms for $\mathcal{O}_X$ and the fact that $\mathcal{O}_X(X)_f$ is flat over $\mathcal{O}_X(X)$, we obtain the following commutative diagram with exact rows (note that localisation at f commutes with finite products), where $V_i = \mathcal{U}_i \cap X_f$:

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{O}_X(X) \arrow[r] \arrow[d] & \prod_i \mathcal{O}_X(\mathcal{U}_i) \arrow[r] \arrow[d] & \prod_{i,j} \mathcal{O}_X(\mathcal{U}_i \cap \mathcal{U}_j) \arrow[d] \\
0 \arrow[r] & \mathcal{O}_X(X)_f \arrow[r] \arrow[d, hook] & \prod_i \mathcal{O}_X(\mathcal{U}_i)_f \arrow[r] \arrow[d, "\simeq"'] & \prod_{i,j} \mathcal{O}_X(\mathcal{U}_i \cap \mathcal{U}_j)_f \arrow[d, hook] \\
0 \arrow[r] & \mathcal{O}_X(X_f) \arrow[r] & \prod_i \mathcal{O}_X(V_i) \arrow[r] & \prod_{i,j} \mathcal{O}_X(V_i \cap V_j)
\end{tikzcd}$$

As $\mathcal{U}_i$ is affine and we know the result for affine schemes, we obtain that the map $\prod_i \mathcal{O}_X(\mathcal{U}_i)_f \longrightarrow \prod_i \mathcal{O}_X(V_i)$ is an isomorphism. Using this, we easily show that the map $\mathcal{O}_X(X)_f \longrightarrow \mathcal{O}_X(X_f)$ is injective. The same proof now works if we replace X by $\mathcal{U}_i \cap \mathcal{U}_j$ for any i, j (we only used the fact that X has a finite affine open cover); thus, the map $\prod_{i,j} \mathcal{O}_X(\mathcal{U}_i \cap \mathcal{U}_j)_f \longrightarrow \prod_{i,j} \mathcal{O}_X(V_i \cap V_j)$ is also injective. Now, a standard diagram chase shows that the map $\mathcal{O}_X(X)_f \longrightarrow \mathcal{O}_X(X_f)$ is surjective, so it is an isomorphism. □

Proposition 4.3.3 (Affineness Criterion)**.** *Let X be a scheme. Assume that there exist $f_1, \ldots, f_n \in \mathcal{O}_X(X)$ s.t.*

(i) *For all $i \in \{1, \ldots, n\}$, $X_{f_i} = \{x \in X,\ f_i(x) \neq 0\}$ is affine.*

(ii) *$f_1, \ldots, f_n$ generate the ring $\mathcal{O}_X(X)$.*

Then X is affine.

Proof. Let $A = \mathcal{O}_X(X)$. We want to show that $X \simeq \mathrm{Spec}(A)$. By (ii), there exist $g_1, \dots, g_n \in A$ s.t.

$$1 = f_1 g_1 + \cdots + f_n g_n.$$

Therefore, for $x \in X$, we have $f_1(x)g_1(x) + \cdots + f_n(x)g_n(x) = 1$ in $\kappa(x)$, so there exists $i \in \{1, \dots, n\}$ s.t. $f_i(x) \neq 0$ in $\kappa(x)$, which means that $x \in X_{f_i}$. Therefore, we have a finite affine open cover:

$$X = \bigcup_{i=1}^{n} X_{f_i}.$$

This open cover satisfies the condition $(*)$ of Lemma 4.3.2 because, for $i, j \in \{1, \dots, n\}$, $X_{f_i} \cap X_{f_j} = (X_{f_i})_{f_j}$ is affine. Therefore, the lemma implies that:

$$\forall i \in \{1, \dots, n\},\ X_{f_i} \simeq \mathrm{Spec}\left(A_{f_i}\right).$$

Now, for each $i \in \{1, \dots, n\}$, we denote by $u_i : X_{f_i} \to \mathrm{Spec}(A)$ the morphism of schemes induced by the natural map $A \to A_{f_i}$. For $i, j \in \{1, \dots, n\}$, $u_{i|X_{f_i} \cap X_{f_j}} : X_{f_i f_j} \to \mathrm{Spec}(A)$ is the morphism induced by the natural map $A \to A_{f_i f_j}$, therefore $u_{i|X_{f_i} \cap X_{f_j}} = u_{j|X_{f_i} \cap X_{f_j}}$. Hence, we can glue the morphisms $u_1, \dots, u_n$ to a morphism of schemes $u : X \to \mathrm{Spec}(A)$ s.t. $u_{|X_{f_i}} = u_i$ for all i. To see that u is an isomorphism, note that, for $i \in \{1, \dots, n\}$, $u_{|X_{f_i}} = u_i$ induces an isomorphism $X_{f_i} \to D_{f_i}$, and we have $u^{-1}\left(D_{f_i}\right) = X_{f_i}$. ☐

Theorem 4.3.4. *Let A be a ring and let $X = \mathrm{Spec}(A)$. Consider a closed immersion $j : Z \to X$. Then Z is affine. Moreover, there is a unique ideal $J \trianglelefteq A$ s.t. j is isomorphic to the natural closed immersion $\mathrm{Spec}(A/J) \to \mathrm{Spec}(A)$, i.e. we have a commutative diagram as below:*

$$\begin{tikzcd} Z \arrow[rr, "\simeq"] \arrow[dr, "j"'] & & \mathrm{Spec}(A/J) \arrow[dl] \\ & \mathrm{Spec}(A) & \end{tikzcd}$$

Proof. Since Z is a scheme with the structure induced by the map $j : Z \to X$, there exist open subschemes $(\mathcal{U}_\lambda)_{\lambda \in \Lambda}$ of X s.t.

$$Z = \bigcup_{\lambda \in \Lambda} j^{-1}\left(\mathcal{U}_\lambda\right),$$

and $j^{-1}\left(\mathcal{U}_\lambda\right)$ is an open affine subscheme of Z for all $\lambda \in \Lambda$. As principal open subschemes form a basis of open subsets, we can write $\mathcal{U}_\lambda = \bigcup_{\mu \in M} \mathcal{U}_{\lambda,\mu}$, where $\mathcal{U}_{\lambda,\mu}$ is a principal open affine subscheme of X, say $\mathcal{U}_{\lambda,\mu} = D_{h_{\lambda,\mu}}$ with $h_{\lambda,\mu} \in A$. Therefore:

$$Z = \bigcup_{(\lambda,\mu) \in \Lambda \times M} Z_{h_{\lambda,\mu}},$$

where $Z_{h_{\lambda,\mu}} = \{x \in Z,\ h_{\lambda,\mu}(x) \neq 0\}$. The non-vanishing locus $Z_{h_{\lambda,\mu}}$ is actually a principal open affine subscheme of $j^{-1}\left(\mathcal{U}_\lambda\right)$, so it is affine. Moreover, Z is quasi-compact as a closed subset of the quasi-compact set X, so there exist $h_1, \dots, h_L \in \{h_{\lambda,\mu},\ (\lambda, \mu) \in \Lambda \times M\}$ s.t.

$$Z = \bigcup_{i=1}^{L} Z_{h_i}.$$

Moreover, since $j^\sharp : \mathcal{O}_X \to j_* \mathcal{O}_Z$ is surjective, there exist $g_1, \dots, g_L \in A$ s.t. $h_i = g_{i|Z}$. We see that $A = (g_1, \dots, g_L)$, and therefore $\mathcal{O}_Z(Z) = (h_1, \dots, h_L)$. The Affineness Criterion (Proposition 4.3.3) implies that Z is affine. Write $B = \mathcal{O}_Z(Z)$, so that $Z \simeq \mathrm{Spec}(B)$. Let $\varphi : A \to B$ be the map corresponding to the morphism $j : \mathrm{Spec}(B) \to \mathrm{Spec}(A)$. Let $I = \mathrm{Ker}\,\varphi$. Then there exists $\overline{\varphi} : A/I \to B$ s.t. $\varphi = \overline{\varphi} \circ \pi$, where $\pi : A \to A/I$ is the projection. We shall show that $\overline{\varphi}^* : \mathrm{Spec}(B) \to \mathrm{Spec}(A/I)$ is an isomorphism, which will conclude the proof. Firstly, $\overline{\varphi}$ is injective, from which we deduce that $\overline{\varphi}^*$ has dense image, so it is a homeomorphism. Furthermore, $(\overline{\varphi}^*)^\sharp$ is injective because $\overline{\varphi}$ is. And it is surjective because $(\varphi^*)^\sharp = j^\sharp$ is. ☐

4.4 Finiteness conditions

Definition 4.4.1 (Quasi-compact morphism of schemes). *A morphism of schemes $f : X \to Y$ is said to be* quasi-compact *if for any open affine subscheme V of Y, $f^{-1}(V)$ is quasi-compact.*

Lemma 4.4.2. *A scheme X is quasi-compact* iff *it can be covered by finitely many affine open subschemes.*

Proposition 4.4.3. *Let $f : X \to Y$ be a morphism of schemes. If there exists an affine open cover $Y = \bigcup_{i\in I} V_i$ s.t. $f^{-1}(V_i)$ is quasi-compact for all $i \in I$, then f is quasi-compact.*

Proof. Let $V \subseteq Y$ be an affine open subscheme. We have $V = \bigcup_{i\in I} (V \cap V_i)$. For $i \in I$, we can write $V \cap V_i = \bigcup_{j\in J} V_{ij}$, where V_{ij} is a principal affine open subset of V_i. Therefore, $V = \bigcup_{(i,j)\in I\times J} V_{ij}$, with V_{ij} affine for all i, j. Now, V is affine so it is quasi-compact. We may therefore assume that $I \times J$ is finite. It suffices to prove that $f^{-1}(V_{ij})$ is quasi-compact for all i, j. For $i \in I$, $f^{-1}(V_i)$ is quasi-compact, so by Lemma 4.4.2, there exists a finite affine open cover $f^{-1}(V_i) = \bigcup_{k\in K} U_{ik}$. Now, for $j \in J$, recall that V_{ij} is a principal open subset of V_i, so one can write $V_{ij} = (V_i)_{g_{ij}}$, with $g_{ij} \in \mathcal{O}_X(V_i)$. Therefore, $f^{-1}(V_{ij}) = (f^{-1}(V_i))_{f^*(g_{ij})} = \bigcup_{k\in K} (U_{ik})_{f^*(g_{ij})}$. But for all $k \in K$, $(U_{ik})_{f^*(g_{ij})}$ is affine as a principal open subset of an affine scheme; therefore $f^{-1}(V_{ij})$ is quasi-compact by Lemma 4.4.2. □

Corollary 4.4.4. *A scheme X is quasi-compact* iff *the canonical morphism $X \to \mathrm{Spec}(\mathbb{Z})$ is quasi-compact.*

Definition 4.4.5 (Morphism of schemes locally of finite type). *A morphism of schemes $f : X \to Y$ is said to be* locally of finite type *if for any affine open subscheme V of Y and for any affine open subscheme U of X s.t. $f(U) \subseteq V$, the morphism $\mathcal{O}_Y(V) \to O_X(U)$ is of finite type (i.e. $\mathcal{O}_X(U)$ is a finitely generated $\mathcal{O}_Y(V)$-algebra).*

Lemma 4.4.6. *Let X be a scheme.*

(i) *If X is affine and $\mathcal{U}$ is a principal open affine subscheme of X (so $\mathcal{U}$ is affine), then any principal open affine subscheme of $\mathcal{U}$ is also principal in X.*

(ii) *Let $\mathcal{U}, \mathcal{V}$ be two open affine subschemes of X. Then for all $x \in \mathcal{U} \cap \mathcal{V}$, there exists an open subscheme $\mathcal{W} \subseteq \mathcal{U} \cap \mathcal{V}$ containing x s.t. $\mathcal{W}$ is principal in both $\mathcal{U}$ and $\mathcal{V}$.*

Lemma 4.4.7. *Let $\varphi : A \to B$ be a ring homomorphism.*

(i) *Let $a \in A$. If B is a finitely generated A-algebra, then $B_{\varphi(a)}$ is a finitely generated A_a-algebra.*

(ii) *Let $b_1, \dots, b_r \in B$ s.t. $B = (b_1, \dots, b_r)$ and, for all $i \in \{1, \dots, r\}$, B_{b_i} is a finitely generated A-algebra. Then B is a finitely generated A-algebra.*

Proof. (i) Clear. (ii) By assumption, there exist $x_1, \dots, x_r \in B$ s.t. $1 = \sum_{i=1}^r b_i x_i$. For $i \in I$, let $\left(\frac{a_{ij}}{b_i^{n_i}}\right)_{1\le j\le m}$ be a system of generators of B_{b_i} over A, with $a_{ij} \in B$ and $n_i \in \mathbb{N}$. Consider the sub-A-algebra C of B generated by all the $(a_{ij})_{\substack{1\le i\le r\\1\le j\le m}}$ and the $(b_i)_{1\le i\le r}$. Hence, for $i \in I$, we have $C_{b_i} = B_{b_i}$. Let D be the sub-C-algebra of B generated by all the $(x_i)_{1\le i\le r}$. Note that D is finitely generated over C, which is finitely generated over A, so D is finitely generated over A. It remains to show that $D = B$. For $b \in B$, we have $C_{b_i} = B_{b_i}$ for all i, so there exist $c_i \in C$ and $k_i \in \mathbb{N}$ s.t. $b_i^{k_i} b = b_i^{k_i} c_i$. As i varies in a finite set, we may assume that $k = k_i$ is independent of i. Now, recall that $1 = \sum_{i=1}^r b_i x_i$. By raising this equality to the power rk, we obtain:

$$1 = \sum_{i=1}^r b_i^k d_i,$$

for some $d_1, \dots, d_r \in D$. Thus, $b = \sum_{i=1}^r b_i^k b d_i = \sum_{i=1}^r b_i^k c_i d_i \in D$. □

Proposition 4.4.8. *Let $f : X \to Y$ be a morphism of schemes. The following assertions are equivalent:*

(i) *f is locally of finite type.*

(ii) *For all open affine subscheme V of Y, there exists an open cover of $f^{-1}(V)$ by open affine subschemes $(U_i)_{i\in I}$ s.t. for all $i \in I$, $\mathcal{O}_{U_i}(U_i)$ is a finitely generated $\mathcal{O}_V(V)$-algebra.*

(iii) *There exists an open cover of Y by open affine subschemes $(V_i)_{i\in I}$ s.t. for all $i \in I$, there exists an open cover of $f^{-1}(V_i)$ by open affine subschemes $(U_{ij})_{j\in J_i}$ s.t. for all $j \in J_i$, $\mathcal{O}_{U_{ij}}(U_{ij})$ is a finitely generated $\mathcal{O}_{V_i}(V_i)$-algebra.*

Proof. (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) Clear. (ii) $\Rightarrow$ (i) One can assume that $V = Y$ is affine (and $X = f^{-1}(V)$). Therefore, we can write $X = \bigcup_{i\in I} U_i$, where U_i is an open affine subscheme of X with $\mathcal{O}_{U_i}(U_i)$ of finite type over $\mathcal{O}_Y(Y)$. Let $\mathcal{U} \subseteq X$ be an open affine subscheme; we want to show that $\mathcal{O}_{\mathcal{U}}(\mathcal{U})$ is of finite type over $\mathcal{O}_Y(Y)$. By Lemma 4.4.6, for $i \in I$, there exists an affine open cover of $\mathcal{U} \cap U_i$ by open subschemes $(W_{ij})_{j\in J}$ s.t. W_{ij} is principal in both $\mathcal{U}$ and U_i for all $j \in J$. Thus, $\mathcal{U} = \bigcup_{(i,j)\in I\times J} W_{ij}$. But $\mathcal{U}$ is affine hence quasi-compact, so we may assume that $I \times J$ is finite. Now, for $(i, j) \in I \times J$, W_{ij} is principal in U_i, so $\mathcal{O}_{W_{ij}}(W_{ij})$ is of finite type over $\mathcal{O}_{U_i}(U_i)$, and thus over $\mathcal{O}_Y(Y)$ (because $\mathcal{O}_{\mathcal{U}_i}(\mathcal{U}_i)$ is of finite type over $\mathcal{O}_Y(Y)$). Now, using Lemma 4.4.7, we see that $\mathcal{O}_{\mathcal{U}}(\mathcal{U})$ is of finite type over $\mathcal{O}_V(V)$. (iii) $\Rightarrow$ (ii) Similar proof. $\square$

Definition 4.4.9 (Morphism of schemes of finite type)**.** *A morphism of schemes is said to be* of finite type *if it is quasi-compact and locally of finite type.*

Example 4.4.10.

(i) *If k is a ring, then natural morphisms* $\operatorname{Spec}(k[T_1, \ldots, T_n]/(f_1, \ldots, f_m)) \to \operatorname{Spec}(k)$ *are of finite type.*

(ii) *An open immersion is locally of finite type, but not necessarily of finite type.*

(iii) *Let A be a ring and let $\mathfrak{p} \in \operatorname{Spec}(A)$. Then $A_{\mathfrak{p}}$ is not necessarily finitely generated over A, so the natural morphism* $\operatorname{Spec}(A_{\mathfrak{p}}) \to \operatorname{Spec}(A)$ *is not necessarily locally of finite type.*

Proposition 4.4.11. *Let $f : X \to Y$ and $g : Y \to Z$ be morphisms of schemes.*

(i) *If f and g are locally of finite type (resp. of finite type, quasi-compact), then $g \circ f$ is locally of finite type (resp. of finite type, quasi-compact).*

(ii) *If $g \circ f$ is locally of finite type, then f is locally of finite type.*

Definition 4.4.12 (Affine and finite morphisms of schemes)**.** *Let $f : X \to Y$ be a morphism of schemes.*

(i) *f is called* affine *if for all open affine subscheme $V \subseteq Y$, $f^{-1}(V)$ is affine.*

(ii) *f is called* finite *if f is affine and for all open affine subscheme $V \subseteq Y$, $\mathcal{O}_{f^{-1}(V)}(f^{-1}(V))$ is a finite $\mathcal{O}_V(V)$-module.*

Proposition 4.4.13. *Let $f : X \to Y$ be a morphism of schemes.*

(i) *f is affine* iff *there exists an open cover of Y by open affine subschemes $(V_i)_{i\in I}$ s.t. $f^{-1}(V_i)$ is affine for all $i \in I$.*

(ii) *f is finite* iff *there exists an open cover of Y by open affine subschemes $(V_i)_{i\in I}$ s.t. $f^{-1}(V_i)$ is affine and $\mathcal{O}_{f^{-1}(V_i)}(f^{-1}(V_i))$ is a finite $\mathcal{O}_{V_i}(V_i)$-module for all $i \in I$.*

Proof. (i) ($\Rightarrow$) Clear. ($\Leftarrow$) Let V be an affine open subscheme of Y; we want to show that $f^{-1}(V)$ is affine. For $i \in I$, choose an open cover $V_i \cap V = \bigcup_{j \in J} W_{ij}$, where W_{ij} is principal w.r.t. both V_i and V. Thus $V = \bigcup_{(i,j) \in I \times J} W_{ij}$, and V is quasi-compact, so we may assume that $I \times J$ is finite. Now, for $(i,j) \in I \times J$, V_{ij} is principal in V_i, so we can write $V_{ij} = (V_i)_{g_{ij}}$ for some $g_{ij} \in \mathcal{O}_Y(V_i)$. Thus:

$$f^{-1}(V_{ij}) = f^{-1}\left((V_i)_{g_{ij}}\right) = f^{-1}(V_i)_{\widetilde{g_{ij}}}.$$

This implies that $f^{-1}(V) = \bigcup_{(i,j) \in I \times J} f^{-1}(V_i)_{\widetilde{g_{ij}}}$. Using the Affineness Criterion (Proposition 4.3.3), we see that $f^{-1}(V)$ is affine. (ii) ($\Rightarrow$) Clear. ($\Leftarrow$) The proof is similar to (i). $\square$

Example 4.4.14.

(i) *By Theorem 4.3.4, closed immersions are finite.*

(ii) *Open immersions are not necessarily affine (for instance, consider the inclusion $\mathbb{A}_k^2 \backslash \{0\} \subseteq \mathbb{A}_k^2$).*

(iii) *If $A \to B$ is finite, then* $\operatorname{Spec}(B) \to \operatorname{Spec}(A)$ *is also finite.*

4.5 Noetherian property

Definition 4.5.1 (Noetherian topological space)**.** *A topological space X is called* noetherian *if any descending chain of closed subsets is eventually constant.*

Example 4.5.2. *If A is a noetherian ring, then* $\operatorname{Spec}(A)$ *is a noetherian topological space.*

Proposition 4.5.3. *Let X be a topological space.*

(i) *If X is noetherian, then any open or closed subspace of X is also noetherian.*

(ii) *X is noetherian* iff *all open subspaces of X are quasi-compact.*

(iii) *If X is a finite union of noetherian topological spaces, then X is also noetherian.*

Definition 4.5.4 (Irreducible topological space)**.** *A nonempty topological space X is called* irreducible *if for all closed subsets $X_1, X_2 \subseteq X$ s.t. $X = X_1 \cup X_2$, we have $X = X_1$ or $X = X_2$.*

Proposition 4.5.5. *Let X be a topological space.*

(i) *If X is irreducible, then any nonempty open subspace of X is irreducible.*

(ii) *If $Y \subseteq X$ is an irreducible subspace of X, then $\overline{Y}$ is also irreducible.*

(iii) *If $Y \subseteq X$ is a dense irreducible subspace of X, then X is also irreducible.*

Proposition 4.5.6. *Let X be a noetherian topological space. For any closed subspace $Y \subseteq X$, there exists a finite number of irreducible subspace $Y_1, \dots, Y_n \subseteq Y$ s.t.*

$$Y = \bigcup_{i=1}^{n} Y_i,$$

and $Y_i \not\supseteq Y_j$ if $i \neq j$. The decomposition is unique up to reindexing, and the subspaces $Y_1, \dots, Y_n$ are called the irreducible components *of Y.*

Definition 4.5.7 ((Locally) noetherian schemes)**.** *Let X be a scheme.*

(i) *X is called* locally noetherian *if there exists an affine open cover $X = \bigcup_{i \in I} U_i$ s.t. $\mathcal{O}_{U_i}(U_i)$ is a noetherian ring for all $i \in I$.*

(ii) *X is called* noetherian *if it is locally noetherian and quasi-compact (in other words, there exists a finite affine open cover $X = \bigcup_{i \in I} U_i$ s.t. $\mathcal{O}_{U_i}(U_i)$ is a noetherian ring for all $i \in I$).*

Remark 4.5.8. *The underlying topological space of a noetherian scheme is noetherian. However, the converse is false.*

Proposition 4.5.9. *Let X be a noetherian (resp. locally noetherian) scheme and let $\mathcal{U} \subseteq X$ be an open subscheme. Then $\mathcal{U}$ is noetherian (resp. locally noetherian).*

Proof. Assume that X is locally noetherian. Consider an affine open cover $X = \bigcup_{i \in I} U_i$, with $U_i = \operatorname{Spec}(A_i)$ where A_i is a noetherian ring. For $i \in I$, choose an open cover $\mathcal{U} \cap U_i = \bigcup_{j \in J} V_{ij}$, with $V_{ij} = (U_i)_{f_{ij}} = \operatorname{Spec}\left((A_i)_{f_{ij}}\right)$ principal in U_i, $f_{ij} \in A_i$. Since A_i is noetherian, so is $(A_i)_{f_{ij}}$, and $\mathcal{U} = \bigcup_{(i,j) \in I \times J} V_{ij}$, and therefore $\mathcal{U}$ is locally noetherian. If X is noetherian, then it is quasi-compact, and so is $\mathcal{U}$, so we may assume that $I \times J$ is finite in the previous reasoning. $\square$

Proposition 4.5.10. *Let A be a ring. Then* $\operatorname{Spec}(A)$ *is a noetherian scheme* iff *A is a noetherian ring.*

4.6 Integral schemes

Definition 4.6.1 (Reduced rings)**.** *A ring A is said to be* reduced *if it has no nonzero nilpotent element, i.e. if $\sqrt{(0)} = (0)$.*

Definition 4.6.2 (Reduced and integral schemes)**.** *Let X be a scheme.*

(i) *X is called* reduced *if for all $x \in X$, the local ring $\mathcal{O}_{X,x}$ is reduced.*

(ii) *X is called* integral *if it is irreducible and reduced.*

Proposition 4.6.3. *A scheme X is reduced* iff *for any open subset $\mathcal{U} \subseteq X$, $\mathcal{O}_X(\mathcal{U})$ is a reduced ring.*

Example 4.6.4.

(i) *If A is a reduced ring, then* $\operatorname{Spec}(A)$ *is a reduced scheme.*

(ii) *If A is an integral domain, then* $\operatorname{Spec}(A)$ *is an integral scheme.*

(iii) $\operatorname{Spec}\left(k[T]/\left(T^2\right)\right)$ *is not a reduced scheme.*

(iv) $\operatorname{Spec}\left(k\left[T_1, T_2\right]/\left(T_1 T_2\right)\right)$ *is not irreducible because it can be written as the union of $V(T_1)$ and $V(T_2)$.*

(v) *If A is a ring, we may define a reduced ring $A_{\mathrm{red}} = A/\sqrt{(0)}$. Then we have a closed immersion $f : \operatorname{Spec}(A_{\mathrm{red}}) \to \operatorname{Spec}(A)$. The underlying continuous map of this closed immersion is a homeomorphism (but not an isomorphism); the scheme* $\operatorname{Spec}(A_{\mathrm{red}})$ *is called the* reduced scheme structure *on* $\operatorname{Spec}(A)$. *Moreover, this construction glues: if X is any scheme, then there exists a unique closed immersion $f : X_{\mathrm{red}} \to X$ s.t. X_{red} is reduced and f induces a homeomorphism.*

Proposition 4.6.5. *Let $X = \operatorname{Spec}(A)$ be an affine scheme.*

(i) *If $I \trianglelefteq A$ is an ideal, then the closed subscheme $V(I)$ is irreducible* iff *$\sqrt{I}$ is prime.*

(ii) *X is an integral scheme* iff *A is an integral domain.*

(iii) *If A is noetherian, then the irreducible components of X are the closed subschemes $V(\mathfrak{p})$, where $\mathfrak{p}$ is a minimal prime ideal of A.*

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