On the Variety generated by all
Nilpotent Lattice-ordered Groups.

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To Valerie Kopytov on his 26th Birthday.

Abstract: In 1974, J. Martínez introduced the variety $\mathcal{W}$ of all weakly Abelian lattice-ordered groups; it is defined by the identity

$$x^{-1}(y \lor 1)x \lor (y \lor 1)^2 = (y \lor 1)^2.$$ 

We prove

**Theorem A** There is a centre-by-metabelian weakly Abelian ordered group that does not belong to the variety of lattice-ordered groups generated by all nilpotent lattice-ordered groups.

This answers two questions of V. M. Kopytov.

We extend our techniques to show

**Theorem B** The quasivariety generated by all nilpotent lattice-ordered groups is the same as the variety generated by all nilpotent lattice-ordered groups.

Our proof also gives a set of defining identities for this variety.

In contrast to Theorem A we show:

**Theorem C** Every Abelian-by-nilpotent weakly Abelian lattice-ordered group belongs to the variety of lattice-ordered groups generated by all nilpotent lattice-ordered groups.

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1 Basic Definitions and Facts

We will use the shorthand \( \mathbb{Z} \) for the group of integers under addition with the usual ordering, \( \mathbb{N} \) for the set of non-negative integers, and \( \mathbb{R} \) for the additive group of real numbers with the usual ordering.

We will use standard group-theoretic notation, as in, e.g., [7] and [14]. In particular, if \( G \) is any group, we write \( \zeta(G) \) for the centre of \( G \); and if \( f, g, h \in G \), we write \( f^g \) for \( g^{-1}fg \) and \( [f, g] \) for \( f^{-1}g^{-1}fg = f^{-1}f^g \); and \([f, g, h] \) as a shorthand for \([[f, g], h]] \), etc. If \( H \) and \( K \) are subgroups of \( G \) we write \( [H, K] \) for the subgroup generated by \( \{[h, k] : h \in H, k \in K \} \) and define the lower central series \( \gamma_m(G) \) of \( G \) inductively:

\[
\gamma_1(G) = G, \quad \gamma_{m+1}(G) = [\gamma_m(G), G]
\]

(for all \( m = 1, 2, 3, \ldots \)). Each \( \gamma_m(G) \) is an invariant subgroup of \( G \) (and hence a normal subgroup of \( G \)). Moreover, \( G \) is nilpotent class \( c \) if and only if \( \gamma_{c+1}(G) = \{1\} \); i.e., if \( [g_1, \ldots, g_{c+1}] = 1 \) for all \( g_1, \ldots, g_{c+1} \in G \).

Thus \( G/\gamma_{c+1}(G) \) is nilpotent of class \( c \) for all groups \( G \). Throughout, let \( I_k(G) \) be the isolator subgroup of \( \gamma_k(G) \) in \( G \) \( (k \in \mathbb{Z}_+) \), and \( \Gamma(G) \) be \( \bigcap_{k \in \mathbb{Z}_+} I_k(G) \).

If \( \mathfrak{X} \) is a class of groups closed under isomorphisms, then a group \( G \) is called residually \( \mathfrak{X} \) if there is a family \( \{N_i : i \in I\} \) of normal subgroups of \( G \) with \( \bigcap_{i \in I} N_i = \{1\} \) and each \( G/N_i \in \mathfrak{X} \); i.e., \( G \) is a subdirect product of the \( \mathfrak{X} \)-groups \( G/N_i \) \( (i \in I) \).

A group equipped with lattice operations such that \( x(y \vee z)t = xyt \vee xzt \) and \( x(y \wedge z)t = xyt \wedge xzt \) for all \( x, y, z, t \) is called a lattice-ordered group (or \( \ell \)-group, for short). If the lattice order is total (i.e., for any pair of elements \( x \& y \), either \( x \vee y = x \) or \( x \vee y = y \)), then the \( \ell \)-group is said to be an ordered group (or \( o \)-group for short). As usual, we write \( x \leq y \) as a shorthand for either \( x \vee y = y \) or \( x \wedge y = x \).

Let \( G \) be a partially ordered group (or p.o.group, for short), \( G^+ = \{g \in G : g \geq 1\} \) and \( G_+ = G^+ \setminus \{1\} \). If \( X \) is a subset of a group \( G \), let \( N_G(X) \) be the normal subsemigroup of \( G \) generated by \( X \). As is well known (see, e.g., [12, page 2]),

**Lemma 1.1** \( G \) can be made into a p.o.group with \( G_+ = N_G(X) \) if and only if \( 1 \not\in N_G(X) \).
Recall [12, Theorems 3.1.5 and 3.1.7]

**Lemma 1.2** If $G$ is a nilpotent or metabelian p.o.group, then the order can be extended so that $G$ becomes an o-group.

As is well known (see, e.g., [4, Chapter 2]), every element of an $\ell$-group $G$ can be written in the form $fg^{-1}$ for some $f, g \in G^+$; moreover, $\{|g| : g \in G\} = G^+ = \{g \vee 1 : g \in G\}$ where we write $|g|$ for $g \vee g^{-1}$. Therefore $G^+$ completely determines the order on $G$.

A subgroup $C$ of an o-group $G$ is called convex if for all $c_1, c_2 \in C$, $g \in G$, $(c_1 \leq g \leq c_2$ implies $g \in C)$. The set of convex subgroups of an o-group is totally ordered by inclusion [4, Lemma 3.1.2]. The set is closed under unions and intersections. Hence, given any non-identity element $g$ of an o-group $G$, there is a unique subgroup $C_g$ maximal with respect to not containing $g$. It is called the value of $g$ and is strictly contained in the convex subgroup $C(g)$ of $G$ generated by $g$. Further, $C_g \triangleleft C(g)$. The pair $(C_g, C(g))$ is called a convex jump, and $C(g)/C_g$ is isomorphic (as an o-group) to an additive subgroup of $\mathbb{R}$ (see, e.g., [4, Chapters 3 and 4]).

As is also standard, we write $f \ll g$ for: $f^n \leq |g|$ for all $n \in \mathbb{Z}$ (and say that $f$ is very much less than $g$). In any o-group, $f_1f_2 \ll g$ whenever $f_1 \ll g$ & $f_2 \ll g$ (see, e.g., [4, Chapter 3]).

Residually ordered $\ell$-groups form a variety $\mathcal{R}$ in the class of all $\ell$-groups (see, e.g., [4, Section 3.8] or [9, Section 9.3]). Hence any $G \in \mathcal{R}$ is $\ell$-isomorphic to a subdirect product of o-groups. So any subvariety of $\mathcal{R}$ is generated by all its o-groups. But any o-group is o-isomorphic to a subgroup of an ultraproduct of its finitely generated subgroups. Consequently, we get:

**Lemma 1.3** Any subvariety of $\mathcal{R}$ is generated by its finitely generated o-groups.

This is one of the essential tools in our proofs.

A lattice-ordered group $G$ is called weakly Abelian if $g^{-1}|f||g| \leq |f|^2$ for all $f, g \in G$ [11] (or see [4, Section 6.4] or [9, Sections 6.2 and 9.4]). Equivalently, with the above shorthand, this law can be written in the form

$$f^g \leq f^2$$

for all $f \in G^+$, $g \in G$. 

3
Throughout we will use the following well-known result [11] (or see, e.g., [4, Lemma 6.4.1]):

**Lemma 1.4** An $\ell$-group $G$ is weakly Abelian if and only if $|[f, g]| \ll |f|$ for all $f, g \in G$.

Hence every weakly Abelian $\ell$-group is residually ordered; i.e., a subdirect product of (weakly Abelian) o-groups [ibid.]. Weakly Abelian o-groups are called centrally ordered and have recently been studied by the authors with Rhemtulla (see [2] and [3]). Note that in any centrally ordered group, $C(g)$ and $C_g$ are normal subgroups of $G$ for all $g \in G \setminus \{1\}$. Moreover, $g$ is central in $G/C_g$ and $[G, C(g)] \subseteq C_g$ [op. cit.].

We will need one fact from [3]:

**Lemma 1.5** [3, Theorem D] If $G$ is a finitely generated centrally ordered Abelian-by-nilpotent group and $g$ is a non-identity element of $G$, then $G/C_g$ is residually torsion-free-nilpotent.

Any locally nilpotent $\ell$-group is weakly Abelian ([8], [13] or see [4, Theorem 6.D] or [9, Theorem 9.4.1]); so the variety of $\ell$-groups generated by all nilpotent $\ell$-groups is contained in $\mathcal{W}$. In 1984, V. M. Kopytov asked if the converse were true [The Black Swamp Problem Book, Question 40] (c.f., Kopytov’s stronger question [10, Problem 5.23]).

Since weakly Abelian $\ell$-groups are residually ordered, the question is equivalent to:

\[\text{Does every centrally ordered group belong to the variety of }\]
\[\text{lattice-ordered groups generated by all nilpotent lattice-ordered groups?}\]

Theorem A provides negative answers to Kopytov’s questions.

For prior work on this topic, see [5] and [6].

Let $\mathfrak{N}$ be the class of all lattice-ordered groups that are nilpotent.

Throughout, let $\mathfrak{N}$ denote the variety of lattice-ordered groups generated by $\mathfrak{N}$. So $\mathfrak{N}$ is defined by all identities that hold in all nilpotent $\ell$-groups. Every element of $\mathfrak{N}$ is an $\ell$-homomorphic image of a subdirect product of nilpotent o-groups.
We will write \( q(\mathfrak{N}) \) for the quasi-variety of lattice-ordered groups generated by \( \mathfrak{N} \). It is the smallest class of \( \ell \)-groups that is closed under \( \ell \)-isomorphisms, sublattice subgroups, direct products and ultraproducts and contains \( \mathfrak{N} \). It is the class of \( \ell \)-groups defined by all implications of the form

\[
(\forall x_1, \ldots, x_n)((u_1(x_1, \ldots, x_n) = 1 \& \ldots \& u_k(x_1, \ldots, x_n) = 1) \rightarrow w(x_1, \ldots, x_n) = 1)
\]

that hold in all \( \ell \)-groups belonging to \( \mathfrak{N} \), where \( u_1(x_1, \ldots, x_n) \), \ldots, \( u_k(x_1, \ldots, x_n) \), \( w(x_1, \ldots, x_n) \) are \( \ell \)-group words.

Clearly, \( q(\mathfrak{N}) \subseteq \hat{\mathfrak{N}} \).

2 The proof of Theorem A

Let \( G \) be a group with generators \( a_1, a_2, y, c \) and relations:

\[
[a_1, a_2] = [a_1, y] = [a_2, y] = 1, \quad (1)
\]

\[
[a_i^{\pm 1}, c, c] = [a_i^2, c, c] = 1, \quad i = 1, 2, \quad (2)
\]

\[
c^2c^{a_2} = c^{2y}, \quad (3)
\]

\[
[[a_1^2, c], [a_2, c]] = [[a_1^2, c], [a_2^{-1}, c]], \quad (4)
\]

\[
[d, a_1] = [d, a_2] = [d, y] = [d, c] = 1, \quad (5)
\]

where \( d := [[a_1^2, c], [a_2, c]] \).

This is a finitely presented group. By (5),

\[
d \in \zeta(G) \quad (6)
\]

To prove Theorem A, we first establish two facts:

(i) if \( G^\phi \) is a torsion-free nilpotent homomorphic image of \( G \), then \( d^\phi = 1 \) (and so \( d \in \Gamma(G) \));

(ii) \( G \) has a centrally orderable homomorphic image \( G^\varphi \) with \( d^\varphi \neq 1 \).

To prove (i) we need
Lemma 2.1 In $G$, 
\[ c \in \zeta(c, c(a^2_1), c(a_2^{+1})) , \] (7)

and for all $m \in \mathbb{Z}_+$ 
\[ [[a_1^2, c], [a_2^{2m+1}, c]] = d. \] (8)

Proof: Trivially, in any group $L$, 
\[ [x, y] = 1 \text{ implies } [x^p, y^q] = 1 \text{ for all } p, q \in \mathbb{Z}. \] (9)

By standard commutator calculus and induction on $\ldots$ 
\[ [x, y] = 1 \text{ implies } [x, y^k] = [x, y]^k \text{ for all } k \in \mathbb{Z}. \] (10)

Moreover, 
\[ [x, y] = [y^{-x}, y] = [y^{-x}, y]^y. \]

Thus 
\[ [x, y] = 1 \text{ if and only if } [y^x, y] = 1. \] (11)

This gives (7).

We prove (8) by induction on $m \in N \cup \{-1\}$.

By (4), $d = [[a_1^2, c], [a_2^{+1}, c]]$, so (8) is true if $m = -1$ or 0.

We now induct on $m$ using $y$.

We have $d = [c^{a_1^2}, c^{a_2^2}] = [c^{a_1^2}, c^{a_2^{+1}}]$. Assume that 
\[ d^k = [c^{2a_1^2}, c^{2a_2^{2k+1}}] = [c^{2a_1^2}, c^{2a_2^{2k-1}}]. \]
Then 
\[ d^{k+1} = [c^{2a_1^2}, c^{2a_2^{2k+1}}]^g = [c^{a_1^2}, c^{a_1^2}, c^{a_2^{2k+1}}] = 
[c^{a_1^2}, c^{a_2^{2k+1}}, c^{a_1^2}, c^{a_2^{2k+3}}] = 
[c^{a_1^2}, c^{a_2^{2k+1}}, c^{a_1^2}, c^{a_2^{2k+3}}, c^{a_2^{+3}}] = 
[c^{a_1^2}, c^{a_2^{2k+3}}, c^{a_1^2}, c^{a_2^{2k+3}}, c^{a_2^{+3}}, c^{a_2^{+5}}] = 
[c^{a_1^2}, c^{a_2^{2k+3}}, c^{a_1^2}, c^{a_2^{2k+3}}, c^{a_2^{+3}}, c^{a_2^{+5}}, c^{a_2^{+7}}] = 
\ldots = d^{k+1}. \]

Thus $[c^{a_1^2}, c^{a_2^{2k+3}}, c^{a_2^{+5}}] \in \zeta(H)$ and $[c^{a_1^2}, c^{a_2^{2k+3}}] = d$. So $[c^{a_1^2}, c^{a_2^2}] = d$ for all odd $n \in \mathbb{Z}_+$. But $[[a_1^2, c], [a_2^{+1}, c]] = [c^{a_1^2}, c^{a_2^2}]$ using (7) and standard commutator calculus. (8) now follows. //

We now use Lemma 2.1 to establish (i).
Let $p$ be an odd prime and $2m + 1$ range over all powers of $p$ in (8). By Gruenberg's Theorem [14, Theorem 5.2.21], every finitely generated torsion-free nilpotent group is a residually finite $p$-group. By (8), $d = 1$ in any finite $p$-group that is an image of $G$. Therefore, $d = 1$ in every torsion-free nilpotent homomorphic image of $G$. 

We now prove (ii): the initial group, $G$, has a centrally ordered homomorphic image $G^*$ with $d^* = [[a_1^2, c], [a_2, c]]^p \neq 1$. For this we construct an example. The first examples we considered were based on [1, Statement 6]; what is presented here is several modifications later.

**Example 2.2** Let $C$ be the free nilpotent class 2 group with free generators $\{c_{m,n} : m, n \in \mathbb{Z}\}$. The centre of $C$ is the free Abelian group with free generators $[c_{m,n}, c_{p,q}]$ ($m, n, p, q \in \mathbb{Z}$) where $(m, n) > (p, q)$ lexicographically. We embed $C$ in a divisible nilpotent class 2 group $C^\#$; so each element of $C^\#$ can be written uniquely (to within the order that the commutators appear) in the form

$$c_{m_1,n_1}^{r_1} \cdots c_{m_k,n_k}^{r_k} \prod_{i=1}^\ell [c_{p_i,q_i}, c_{p'_i,q'_i}]^{t_i},$$

where $(m_1, n_1) > \ldots > (m_k, n_k)$ lexicographically, $p_i, p'_i, q_i, q'_i \in \mathbb{Z}$ with $(p_i, q_i) > (p'_i, q'_i)$ lexicographically ($i = 1, \ldots, \ell$) and $r_1, \ldots, r_k, t_1, \ldots, t_\ell \in \mathbb{Q}$ (see [15, Theorem 8.5]). We add relations

$$[c_{m,n}, c_{p,q}] = [c_{m-p,0}, c_{0,q-n}] \quad m, n, p, q \in \mathbb{Z},$$

$$[c_{m,0}, c_{0,2q}] = [c_{2m+1,0}, c_{0,2q+1}] = 1 \quad m, q \in \mathbb{Z},$$

$$[c_{2m,0}, c_{0,2q+1}] = [c_{2,0}, c_{0,1}]^m \quad m, q \in \mathbb{Z},$$

and quotient out from $C^\#$ the divisible normal subgroup determined by these relations. We obtain a factor group $C^\#$ with centre $D$, the divisible closure of $\langle d \rangle$ in $C^\#$, where $d := [c_{2,0}, c_{0,1}]$. So $D$ is a rank 1 Abelian group.

Note that $C^\#$ is a divisible nilpotent class 2 group and each element of $C^\#$ can be uniquely written in the form

$$c_{m_1,n_1}^{r_1} \cdots c_{m_k,n_k}^{r_k} d^\ell,$$

where $(m_1, n_1) > \ldots > (m_k, n_k)$ lexicographically and $r_1, \ldots, r_k, t \in \mathbb{Q}$ (see [15, Theorem 8.5]).
Let \( a, b \) be automorphisms of \( C^\# \) determined by:

\[
a : c_{m,n} \mapsto c_{m+1,n} \quad b : c_{m,n} \mapsto c_{m,n+1}
\]  

(15)

Note that \( a, b \) induce automorphisms of \( C^\# \).

**Proof:** We need only show that \( a, b \) respect the relations (12) - (14).

This is immediate since

\[
[c_{m,0}, c_{0,q}]^a = [c_{m+1,0}, c_{1,q}] = [c_{m,0}, c_{0,q}] = [c_{m,0}, c_{0,q+1}] = [c_{m,0}, c_{0,q}]^b.
\]

Therefore \( a \) and \( b \) induce well-defined automorphisms of \( C^\# \).

We observe that automorphisms \( a \) and \( b \) commute and fix \( D \) point-wise.

So we can construct a splitting extension \( K = C^\# \rtimes \langle a, b \rangle \) with Abelian top group \( \langle a, b \rangle \). Note that \( K/D \cong (C^\#/D) \rtimes \langle a, b \rangle \) and \( C^\#/D \) is Abelian. Thus \( K/D \) is metabelian with torsion-free Abelian quotients \( \gamma_n(K/D)/\gamma_{n+1}(K/D) \) and so is residually torsion-free-nilpotent. Further, \( \zeta(K) = D \).

Each element of \( K \) can be written uniquely in normal form

\[
a^p b^q c^r_{m_1, n_1} \cdots c^r_{m_k, n_k} d^t,
\]

where \( p, q, m_1, n_1, \ldots, m_k, n_k \in \mathbb{Z} \) and \( r_1, \ldots, r_k, t \in \mathbb{Q} \), and \( (m_1, n_1) > \ldots > (m_k, n_k) \) in the lexicographic order on \( \mathbb{Z} \times \mathbb{Z} \).

Let \( L = \langle a, b, c_{0,0} \rangle \). Now \( d = [c_{2,0}, [c_{0,0}, b^{2m+1}]] \), so by applying Gruenberg’s Theorem to \( L/I_n(L) \), we deduce that \( d \in I_n(L) \) for all \( n \in \mathbb{Z}_+ \).

Hence \( d \in \Gamma(L) \subseteq \Gamma(K) \), whence \( D \subseteq \Gamma(K) \). Since \( \Gamma(K/D) = 1 \), we have \( \Gamma(K) \subseteq D \). Therefore \( D = \Gamma(K) \). Since \( C^\# \) is divisible and \( [C^\#, C^\#] \leq \zeta(C^\#) = D \), by the normal form for elements of \( K \) we obtain \( D = \Gamma(K) = \bigcap_{n=1}^{\infty} \gamma_n(K) \). It follows that \( \gamma_n(K)/\gamma_{n+1}(K) \cong \gamma_n(K/D)/\gamma_{n+1}(K/D) \) for all \( n \in \mathbb{Z}_+ \). Thus \( K \) has a central order with a series of convex subgroups

\[
\{1\} < D = \bigcap_{i=1}^{\infty} \gamma_i(K) \ldots < \gamma_{n+1}(K) < \gamma_n(K) < \ldots < \gamma_1(K) = K.
\]

By (12) - (14), for all \( m, q \in \mathbb{Z} \) we get (in \( C^\# \))

\[
[c_{m+2,0}, c_{0,q}] [c_{m-2,0}, c_{0,q}] = [c_{m,0}, c_{0,q}]^2 = [c_{m,0}, c_{0,q+2}] [c_{m,0}, c_{0,q-2}].
\]  

(16)
In $C^\#$, $c_{m,n}$ commutes with $c_{m+2,n}$ and $c_{m,n+2}$ ($m, n \in \mathbb{Z}$). These lead us to the following:

Consider the injective endomorphism $\beta$ of $C^\#$ determined by

$$\beta : c_{m,n} \mapsto c_{m,n+2}^{1/2}c_{m,n}^{1/2}.$$  

(17)

A tedious but thoroughly routine verification shows that $\beta$ respects the relations (12) - (14). Thus $\beta$ induces an injective endomorphism of $C^\#$.

We extend $\beta$ to an injective endomorphism of $K$ by:

$$\beta : a \mapsto a; \quad \beta : b \mapsto b.$$  

(18)

Let $K^\beta = \bigcup \{K^{\beta^{-n}} : n \in \mathbb{Z}_+\}$ with central order inherited from $K$, and $H = \langle K^\beta, \beta : (17), (18) \rangle$ be the ascending HNN-extension of $K$ with respect to $\beta$. By the normal form for HNN-extensions, we see that $H/D$ is a metabelian residually torsion-free-nilpotent group and $D = \zeta(H)$. Equations (17) and (18) show that $\beta$ acts as the identity on $D$, and also on each $\gamma_n(K)/\gamma_{n+1}(K)$ ($n \in \mathbb{Z}_+$). The latter follows at once from $c_{m,n}^{\beta} = c_{m,n}[c_{m,n}^{1/2}, b^{2}]$. Hence the central order on $K$ is preserved by the injective endomorphism $\beta$, and we can extend this central order to $H$ by the spelling for HNN-extensions, with

$$K^\beta \leq \beta.$$  

Let $K_0$ be the subgroup of $H$ generated by $a, b, c_{0,0}, \beta$. Then $K_0$, the “Kopytov group”, is centre-by-metabelian. By construction, the relations (1) - (5) hold in $K_0$ under the substitution $a, b, \beta, c_{0,0}$ for $a_1, a_2, y, c$, respectively, and $d \neq 1$ by the normal form of the HNN-extension. Moreover, $K_0$ is centrally orderable since $H$ is.

Thus $K_0$ is a centrally orderable homomorphic image of $G$ satisfying (ii). //

We must now show that (i) and (ii) imply Theorem A.

This follows immediately from the more general theorem:

**Theorem D** Let $G_0$ be a lattice-ordered group and $G_0 \in \hat{\mathfrak{N}}$. If there is a group homomorphism $\varphi$ from a finitely presented group $G$ into $G_0$, then $\Gamma(G) \subseteq \ker(\varphi)$.  

9
For if $G$ is defined by (1) – (5) and $K_0$ is the centrally ordered group constructed in Example 2.2, then it follows from (i) and (ii) that $d \in \Gamma(G) \setminus \ker(\varphi)$ where $G^\varphi = K_0$; hence $K_0 \in \mathcal{W} \setminus \mathfrak{W}$ and Theorem A is established. ✓

To prove Theorem D, we will transform certain types of group implications into $\ell$-group identities. To achieve this we will need two lemmata.

Let $H$ be a finitely generated group with generators $h_1, \ldots, h_n$. Let $U = \langle u_1, \ldots, u_k \rangle$ be a finitely generated subgroup of $H$. We write $\Gamma_U(H) = \bigcap_{j=1}^\infty I(U^H, x_j(H))$, where $I(G)$ is the isolator of $G$. So $\Gamma_1(H) = \Gamma(H)$. By construction,

$$\{1\} \subseteq U \subseteq U^H \subseteq \Gamma_U(H) \subseteq H.$$  

**Lemma 2.3** Let $H$ be a finitely generated group with generators $h_1, \ldots, h_n$, and $U = \langle u_1, \ldots, u_k \rangle$ be a finitely generated subgroup of $H$. Then for any $w \in \Gamma_U(H)$, there are $m_0, \ldots, m_k \in \mathbb{Z}$ with $m_0 \in \mathbb{Z}_+$ such that

$$w^{m_0} u_1^{m_1} \ldots u_k^{m_k} \in \Gamma_{[U,H]}(H). \quad (19)$$

**Lemma 2.4** Let $F = F(x_1, \ldots, x_n)$ be the free group and $U = \langle u_1, \ldots, u_k \rangle$ be a finitely generated subgroup of $F$. If $w \in \Gamma_{[U,F]}(F)$, then

$$|w(y_1, \ldots, y_n)| \ll \bigvee_{j=1}^k |u_j(y_1, \ldots, y_n)| \quad (**)$$

holds in every nilpotent lattice-ordered group.

Actually, (**) is an infinite set of identities since $w_1 \ll w_2$ is a shorthand for $w_1^m \leq w_2$ for all $m \in \mathbb{Z}$.

We first show that the lemmata indeed imply Theorem D.

**Proof:** Let $G = \langle g_1, \ldots, g_n : u_1(g_1, \ldots, g_n) = 1, \ldots, u_k(g_1, \ldots, g_n) = 1 \rangle$. Let $F$ be the free group on $x_1, \ldots, x_n$ and $u_j = u_j(x_1, \ldots, x_n)$ be the result of replacing each occurrence of $g_i$ in $u_j(g_1, \ldots, g_n)$ by $x_i$ (1 $\leq i \leq n$; 1 $\leq j \leq k$). Then the natural homomorphism $\psi : F \to G$ determined by $x_i \mapsto g_i$ (1 $\leq i \leq n$) has kernel $U^F$ where $U = \langle u_1, \ldots, u_k \rangle$. If $w(g_1, \ldots, g_n) \in$
\[ \Gamma(G) \setminus \ker(\varphi), \text{then } w(g_1, \ldots, g_n)^{\varphi} \neq 1 \text{ in } G_0. \] But \( w(x_1, \ldots, x_n) \in \Gamma_U(F) \) by definition. By Lemma 2.3, there are \( m_0, \ldots, m_k \in \mathbb{Z} \) such that

\[ \hat{w} := w^{m_0}u_1^{m_1} \cdots u_k^{m_k} \in \Gamma_{[U,F]}(F). \]

By Lemma 2.4,

\[ 1 \neq |w(g_1, \ldots, g_n)^{\varphi}|^{m_0} = |\hat{w}(g_1, \ldots, g_n)^{\varphi}| \leq \sum_{j=1}^{k} |u_j(g_1, \ldots, g_n)^{\varphi}| = 1, \]

since \( u_1(x_1, \ldots, x_n), \ldots, u_k(x_1, \ldots, x_n) \in \ker(\varphi) \). This contradiction establishes the theorem. \\

We now prove Lemma 2.3.

**Proof:** Let \( w \in \Gamma_U(H) \). Then for every \( j \in \mathbb{Z}_+ \), there is a positive integer \( t_j \) such that \( w^{t_j} \in U^H \gamma_j(H) \); say,

\[ w^{t_j} \equiv u_1^{m_{1,j}} \cdots u_k^{m_{k,j}} \pmod{[U,H]^H \gamma_j(H)} \quad (*) \]

with \( m_{1,j}, \ldots, m_{k,j} \in \mathbb{Z} \). We call \((*)\) a \( j^{th}\)-representation for \( w \); it is not usually unique.

If \( t_j' \in \mathbb{Z}_+ \) and \( m_{1,j}', \ldots, m_{k,j}' \in \mathbb{Z} \) with

\[ (w')^{t_j'} \equiv u_1^{m_{1,j}'} \cdots u_k^{m_{k,j}'} \pmod{[U,H]^H \gamma_j(H)}, \quad (*)' \]

then we say that the \((*)'\) representation is less than the \((*)\) representation if

\[ (|m_{1,j}|, \ldots, |m_{k,j}|) < (|m_{1,j}'|, \ldots, |m_{k,j}'|), \]

in the lexicographic ordering on \( \mathbb{N}^k \). This is a well-ordering.

For each \( j \in \mathbb{Z}_+ \), among all the representations \((*)\) for positive powers of \( w \), choose one so that the right hand side of \((*)\) is minimal. We will assume that the sequence \( \{t_j : j \in \mathbb{Z}_+\} \) has been chosen so that, for each \( j \in \mathbb{Z}_+ \), \((*)\) is minimal for all positive powers of \( w \).

Fix \( j_1 \) and let \( j \geq j_1 \). We first establish

**Claim:** If \( m_{1,j_1} \neq 0 \), then \( m_{1,j_1} \) divides \( m_{1,j} \) and \( t_j/t_{j_1} = m_{1,j}/m_{1,j_1} \).

**Proof:** Let \( j \geq j_1 \). Since \( m_{1,j_1} \neq 0 \), we can write \( m_{1,j} = qm_{1,j_1} + r \) where \( q, r \in \mathbb{Z} \) with \( 0 \leq r < |m_{1,j_1}| \). Now

\[ w^{t_j-qt_{j_1}} \equiv u_1^r u_2^{m_{2,j_1}-qm_{2,j_1}} \cdots u_k^{m_{k,j_1}-qm_{k,j_1}} \pmod{[U,H]^H \gamma_{j_1}(H)}. \]
This contradicts the minimality of \((*_{j_1})\) unless \(t_j = qt_{j_1}\). Hence
\[
1 \equiv u_1^{r^{m_{2,j_1} - qm_{2,j}}} \cdots u_k^{m_{k,j_1} - qm_{k,j_1}} \pmod{[U, H]^{H \gamma_{j_1}(H)}}.
\]
If \(r \neq 0\), then
\[
u_r \equiv u_2^{q m_{2,j_1} - m_{2,j}} \cdots u_k^{q m_{k,j_1} - m_{k,j}} \pmod{[U, H]^{H \gamma_{j_1}(H)}}.
\]
Raising both sides of \((*_{j_1})\) to the \(r\)-th power and substituting the above gives a \((*_{j_1})\) representation of a positive power of \(w\) with the exponent on \(u_1\) being \(0 < j_{m_1,j_1}j_1\). This contradicts the minimality of \((*_{j_1})\). Thus \(r = 0\) and we have \(m_{1,j_1} = m_{1,j_1} = q = t_j/t_{j_1}\).

The claim also implies that \(m_{1,j} \neq 0\) if \(j \geq j_1\) and \(m_{1,j_1} \neq 0\) (since \(t_j \in \mathbb{Z}_+\)).

Either \(m_{1,j} = 0\) for all \(j \in \mathbb{Z}_+\) or there is a least \(j_1 \in \mathbb{Z}_+\) with \(m_{1,j_1} \neq 0\). Then for all \(j \geq j_1\), there are \(q_j \in \mathbb{Z} \setminus \{0\}\) such that the minimal \(j\)-th representation for all positive powers of \(w\) is
\[
w^{j_1,q_j} \equiv u_1^{m_{1,j_1}q_j} \cdots u_k^{m_{k,j}} \pmod{[U, H]^{H \gamma_j(H)}}.
\]
Let \(w_1 = w^{j_1}u_1^{-m_{1,j_1}}\) if such a \(j_1\) exists; if no such \(j_1\) exists, let \(j_1 = 1\) and \(w_1 = w^{t_1}u_1^{-m_{1,1}}\).

Repeating the above argument with \(w_1\) in place of \(w\), we can find \(j_2 \geq j_1\) and \(q_j \in \mathbb{Z}\) such that
\[
w_1^{j_2,q_j} \equiv u_2^{m_{2,j_2}q_j} \cdots u_k^{m_{k,j}} \pmod{[U, H]^{H \gamma_j(H)}}
\]
for all \(j \geq j_2\).

Let \(w_2 = w_1^{j_2}u_2^{-m_{2,j_2}}\).

Continuing in this way, we obtain \(m_1, \ldots, m_k \in \mathbb{Z}\) and \(m_0, q_j \in \mathbb{Z}_+\) such that
\[
(w^{m_0}u_1^{m_1} \cdots u_k^{m_k})q_j \in [U, H]^{H \gamma_j(H)}
\]
for all sufficiently large \(j \in \mathbb{Z}_+\). This completes the proof of the lemma.

We now prove Lemma 2.4.
Proof: Assume the hypotheses of the lemma. Let \( G \) be a nilpotent class \( c \) o-group and \( g_1, \ldots, g_n \in G \). Let \( \varphi : F \to G_1 = \langle g_1, \ldots, g_n \rangle \) be determined by \( x_i \mapsto g_i \) (\( 1 \leq i \leq n \)). Let \( v = v(g_1, \ldots, g_n) = \bigvee_{j=1}^k |u_j(g_1, \ldots, g_n)| \), and \( C_v \) be its value in \( G \). Hence \( u_1(g_1, \ldots, g_n), \ldots, u_k(g_1, \ldots, g_n) \in C(v) \). By Lemma 1.4, \( U^c \subseteq C(v) \), the cover of \( C_v \) in \( G \). Since \( G \) is weakly Abelian, \( U^c \subseteq C(v) \); thus \( ([U, F]^c, G_1) \subseteq [C(v), G] \subseteq C_v \). So if \( w(x_1, \ldots, x_n) \in \Gamma_{[U, F]}(F) \), then \( w(x_1, \ldots, x_n)^m \in [U, F]^c, \gamma_{c+1}(F) \) for some \( m = m(c) \in \mathbb{Z}^+ \). Thus \( w(g_1, \ldots, g_n)^m \in C_v \). Therefore \( w(g_1, \ldots, g_n) \in C_v \) (convex subgroups are isolated). Consequently,

\[
|w(g_1, \ldots, g_n)| \ll \bigvee_{j=1}^k |u_j(g_1, \ldots, g_n)|.
\]

Since \( g_1, \ldots, g_n \) were arbitrary in \( G \), we get that (**) holds in \( G \). Hence (**) holds in all nilpotent o-groups (it is independent of the nilpotency class of \( G \)), and so in \( \hat{N} \). //

We now have, for example, by Theorem D:

**Corollary 2.5** If a lattice-ordered group \( G \in \hat{N} \) is finitely presented as an abstract group, then \( G \) is residually torsion-free-nilpotent.

Finally in this section, we can now exhibit a central order on a free group that does not belong to \( \hat{N} \); c.f., [5].

**Corollary 2.6** There is a central order on \( F_4 \), the free group on 4 generators, so that \( (F_4, \leq) \notin \hat{N} \).

**Proof:** Let \( K_0 \) be the centrally ordered Kopytov group of Example 2.2 and \( a, b, c, \beta \) be the free generators of \( F_4 \). Let \( \theta : F_4 \to K_0 \) be the homomorphism given by \( a \mapsto a, b \mapsto b, c \mapsto c_{0,0} \), and \( \beta \mapsto \beta \). Let \( C = ker(\theta) \). Then \( C \) is a free group and so can be centrally ordered using the series \( C \cap \gamma_j(F_4) \) (\( j \in \mathbb{Z}^+ \)). Then \( F_4 \) is centrally ordered by: \( f > 1 \) if either \( (f \notin C \) and \( f^\theta \in (K_0)_+) \) or \( f \in C_+ \). Since \( K_0 \notin \hat{N} \), we have \( (F_4, \leq) \notin \hat{N} \). //
3 Identities in nilpotent $\ell$-groups

In Lemma 2.4, we found a set of identities that hold in $\hat{\mathfrak{N}}$. We now show that these are sufficient to define $\hat{\mathfrak{N}}$.

Let $\mathfrak{N}$ be the variety of $\ell$-groups defined by the identities (***) in Lemma 2.4. That is, the defining identities for $\mathfrak{N}$ are:

for each $k, m, n \in \mathbb{Z}_+$,
$$u_1(x_1, \ldots, x_n), \ldots, u_k(x_1, \ldots, x_n) \in F = F(x_1, \ldots, x_n)$$
and
$$w(x_1, \ldots, x_n) \in \Gamma_{[U,F]}(F),$$
where $U = \langle u_1(x_1, \ldots, x_n), \ldots, u_k(x_1, \ldots, x_n) \rangle$.

By Lemma 2.4,
$$\hat{\mathfrak{N}} \subseteq \hat{\mathfrak{N}} \subseteq \mathcal{W} \subseteq \mathcal{R}.$$  

For, putting $k = 1, n = 2, u_1 = x_1$, and $w = [x_1, x_2] \in \langle x_1 \rangle, F \subseteq \Gamma_{[x_1,F]}(F)$, we obtain the identities $|w(x_1, x_2)|^m \leq \sum_{j=1}^{k} |u_j(x_1, \ldots, x_n)|$, which define the weakly Abelian variety by Lemma 1.4.

The proof of Theorem D from Lemmata 2.3 and 2.4 (see Section 2) applies equally to $\mathfrak{N}$; that is,

**Lemma 3.1** Let $G_0$ be a lattice-ordered group and $G_0 \in \mathfrak{N}$. If there is a group homomorphism $\varphi$ from a finitely presented group $G$ into $G_0$, then $\Gamma(G) \subseteq \ker(\varphi)$.

The main result of this section is

**Theorem E** $q(\mathfrak{N}) = \hat{\mathfrak{N}} = \mathfrak{N}$.

Since we have provided an explicit set of defining laws for $\mathfrak{N}$, Theorem E gives both Theorem B and a set of defining identities for $\mathfrak{N}$.

To prove the theorem, we will need one extra technical fact.

**Lemma 3.2** Let $F = F(x_1, \ldots, x_n)$ be a free group of rank $n$ and $H = \langle h_1, \ldots, h_n : u_1(h_1, \ldots, h_n) = 1, \ldots, u_p(h_1, \ldots, h_n) = 1 \rangle$. Let $G \in \mathfrak{N}$ be an $\alpha$-group that is a homomorphic image of $H$ (qua group); say, $G = H^\alpha$. 

14
with \( g_i = h_i^2 \) (\( i = 1, \ldots, n \)). Let \( \{w_1(x_1, \ldots, x_n), \ldots, w_k(x_1, \ldots, x_n)\} \subseteq F \) be such that \( \{w_1(g_1, \ldots, g_n), \ldots, w_k(g_1, \ldots, g_n)\} \subseteq G_+ \). Then there exists \( j_0 \in \mathbb{Z}_+ \) and a total order on \( H/I_{j_0}(H) \) such that

\[
\{I_{j_0}(H)w_1(h_1, \ldots, h_n), \ldots, I_{j_0}(H)w_k(h_1, \ldots, h_n)\} \subseteq (H/I_{j_0}(H))+. 
\]

We now deduce Theorem \ref{maintheorem} from Lemma \ref{mainlemma}.

**Proof:** By Lemma \ref{ultimatelemma}, it suffices to show that \( \mathfrak{n} \subseteq q(\mathfrak{m}) \). Since \( \mathfrak{n} \subseteq \mathcal{R} \), by Lemma \ref{necessarylemma} it is enough to prove that every finitely generated o-group \( G \in \mathfrak{n} \) belongs to \( q(\mathfrak{m}) \).

Let \( G \) be such an o-group generated by \( g_1, \ldots, g_n \). Let \( F \) be the free group on \( x_1, \ldots, x_n \). Let \( \{R_i(x_1, \ldots, x_n) : i \in \mathbb{Z}_+\} \) be the set of all words in \( F \) such that \( R_i(g_1, \ldots, g_n) = 1 \) in \( G \). Then \( G \cong F/K \), where \( K = \langle R_i(x_1, \ldots, x_n) : i \in \mathbb{Z}_+ \rangle^F \). Let \( \varphi : F \to G \) be the group homomorphism determined by \( \varphi : x_\ell \mapsto g_\ell \) (\( \ell = 1, \ldots, n \)). Let \( \{c_i(g_1, \ldots, g_n) : i \in \mathbb{Z}_+\} \) be an enumeration of \( G_+ \).

For each \( m \in \mathbb{Z}_+ \), let \( K_m = \langle R_i(x_1, \ldots, x_n) : 1 \leq i \leq m \rangle^F \), and \( G_m \cong F/K_m \); so \( G_m = F^{\varphi_m} \), say. By construction, each \( G_m \) is a finitely presented group and the o-group \( G \) is a group homomorphic image of it.

By Lemma \ref{mainlemma}, for each \( m \in \mathbb{Z}_+ \) there is \( j = j(m) \in \mathbb{Z}_+ \) and a total order on the torsion-free nilpotent group \( N_{m,j(m)} := G_m/I_{j(m)}(G_m) \) such that elements

\[
I_{j(m)}(G_m)c_1(x_1^{\varphi_m}, \ldots, x_n^{\varphi_m}), \ldots, I_{j(m)}(G_m)c_m(x_1^{\varphi_m}, \ldots, x_n^{\varphi_m})
\]

all belong to \( (N_{m,j(m)})+ \).

Let \( \mathcal{U} \) be an ultrafilter on \( \mathbb{Z}_+ \), and \( H = (\prod_{m \in \mathbb{Z}_+} N_{m,j(m)})/\mathcal{U} \) be the resulting ultraproduct. Since each \( N_{m,j(m)} \) is an o-group, so is \( H \). Moreover, since each \( N_{m,j(m)} \) is nilpotent, we get \( H \in q(\mathfrak{m}) \). Then \( G \) can be mapped into \( H \) by mapping \( g_\ell \) to the \( \mathcal{U} \)-equivalence class of the element whose \( m \text{th} \) coordinate is \( I_{j(m)}(G_m)x_\ell^{\varphi_m} \) (\( m \in \mathbb{Z}_+ \)), where \( \ell = 1, \ldots, n \). Call this map \( \theta \). It is well-defined since for each \( i \in \mathbb{Z}_+ \), \( \{m \in \mathbb{Z}_+ : i \leq m \} \in \mathcal{U} \) (so \( R_i(g_1^\theta, \ldots, g_n^\theta) = 1 \)). Thus \( \theta \) is a group homomorphism that preserves order since \( \{c_i(g_1, \ldots, g_n) : i \in \mathbb{Z}_+\} \) is an enumeration of \( G_+ \) (again use \( \{m \in \mathbb{Z}_+ : i \leq m \} \in \mathcal{U} \)). Hence \( G \) is \( \ell \)-isomorphic to a (sublattice) subgroup of the o-group \( H \in q(\mathfrak{m}) \). Consequently, \( G \in q(\mathfrak{m}) \). \( \square \)

It only remains to prove Lemma \ref{mainlemma}.
Proof: If the lemma were false, choose a counter-example $G$ with $k$
minimal. By the minimality and rechristening, we may assume that
(I) $w_1(g_1, \ldots, g_n) > \ldots > w_k(g_1, \ldots, g_n) > 1$ in $G$, and 
(II) there is $j_0 \in \mathbb{Z}_+$ and a total order on $H/I_{j_0}(H)$ in which

$I_{j_0}(H)w_2(h_1, \ldots, h_n), \ldots, I_{j_0}(H)w_k(h_1, \ldots, h_n) \in (H/I_{j_0}(H))_+.$

Whenever $j \geq j_0$, there is a natural homomorphism from $H/I_{j_0}(H)$
onto $H/I_{j_0}(H)$ determined by $I_j(H)h \mapsto I_{j_0}(H)h$. Since $H/I_{j_0}(H)$ is a
torsion-free nilpotent group, by Lemma 1.2 we can lift any total order
from $H/I_{j_0}(H)$ to $H/I_j(H)$ (let $I_j(H)h \in (H/I_j(H))_+$ iff $I_{j_0}(H)h \in
(H/I_{j_0}(H))_+$; extend this partial order to a total order on $H/I_j(H)$ by
Lemma 1.2).

By Lemma 1.1, if $j \geq j_0$, we may assume that $b_2, \ldots, b_k \in (H/I_j(H))_+$
but

$1 \in N_{H/I_j(H)}(b_1, \ldots, b_k),$

where $b_i = I_j(H)w_i(h_1, \ldots, h_n)$ ($i = 1, \ldots, k$).

Let $d_i = w_i(g_1, \ldots, g_n) \in \mathbb{Z}_+ (i = 1, \ldots, k)$, and $C = C_{d_i}$ be the value
of $d_i$ in $G$. By (I), we have $d_1, \ldots, d_k \in C(d_1) = C(d_1)/C \subseteq \zeta(G/C).

Let $K = \langle [b_i, h_{\ell}] : i = 1, \ldots, k, \ell = 1, \ldots, n \rangle^H$. Let $\bar{H} = H/K$
and $\bar{G} = G/C$; write $\bar{h}$ for $Kh$ and $\bar{g}$ for $Cg$ ($h \in H; g \in G$). Since
$d_i \in \zeta(\bar{G})$ ($i = 1, \ldots, k$), $\bar{G}$ is a homomorphic image of $\bar{H}$ under the
naturally induced map; say $\bar{G} = \bar{H}^{\bar{\phi}}$. Note that $\bar{H}$ is a finitely presented
group and $\bar{G} \in \bar{\mathcal{N}}$.

Now $\bar{d}_1 \geq \ldots \geq \bar{d}_k \geq 1$ with $\bar{d}_i \in \bar{G}_+$. Since $\bar{b}_i \in \zeta(\bar{H})$ ($i = 1, \ldots, k$)
and $1 \in N_{H/I_j(H)}(b_1, \ldots, b_k)$ for all $j \geq j_0 \in \mathbb{Z}_+$, we have (for such $j$)

$\bar{b}_1^{m_{1,j}} \cdots \bar{b}_k^{m_{k,j}} \equiv 1 \pmod{I_j(\bar{H})}$ (20)

with $m_{1,j} \in \mathbb{Z}_+$ and $m_{2,j}, \ldots, m_{k,j} \in \mathbb{N}$.

For each $j \geq j_0$, we define $\text{rank}_j(\bar{b}_1, \ldots, \bar{b}_k)$ as the minimum of
$|\{i \in \{1, \ldots, k\} : m_{i,j} \neq 0\}|$ ranging over all equivalences that can occur
in (20).

If $j \geq j_0$, the natural homomorphism from $\bar{H}/I_j(\bar{H})$ onto $\bar{H}/I_{j_0}(\bar{H})$
gives that $\bar{b}_1^{m_{1,j}} \cdots \bar{b}_k^{m_{k,j}} \equiv 1 \pmod{I_{j_0}(\bar{H})}$ whenever
$\bar{b}_1^{m_{1,j}} \cdots \bar{b}_k^{m_{k,j}} \equiv 1 \pmod{I_j(\bar{H})}$. Hence $\text{rank}_j(\bar{b}_1, \ldots, \bar{b}_k)$ is an increasing
positive-integer-valued function of $j$, bounded above by $k$. Thus there
is $j_1 \geq j_0$ such that $\text{rank}_j(\bar{b}_1, \ldots, \bar{b}_k)$ is constant on all integers greater

16
than or equal to \( j_1 \), with \( m_{i,j} = 0 \) iff \( m_{i,j_1} = 0 \) \((i = 1, \ldots, k)\). By the minimality of \( k \), we have that rank\(_j(\vec{b}_1, \ldots, \vec{b}_k) = k \) for all \( j \geq j_1 \). Thus we can assume that if any of \( m'_{1,j}, \ldots, m'_{k,j} \) is 0, then \( \vec{b}_1^{m'_{1,j}}, \ldots, \vec{b}_k^{m'_{k,j}} \neq 1 \) in (20) \((j \geq j_1)\).

So \( m_{1,j}, \ldots, m_{k,j} \in \mathbb{Z}_+ \) for all \( j \geq j_1 \).

By (20), \( \vec{b}_1 \in \Gamma_{\vec{B}_1}(\vec{H}) \), where \( \vec{B}_1 = (\vec{b}_2, \ldots, \vec{b}_k) \subseteq \vec{H} \).

By Lemma 2.3, there exist \( t_1, t_2, \ldots, t_k \in \mathbb{Z} \) with \( t_1 \in \mathbb{Z}_+ \) such that
\[
\vec{b}_1 := \vec{b}_1^{t_1} \vec{b}_2^{t_2} \cdots \vec{b}_k^{t_k} \in \Gamma_{\vec{B}_1,\vec{H}}(\vec{H}).
\]

But \([\vec{b}_1, \vec{H}] = 1\) for \( i = 1, \ldots, k\) by the definition of \( K \). Hence \( \hat{b}_1 \in \Gamma(\vec{H}) \).

Therefore, for all \( j \geq j_1 \),
\[
\vec{b}_1^{t_1} \vec{b}_2^{t_2} \cdots \vec{b}_k^{t_k} \equiv 1 \pmod{I_j(\vec{H})}. \tag{21}
\]

We now use that rank\(_j(\vec{b}_1, \ldots, \vec{b}_k) = k \) for all \( j \geq j_1 \) to get a contradiction if \( t_i < 0 \) for some \( i = 2, \ldots, k \).

For reductio ad absurdum, assume that \( t_i < 0 \) for some \( i = 2, \ldots, k \). Choose \( r \leq k \) maximal so that there are \( t_{i_2}, \ldots, t_{i_r} < 0 \), where \( i_2, \ldots, i_r \) are distinct members of \( \{2, \ldots, k\} \). Fix \( j \geq j_1 \). By rechristening, we may assume that \(-t_{i_2}/m_{i_2,j}, \ldots, -t_{i_r}/m_{i_r,j}\). By (20) and (21), we get
\[
\vec{b}_1^{-t_{i_2}m_{i_2,j} + t_{i_2}m_{i_2,j}} \cdots \vec{b}_1^{-t_{i_r}m_{i_r,j} + t_{i_r}m_{i_r,j}} \cdots \vec{b}_1^{-t_{i_2}m_{i_2,j} + t_{i_2}m_{i_2,j}} \equiv 1 \pmod{I_j(\vec{H})}.
\]

Now, in this expression, all exponents are non-negative by the maximality of the ratio, and \(-t_{i_2}m_{i_2,j} + t_{i_2}m_{i_2,j} = 0 < -t_{i_2}m_{i_2,j} + t_{i_2}m_{i_2,j}\). This contradicts that rank\(_j(\vec{b}_1, \ldots, \vec{b}_k) = k \). Thus \( t_1, \ldots, t_k \in \mathbb{Z}_+ \).

But \( \vec{G} = \vec{H}^\varphi \) and \( \vec{G} \in \tilde{\mathfrak{u}} \), so \( \vec{b}_1 \in \ker(\varphi) \) by Lemma 3.1. Consequently, \( 1 = \vec{b}_1^\varphi = (\vec{b}_1^{t_1} \vec{b}_2^{t_2} \cdots \vec{b}_k^{t_k})^\varphi = \vec{d}_1^{t_1} \cdots \vec{d}_k^{t_k} \geq \vec{d}_1 \).

This is impossible as \( \vec{d}_1 \in \vec{G}_+ \). The proof of the lemma (and hence Theorem B) is now complete. //
4 The proof of Theorem C

We begin with a lemma:

Lemma 4.1 Let $G$ be a centrally ordered group. Suppose that $G/C$ is residually torsion-free-nilpotent for each value $C$ of $G$. Then $G \in \mu(\mathfrak{M})$.

Proof: Suppose that $G \notin \mu(\mathfrak{M})$. Then $G \notin \mathfrak{N}$ by Theorem E, and so one of the identities (**) fails in $G$. Thus there are $k, m, n \in \mathbb{Z}_+$, $g_1, \ldots, g_n \in G$, $u_1(x_1, \ldots, x_n), \ldots, u_k(x_1, \ldots, x_n) \in F = F(x_1, \ldots, x_n)$, and $w(x_1, \ldots, x_n) \in \Gamma_{[U,F]}(F)$ such that

$$|w(g_1, \ldots, g_n)|^m > \bigvee_{j=1}^k |u_j(g_1, \ldots, g_n)|,$$

where $U := \langle u_1(x_1, \ldots, x_n), \ldots, u_k(x_1, \ldots, x_n) \rangle$.

Let $C = C_w$ be the value of $w = w(g_1, \ldots, g_n)$ in $G$, and $H = G/C$. Let $\bar{g} := Cg \ (g \in G)$, $\bar{v} = Cv(g_1, \ldots, g_n) \ (v \in F)$, and $\bar{U} = \langle \bar{u}_1, \ldots, \bar{u}_k \rangle \subseteq H$. Now $C(w)/C \subseteq \zeta(H)$, so $\bar{w}, \bar{u}_j \in \zeta(H) \ (j = 1, \ldots, k)$. It follows that $[\bar{U}, H] = 1$. Since $w(x_1, \ldots, x_n) \in \Gamma_{[U,F]}(F)$, we get $\bar{w} \in \Gamma_{[\bar{U}, H]}(H) = \Gamma(H)$. This contradicts that $H$ is residually torsion-free-nilpotent. \(/\)

This is enough to prove Theorem C.

Proof: Since weakly Abelian $\ell$-groups are residually ordered, it is enough to prove the theorem for finitely generated centrally ordered Abelian-by-nilpotent groups by Lemma 1.3. The result follows from Lemmata 1.5 and 4.1. \(/\)

Corollary 4.2 (i) There is a weakly Abelian lattice-ordered group with the maximal condition on normal subgroups that does not belong to $\mathfrak{N}$.

(ii) $\mathfrak{N}$ is not closed under central extensions.

Proof: Since Example 2.2 is cyclic-by-(finitely generated metabelian), it satisfies the maximal condition on normal subgroups. (ii) follows from Example 2.2 and Theorem C. \(/\)
5 Concluding remarks

The results we have obtained beg several questions. The easiest to state are:

1. Does $\mathfrak{N}$ have a finite set of defining identities? We suspect not.

2. What varieties of lattice-ordered groups can occur between $\mathfrak{N}$ and $\mathcal{W}$? Is it possible that the latter covers the former (in which case the answer to Kopytov’s question is only just “no”)?

Added in Proof:

The results in Section 3 can be used to obtain the converse of Theorem D. Consequently, $q(\mathfrak{N})$ can be defined by group implications and contains the same groups as the quasi-variety generated by all torsion-free nilpotent groups.

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