Rooted Wreath Products

A. M. W. Glass & Reinhard Winkler

Abstract

We introduce “rooted valuation products” and use them to construct universal Abelian lattice-ordered groups (with prescribed set of components) [CHH] from the more classical theory of [Ha]. The Wreath product construction of [H] and [HMc] generalised the Abelian (lattice-ordered) group ideas to a permutation group setting to respectively give universals for transitive (ℓ-) permutation groups with prescribed set of primitive components. In the case of (not necessarily transitive) sublattice subgroups of order-preserving permutations of totally ordered sets, the set of natural congruences forms a root system. We generalise the rooted valuation product construction to the permutation case when all natural primitive components are regularly obtained; we analogously obtain universals for these permutation groups (for a prescribed set of natural primitive components) which we call “rooted Wreath products”. We identify the rooted valuation product with an appropriate subgroup of the corresponding rooted Wreath product.

The maximal Abelian group actions on the ordered real line were characterised in [W], and their digital representations were consequently obtained. We use the rooted Wreath product construction to get a more general result, and deduce the characterisation in [W] as a consequence.

Subject classification: 06F15, 06F20 (08A35, 26A48)

Key words: Valuation, root system, Hahn group, lattice-ordered group, Wreath product, Abelian group action, order-preserving permutation, normal-valued subgroup.

Acknowledgements: This work was begun at a conference in Dresden in August 2001, continued during a visit to Vienna in June 2002, and completed in Brno in September 2002. We are most grateful
to Manfred Doste for making the initial collaboration possible, and the various funding agencies of that conference; AMWG also thanks the Royal Society and the Österreichische Akademie der Wissenschaften for funding his visit to Vienna, and the Akademie and the Technische Universität Wien for their hospitality; and RW thanks the Austrian Science Foundation FWF for its support through Project no. S8312.

Sadly, during the course of this research, we learnt of the death of F. Sik (Brno, Czech Republic). He was one of the pioneers in the study of the group of partially ordered groups and order-preserving permutations of totally ordered sets. We dedicate this paper to him, in memoriam, as a tribute to his work and courage.
1 Introduction

Classical valuations are assignments of commensurates to a number field $K$. In the Archimedean case, these are maps from $K$ into the additive group of real numbers; and in the discrete case, these are $p$-adic valuations into $\mathbb{Q}_p$ ($p$ prime). In each case, there are naturally associated valuation rings and residue fields [BS]. These concepts have been an important tool in nineteenth and twentieth century number theory.

More generally, valuations from a field $K$ of characteristic 0 into a totally ordered Abelian group $T$ have been considered [J]; the associated residue fields were called components.

In an attempt to better understand H. Hahn’s original work [Ha] of 1907, P. F. Conrad, a student of Reinhold Baer (1902-1979), generalised prior constructions. He replaced the field $K$ by an arbitrary Abelian group $G$ and $T$ by a family $\{R_\gamma : \gamma \in \Gamma\}$ of subgroups of $\mathbb{R}$ for an arbitrary partially ordered set $\Gamma$. As in the field case, Conrad showed that the “$\Gamma$-sum” of the divisible closures of the family $\{R_\gamma : \gamma \in \Gamma\}$ was universal for the class of Abelian groups with these components (see [C]).

In 1963, $G$ was restricted to be an Abelian lattice-ordered group and $\Gamma$ was restricted to a root system (an “upside-down tree”) [CHH] — Abelian lattice-ordered groups correspond to groups of divisibility (that are also Bézout do-

In Section 2 we show how to construct these $\Gamma$-sums for Abelian lattice-ordered groups from the more classical theory for Abelian totally ordered groups $T$ using “rooted valuation products”.

The (non-Abelian) Wreath product construction has been generalised by W. C. Holland [H] to the case that the set of congruences forms a totally ordered set. The ideas he used were loosely based on the classical valuation group construction and the generalisation in [C], and were inspired by [CHH]. They can be regarded as the permutation group analogue of the above: the Abelian group is replaced by an arbitrary permutation group $(G, S)$, and the set $T$ by a partially ordered set. Components in this setting are primitive components of the permutation group and the Wreath product constructed was shown to be universal for them. In [HMc], the sets $S$ and $T$ were restricted to be totally ordered; universals were obtained for the result-
ing systems. The proofs in both [H] and [HMc] critically rely on the extra assumption that \((G, S)\) is transitive.

In the case of (not necessarily transitive) sublattice subgroups of order-preserving permutations of totally ordered sets, the set of natural congruences forms a root system. In Section 3 we take advantage of Holland’s prior analogy and modify the rooted valuation product idea to consider the intransitive permutation group case — provided that all natural primitive components are regularly obtainable. We obtain the “rooted Wreath product”, a universal for certain groups of order-preserving permutations with prescribed set of (natural) primitive components. We conclude the section with a natural identification of a rooted valuation product with the appropriate subgroup of the corresponding rooted Wreath product. Note that intransitive sublattice subgroups were considered in [P], though for an entirely different reason. Pierce’s construction of the orbit Wreath product of an intransitive permutation group does NOT preserve components (it wasn’t designed to do so), only orbit structure (see [G1], Chapter 10).

The maximal Abelian group actions on the ordered real line were characterised in [W], and their digital representations were consequently obtained. In Section 4, we use the rooted Wreath product construction to get a more general result, and deduce the characterisation in [W] as a consequence. We conclude the paper with an example to show that the results in [W] cannot be obtained using transitive permutation groups; intransitive actions are essential. This was our initial motivation for the study of rooted Wreath products.

2 Valuations

2.1 Background

Throughout we will consider groups with partial orders defined on them which are compatible with the group operation (i.e., \( xfy \leq xgy \) whenever \( f \leq g \)). If the partial order is total, we call the structure an ordered group or o-group for short; if the order is a lattice (the least upper bound (\( \vee \)) and greatest lower bound (\( \wedge \)) exist for any pair of elements), then we call the structure a lattice-ordered group or \( \ell \)-group for short. In all these cases, we will write \( G^+ \) for \( \{g \in G : g \geq 1\} \) and \( G^*_\gamma \) for \( G^+ \setminus \{1\} \).

Let \( \Gamma \) be a totally ordered set and for each \( \gamma \in \Gamma \), let \( R_\gamma \) be a subgroup of
the additive group \( \mathbb{R} \). Let \( F \) be the additive group of all functions \( g : \Gamma \to \mathbb{R} \) with \( g(\gamma) \in R_\gamma \) for all \( \gamma \in \Gamma \). For each \( g \in F \), let \( \text{supp}(g) \), the support of \( g \), be the set \( \{ \gamma \in \Gamma : g(\gamma) \neq 0 \} \). Let \( V = V(\Gamma, \{ R_\gamma : \gamma \in \Gamma \}) \) be the set of all \( g \in F \) such that \( \text{supp}(g) \) is either empty or every non-empty subset of \( \text{supp}(g) \) has a maximal element. Then \( V \) is a subgroup of \( F \). It is an Abelian \( o \)-group where \( f < g \iff f(\beta) < g(\beta) \) where \( \beta \) is the greatest element of \( \text{supp}(g - f) \).

If \( G \) is an Abelian \( o \)-group, then the set of convex subgroups forms a chain under inclusion \([G2],[G2]\), Lemma 3.1.2. Thus if \( g \in G \setminus \{ 0 \} \), then there is a unique convex subgroup of \( G \) that is maximal with respect to not containing \( g \). It is called the value of \( g \) and will be denoted by \( V_g \). The intersection of all convex subgroups of \( G \) that contain \( g \) and \( V_g \) is a convex subgroup of \( G \) denoted by \( V_g^* \); the pair \( (V_g, V_g^*) \) is called a covering pair. Let \( \Gamma(G) \) denote the set of all covering pairs totally ordered by inclusion. In 1901, Hölder \([Ho]\) proved that \( V_g^*/V_g \) is isomorphic to an additive subgroup \( R_\gamma \) of \( \mathbb{R} \), and that this isomorphism preserves the natural orders. In 1907, H. Hahn \([Ha]\) obtained the crucial representation that (in modern terminology) every Abelian \( o \)-group is a group of functions; indeed, if \( G \) is an Abelian \( o \)-group, then \( G \) can be embedded in \( V = V(\Gamma(G), \{ \overline{R}_\gamma : \gamma \in \Gamma \}) \), where \( \overline{R}_\gamma \) is the divisible closure of \( R_\gamma \) in \( \mathbb{R} \) (and this embedding preserves order).

Now let \( \Gamma \) be a root system; i.e., a partially ordered set such that \( \gamma \) and \( \delta \) have a common lower bound only if \( \gamma \leq \delta \) or \( \delta \leq \gamma \). For each \( \gamma \in \Gamma \), let \( R_\gamma \) be a subgroup of \( \mathbb{R} \). Let \( F \) be the additive group of all functions \( g : \Gamma \to \mathbb{R} \) with \( g(\gamma) \in R_\gamma \), and \( \text{supp}(g) \) be defined as above. Let \( V = V(\Gamma, \{ R_\gamma : \gamma \in \Gamma \}) \) be the subgroup of all functions \( g \) such that \( \text{supp}(g) \) is either empty or every non-empty totally ordered subset of \( \text{supp}(g) \) has a maximal element. Then \( V \) is an Abelian \( \ell \)-group where \( g > 0 \iff g(\delta) > 0 \) for every maximal element \( \delta \) of \( \text{supp}(g) \).

Let \( G \) be an Abelian \( \ell \)-group. By Zorn’s Lemma, if \( g \in G \setminus \{ 0 \} \), then there is a (not necessarily unique) convex sublattice subgroup of \( G \) that is maximal with respect to not containing \( g \). It is called a value of \( g \) and will be denoted by \( V_g \). The intersection of all convex sublattice subgroups of \( G \) that contain \( g \) and \( V_g \) is a convex sublattice subgroup of \( G \) denoted by \( V_g^* \); the pair \( (V_g, V_g^*) \) is called a covering pair. For fixed \( g \in G \setminus \{ 0 \} \), let \( \Gamma(g) \) be the set of all such pairs \( (V_g, V_g^*) \) with \( V_g \) a value of \( g \) and \( V_g^* \) the cover of \( V_g \). Let \( \Gamma(G) = \bigcup \{ \Gamma(g) : g \in G \setminus \{ 0 \} \} \) denote the set of all covering pairs partially ordered by inclusion. Then \( \Gamma(G) \) is a root system \([G2]\), Corollary
3.5.5) and $\Gamma(G) = \bigcup \{\Gamma(g) : g \in G_+\}$ ([G2], Lemma 2.3.8). Hölder’s proof applies and establishes that $V_g^*/V_g$ is isomorphic to an additive subgroup $R_\gamma$ of $\mathbb{R}$, and that this isomorphism preserves the natural orders.

A subset $\Lambda$ of the root system $\Gamma(G)$ is called a plenary subset if (i) $\lambda \in \Lambda$ implies $\{\gamma \in \Gamma(G) : \gamma \geq \lambda\} \subseteq \Lambda$, and (ii) $\bigcap \{V_g : (V_g, V_g^*) \in \Lambda\} = \{0\}$.

In 1963, Conrad, Harvey and Holland [CHH] extended Hahn’s Theorem and proved that every Abelian $\ell$-group is a group of functions; indeed, if $G$ is an Abelian $\ell$-group and $\Gamma'(G)$ is an arbitrary plenary subset of the root system of all values $\Gamma(G)$ of $G$, then $G$ can be $\ell$-embedded in $V = V(\Gamma'(G), \{R_\gamma : \gamma \in \Gamma'\})$, where $R_\gamma$ is the divisible closure of $R_\gamma$ in $\mathbb{R}$ (an embedding that preserves the group and lattice operations).

A classical example of a connection between these concepts and commutative ring theory is provided by the “divisibility” condition: Let $D$ be a domain, $U(D)$ be its group of units, and $K$ be its field of quotients. Let $K^* = K \setminus \{0\}$. For $a, b \in K^*$, define $a|b$ iff $b/a \in D$. Note that $|$ is reflexive and transitive but not necessarily antisymmetric: $a|b$ and $b|a$ are simultaneously possible, but only if $b = au$ for some $u \in U(D)$. Thus, if we define $aU(D) \leq bU(D)$ if and only if $a|b$, then we obtain a well-defined partial ordering on $G(D) = K^*/U(D)$. The group $G(D)$ is called the group of divisibility and is an Abelian partially ordered group. If $v : K^* \to G(D)$ is the natural map given by $v : a \mapsto aU(D)$, then $x|a, b$ if and only if $v(x) \leq v(a), v(b)$.

Thus (see [Mo]):

$G(D)$ is an Abelian $\ell$-group iff $D$ is an hcf (also called gcd) domain;
$G(D)$ is an Abelian o-group iff $D$ is a valuation ring (with respect to $v$);
$G(D) \cong \mathbb{Z}$ iff $D$ is a discrete valuation ring (with respect to $v$).

### 2.2 Rooted valuation products

Let $\Gamma$ be a root system, $\{R_\gamma : \gamma \in \Gamma\}$ a family of subgroups of $\mathbb{R}$, and $V = V(\Gamma, \{R_\gamma : \gamma \in \Gamma\})$.

Let $\Delta$ be a maximal totally ordered subset of $\Gamma$. Since $\Gamma$ is a root system, if $\gamma \geq \delta \in \Delta$, then $\gamma \in \Delta$. Let

$$K(\Delta) = \{g \in V : \Delta \cap \text{supp}(g) = \emptyset\}.$$ 

Then $K(\Delta)$ is a convex sublattice subgroup of $V$. Now $V/K(\Delta)$ is an $\ell$-group under the naturally induced order: $K(\Delta) + f < K(\Delta) + g$ iff $(\Delta \cap \text{supp}(g-f)) \neq \emptyset$ and $f(\delta) < g(\delta)$ where $\delta$ is the maximal element of $\Delta \cap \text{supp}(g-f))$. Let
\(\nu(\Delta)\) be the natural \(\ell\)-surjection from \(V\) onto \(V/K(\Delta)\). Clearly, \(V/K(\Delta)\) is naturally \(\ell\)-isomorphic to the Hahn group, \(V(\Delta) = V(\Delta, \{R_\delta : \delta \in \Delta\})\). Call this \(\ell\)-isomorphism \(\phi(\Delta)\). So \(\psi(\Delta) := \phi(\Delta)\nu(\Delta) : V \to V(\Delta)\) is an \(\ell\)-surjection.

Let \(\mathcal{M}\) be the set of all totally ordered maximal subsets of \(\Gamma\). Then we can map \(V\) into \(V(\mathcal{M})(\sharp) = \prod_{\Delta \in \mathcal{M}} V(\Delta)\) using the \(\psi(\Delta)\)'s in the natural way:

\[\psi(g)_\Delta = \psi(\Delta)(g)\]

Then \(\psi\) is an \(\ell\)-homomorphism where \(w \in V(\mathcal{M})(\sharp)^+\) iff \((\forall \Delta \in \mathcal{M})(w_\Delta \geq 0)\).

If \(g \in \bigcap\{K(\Delta) : \Delta \in \mathcal{M}\}\), then \(\text{supp}(g) = \emptyset\) (whence \(g = 0\)); thus \(\psi\) is an \(\ell\)-embedding of \(V\) into \(V(\mathcal{M})(\sharp)\).

This construction is akin to writing \(V\) as a subdirect product of o-groups and then using Hahn’s Theorem for each. In that sense, it is wasteful, and we tighten it by using compatibility conditions. This idea was also used in [W]. Of course,

\[\psi(g)_{\Delta_1}(\delta) = \psi(g)_{\Delta_2}(\delta) \quad \forall \Delta_1, \Delta_2 \in \mathcal{M} \text{ and } \delta \in \Delta_1 \cap \Delta_2 \quad (*)\]

Consider \(V(\mathcal{M})\), the set of all elements of \(V(\mathcal{M})(\sharp)\) that enjoy property (*) . Then \(V(\mathcal{M})\) is a sublattice subgroup of \(V(\mathcal{M})(\sharp)\) that contains \(\psi(V)\).

We call \(V(\mathcal{M})\) the \textit{rooted valuation product} of \(V\).

Moreover, for each element \(w \in V(\mathcal{M})\), \(w_{\Delta_1}(\gamma) = w_{\Delta_2}(\gamma)\) for any \(\Delta_1, \Delta_2 \in \mathcal{M}\) to which \(\gamma\) belongs (by (*)). Thus we can obtain an element \(w^b : \Gamma \to \mathbb{R}\) with \(w^b(\gamma) \in R_\gamma\) for all \(\gamma \in \Gamma\). Since \(\text{supp}(w_\Delta)\) is inversely well-ordered for all \(\Delta \in \mathcal{M}\), the element \(w^b\) corresponding to \(w\) belongs to \(V\). Consequently, \(\psi\) maps \(V\) onto \(V(\mathcal{M})\); that is, \(V\) is \(\ell\)-isomorphic to \(V(\mathcal{M})\). Hence

**Theorem 2.2.1** If \(\Gamma\) is a root system, then \(V(\Gamma, \{R_\gamma : \gamma \in \Gamma\})\) is a rooted valuation product.

Every Abelian \(\ell\)-group \(G\) (with a plenary set of values, \(\Gamma'(G)\)) can be regarded as a sublattice subgroup of \(V(\Gamma'(G), \{\overline{R}_\gamma : \gamma \in \Gamma'\})\). We define \(V(G)\) to be \(V(\mathcal{M}')\), where \(\mathcal{M}'\) is the set of all maximal totally ordered subsets of \(\Gamma'(G)\) and \(\overline{R}_\gamma\) is the divisible closure of \(R_\gamma\) in \(\mathbb{R}\). Thus

**Corollary 2.2.2** Every Abelian \(\ell\)-group \(G\) is a sublattice subgroup of a rooted valuation product \(V(G)\).
3 Rooted Wreath products

3.1 Background

Let \( A(\Omega) \) denote the group of all order-preserving permutations of a totally ordered set \((\Omega, \leq)\); i.e., \( A(\Omega) = Aut(\Omega, \leq) \). Under the pointwise ordering, this group of functions (under composition) is an \( \ell \)-group:

\[
\alpha(f \lor g) = \max\{\alpha f, \alpha g\} \quad \text{and} \quad \alpha(f \land g) = \min\{\alpha f, \alpha g\},
\]

where, as is standard in permutation groups, we write \( \alpha f \) for the image of \( \alpha \in \Omega \) under \( f \in A(\Omega) \).

Let \( A(\Omega)^+ = \{ g \in A(\Omega) : (\forall \alpha \in \Omega)(\alpha g \geq \alpha) \} \).

Let \((\Omega, \leq)\) denote the Dedekind completion of \((\Omega, \leq)\); that is, the set obtained by non-empty cuts with the inherited order (as in the construction of \((\mathbb{R}, \leq)\) from \((\mathbb{Q}, \leq)\)). Each element of \( A(\Omega) \) extends uniquely to an element of \( A(\Omega) \) and we will identify \( A(\Omega) \) with this corresponding subgroup of \( A(\Omega) \).

For \( g \in A(\Omega) \), let \( \text{supp}(g) = \{ \alpha \in \Omega : \alpha g \neq \alpha \} \), the support of \( g \), and \( \text{Fix}(g) = \{ \alpha \in \Omega : \alpha g = \alpha \} = \Omega \setminus \text{supp}(g) \). If \( \alpha \in \Omega \), let \( \Delta(g, \alpha) \) be the interval in \( \Omega \) that is the convexification of the orbit of \( \alpha \) under \( g \); so

\[
\Delta(g, \alpha) = \{ \beta \in \Omega : (\exists m, n \in \mathbb{Z})(\alpha g^m \leq \beta \leq \alpha g^n) \}.
\]

If \( \alpha g \neq \alpha \), then \( \Delta(g, \alpha) \) is an open interval in \( \Omega \); otherwise it is a singleton.

Let \( G \) be a subgroup of \( A(\Omega) \) and \( G^+ = G \cap A(\Omega)^+ \). A non-empty convex subset \( X \) of \( \Omega \) is called a convex \( G \)-block if \((\forall g \in G)(Xg = X \text{ or } Xg \cap X = \emptyset)\).

The convex \( G \)-block \( X \) is called an extensive block if for each \( x, y, z \in X \), there are \( f, g \in G \) such that \( y \leq xf \in X \) and \( z \geq xg \in X \).

The convex \( G \)-block \( X \) is called a fat block if \( \{Xg : g \in G \text{ and } Xg > X\} \) has no least element (under \( < \)) and \( \inf(\bigcup\{Xg : g \in G \text{ and } Xg > X\}) = \sup(X) \), and similarly with \( < \) in place of \( > \), where we write \( X < Y \) iff \( x < y \) for all \( x \in X \), \( y \in Y \) and take the supremum and infimum in \( \Omega \).

If \( X \) is a convex \( G \)-block, then \( X^x = \bigcup\{Xg : g \in G\} \) is a \( G \)-invariant set, and \( \{Xg : g \in G\} \) partitions \( X^x \) into convex (in \( \Omega \)) blocks which are fat (or extensive) if \( X \) is.

More generally, let \( Y \subseteq \Omega \) be a \( G \)-invariant set and \( C \) be an equivalence relation on \( Y \). If each \( C \)-class is a fat or extensive block, then \( C \) is a congruence and we call it a natural \( G \)-congruence on \( Y \), or a natural partial \( G \)-congruence (on \( \Omega \)). So equivalence classes of natural partial \( G \)-congruences are convex in \( \Omega \).
We write \( \text{dom}(C) \) for the domain of the natural partial \( G \)-congruence \( C \); that is, all \( \alpha \in \Omega \) such that \( \alpha \beta \) for some \( \beta \in \Omega \) (and so, all \( \alpha \in \Omega \) such that \( \alpha \alpha \)). Note that \( \alpha \in \text{dom}(C) \) implies that \( \alpha g \in \text{dom}(C) \) for all \( g \in G \). As is standard, we write \( \alpha C \) for \( \{ \beta \in \Omega : \alpha \beta \} \).

We will write \( C \subseteq D \) if \( \alpha \beta \) whenever \( \alpha C \beta \). This is clearly equivalent to \( \alpha \subseteq \alpha D \) for all \( \alpha \in \text{dom}(C) \).

Under this partial order \((\subseteq)\), the set of natural partial \( G \)-congruences forms a root system ([Mc] or [G1], Theorem 3B†); so if \( C_1, C_2, C_3 \) are natural partial \( G \)-congruences and \( C_1 \subseteq C_2 \cap C_3 \), then \( C_2 \subseteq C_3 \) or \( C_3 \subseteq C_2 \). Moreover, the union and non-empty intersections of natural partial \( G \)-congruences are natural partial \( G \)-congruences.

As shown in [Mc] (or see [G1], Theorem 3C†), for any distinct \( \alpha, \beta \in \R \) there are natural partial \( G \)-congruences \( C \subseteq C^* \), such that \( \alpha C^* \beta \) and \( \neg(\alpha C \beta) \), where \( C \) and \( C^* \) are natural \( G \)-congruences on the same set \( \{ \alpha C^* \}^\uparrow \) — whence \( G \) is transitive on the set of \( C^* \)-classes and — and no natural \( G \)-congruence on \( \{ \alpha C^* \}^\uparrow \) lies strictly between \( C \) and \( C^* \):

The intersection of all natural partial \( G \)-congruences in which \( \alpha, \beta \) belong to the same class provides \( C^* \). \( C \) is obtained using Zorn’s Lemma: let \( \Lambda = \text{dom}(C^*) \) and consider the set of all natural \( G \)-congruences on \( \Lambda \) (contained in \( C^* \)) in which \( \alpha \) and \( \beta \) belong to separate classes. This set is non-empty (it includes the natural \( G \)-congruence on \( \Lambda \) all of whose classes are singletons) and is closed under unions of chains. \( C \) is any maximal element thereof.

We write \( \text{val}(\alpha, \beta) \) for such a pair \((C, C^*)\) of natural partial \( G \)-congruences. It is further shown that if \( X \) is any \( C^* \)-class, then \( G \) induces a permutation action on \( X \) as follows:

Let \( G_{(X)} = \{ g \in G : X g = X \} \), a subgroup of \( G \) (convex in \( G \) under the pointwise ordering). Let

\[
L(X, G) = \{ g \in G_{(X)} : (\forall x \in X)((x C)g = x C) \}.
\]

Then \( L(X, G) \) is a normal subgroup of \( G_{(X)} \) called the lazy subgroup associated with \((C, C^*)\).

Let \( \hat{G}(X) = G_{(X)} / L(X, G) \). Then \( \hat{G}(X) \) acts faithfully on \( X/C := \{ xC : x \in X \} \). The resulting permutation group \((\hat{G}(X), X/C)\) is called a primitive component of \( G \).

We say that the action of \( G \) on \( X/C \) is coherent if

\[
\hat{G}(X) = \{ 1 \} \text{ or } (\forall a, b \in X)(\exists g \in G_{(X)})(aC < bCg \text{ and } (\forall d \in X)(dC \leq dCg)).
\]
As shown in [Mc] (or see [G1], Theorems 4C and 4A), if \( \hat{G}(X) \neq \{1\} \) and \( G \) is coherent on \( X \), then \( (G(X), X/C) \), satisfies a trichotomy: it is either integral, or transitively derived from a subgroup of \( \mathbb{R} \), or transitively derived from a weakly "locally order-two transive" faithful action on \( X \). That is, either there is a subset of \( X/C \cong \mathbb{Z} \) and \( \hat{G}(X) \) acts as \( \mathbb{Z} \) on this set (and on all of \( X/C \)), or there is a dense subset of \( X/C \) on which \( \hat{G}(X) \) is a right regular subgroup of \( \mathbb{R} \) or for each \( x \in X \), there is \( g \in \hat{G}(X) \) such that \( xC < xCg \) and for all \( y, z \in X \) with \( x < y < z \), there are \( f, h \in \hat{G}(X) \) such that \( xCf = xC \) and \( yCf \geq zC \), and \( zCh = zC \) and \( xCh \geq yC \).

If the action of \( G \) on \( X/C \) is coherent for all such \( X, C \) with \((C, C^*)\) a covering pair of natural partial \( G \)-congruences, then we will say that \( G \) is a coherent subgroup of \( A(\Omega) \). Note that every sublattice subgroup of \( A(\Omega) \) is coherent, but the converse fails: the group of all differentiable functions in \( A(\mathbb{R}) \) is coherent but not a sublattice of \( A(\mathbb{R}) \).

So the definition of coherence generalises the more usual sublattice condition.

If \( H \) is an \( \ell \)-group and \( g \in H \setminus \{1\} \), then there is a convex sublattice subgroup \( V_g \) of \( H \) maximal with respect to not containing \( g \). \( V_g \) is called a value of \( g \). Let \( V^*_g \) be the intersection of all convex sublattice subgroups of \( H \) that contain \( V_g \) and \( g \). If \( V^*_g < V^*_g \) for all \( g \in H \setminus \{1\} \) and values \( V_g \) of \( g \), then we call \( H \) normal-valued. It is well-known that \( H \) is normal-valued iff it satisfies the identity \( |f| |g| \leq |g|^2 |f|^2 \) where \( |h| = h \vee h^{-1} \) (see [G2], Section 4.2 and [G1], Chapter 11). For other equivalent conditions, see op. cit.

This latter condition can be applied to subgroups of \( A(\Omega) \) which are not necessarily sublattices. A coherent subgroup \( G \) of \( A(\Omega) \) will be called normal-valued if it satisfies the condition \( |f| |g| \leq |g|^2 |f|^2 \) for all \( f, g \in G \) (where \( |h| \) belongs to \( A(\Omega) \) for each \( h \in G \), but is not necessarily in \( G \)). Also observe that since \( |g| \geq 1 \) for all \( g \in G \), all Abelian coherent subgroups of \( A(\Omega) \) are normal valued. Many of the proofs of equivalent conditions for \( \ell \)-groups apply equally to the coherent case. In particular, \( G \) is normal-valued iff \( (\Delta(g, \alpha) \subseteq \Delta(f, \alpha) \) or \( \Delta(f, \alpha) \subseteq \Delta(g, \alpha) \) for all \( f, g \in G, \alpha \in \Omega \), [G1], Theorem 11A. That is, (writing \( P(G) \) for \( \{\Delta(g, \alpha) : g \in G, \alpha \in \Omega\} \)),

**Lemma 3.1.1** \( G \) is normal valued iff \( P(G) \) is a root system under inclusion: if \( I, J, K \in P(G) \) then \( K \subseteq I \cap J \) implies \( K \subseteq I \) or \( J \subseteq I \).

Equivalently, \( G \) is normal valued iff each non-trivial primitive component of \( G \) is integral or transitively derived from a right regular representation of
a subgroup of \((\mathbb{R}, +)\) (and so every primitive component is Abelian).

**Note:** The key point is that for normal-valued subgroups \(G\) of \(A(\Omega)\) all primitive components are Abelian and fixed-point free. This suffices for nice representations of Wreath products, and hence the generalisation of \([W]\) (c.f., [W, the Local-Global Principle]). In keeping with this, we will say that the action is “locally Abelian” in this case.

### 3.2 Transitive Wreath products

We recall the main theorem in [H] and [HMc] (or [G1], p.122 ff. or [G2], p.158 ff.). Let \(\Omega\) be a chain and \((G, \Omega)\) be a transitive group of order-preserving permutations of \(\Omega\). Let \(\mathcal{R}_0\) be the set of all natural (in this case, extensive) \(G\)-congruences on \(\Omega\). Then \(\mathcal{R}_0\) is a chain under inclusion. Let \(\mathcal{R}\) be the set of all covering pairs of natural (extensive) \(G\)-congruences in \(\mathcal{R}_0\). So if \(K \in \mathcal{R}\) (say, \(K = \text{val}(x, y)\)), then \((G, \Omega)\) can be embedded in

\[
(W, \hat{\Omega}) = \text{Wr}\{\hat{G}(x\mathcal{C}^K), x\mathcal{C}^K / \mathcal{C}_K : K \in \mathcal{R}\},
\]

the Wreath product of its primitive actions (where we write \((\mathcal{C}_K, \mathcal{C}^K)\) for the covering pair associated with \(K\)).

Specifically, if \(\mathcal{R}\) is the set of covering pairs of natural \(G\)-congruences and the primitive components of \((G, \Omega)\) are \((G_K, \Omega_K)\) \((K \in \mathcal{R})\), then \(\mathcal{R}\) is totally ordered by \((\mathcal{C}, \mathcal{C}^*) < (\mathcal{D}, \mathcal{D}^*)\) iff \(\mathcal{C}^* \subseteq \mathcal{D}\).

Let \(\Omega^1 = \prod \{\Omega_K : K \in \mathcal{R}\}\). Choose an arbitrary fixed reference point in \(\Omega^1\) denoted by \(\hat{0}\). For each \(\alpha \in \Omega^1\), let \(\text{supp}(\alpha) = \{K \in \mathcal{R} : \alpha_K \neq \hat{0}_K\}\). Let

\[
\hat{\Omega} = \{\alpha \in \Omega^1 : \text{supp}(\alpha) \text{ is an inversely well-ordered subset of } \mathcal{R}\}.
\]

Note that if \(\alpha, \beta \in \hat{\Omega}\) are distinct, then \(\emptyset \neq \mathcal{R}(\alpha, \beta) = \{K \in \mathcal{R} : \alpha_K \neq \beta_K\} \subseteq \text{supp}(\alpha) \cup \text{supp}(\beta)\), and so \(\mathcal{R}(\alpha, \beta)\) is also inversely well-ordered. It therefore has a greatest element, say \(K_0\). We make \(\hat{\Omega}\) a totally ordered set via: \(\alpha < \beta\) if and only if \(\alpha_{K_0} < \beta_{K_0}\).

We next define natural equivalence relations on \(\hat{\Omega}\). For each \(K \in \mathcal{R}\), define \(\equiv^K\) and \(\equiv^K\) by:

\[
\alpha \equiv^K \beta \text{ if } \alpha_{K'} = \beta_{K'} \text{ for all } K' > K
\]

\[
\alpha \equiv_K \beta \text{ if } \alpha_{K'} = \beta_{K'} \text{ for all } K' \geq K.
\]
Hence if $\alpha \neq \beta$ and $K_0$ is the largest element of $\mathfrak{R}(\alpha, \beta)$, then $\alpha \equiv^K \beta$ if $K \geq K_0$ and $\alpha \equiv^K \beta$ if $K > K_0$. Clearly, $\equiv^K$ and $\equiv_K$ have convex classes (for all $K \in \mathfrak{K}$). We wish them to be convex congruences; so let $W_1 = \{g \in A(\hat{\Omega}) : (\forall K \in \mathfrak{K})(\forall \alpha, \beta \in \hat{\Omega})[\{(\alpha \equiv^K \beta) \iff \alpha g \equiv^K \beta g\} & (\alpha \equiv_K \beta \iff \alpha g \equiv_K \beta g)\}].$ Then $\equiv^K$ and $\equiv_K$ are convex $W_1$-congruences. Observe that $\{(\alpha(\equiv^K))/(\equiv_K)\}$ is just $\Omega_K$ for each $\alpha \in \hat{\Omega}$ and $K \in \mathfrak{K}$.

For each $K \in \mathfrak{K}$ and $\alpha \in \hat{\Omega}$, let $\alpha^K \in \prod\{\Omega_{K'} : K' > K\}$ with $(\alpha^K)_{K'} = \alpha_{K'}$; i.e., $\alpha^K$ is $\alpha$ above $K$. Note that $\alpha^K = \beta^K$ precisely when $\alpha \equiv^K \beta$.

For each $g \in W_1$, $\alpha \in \hat{\Omega}$ and $K \in \mathfrak{K}, g$ induces an element of $A(\Omega_K)$: Let $\sigma \in \Omega_K$ and define $g_{K,\alpha^K}$ by:

$$
\sigma g_{K,\alpha^K} = (\alpha' g)_{K} \in \Omega_{K}
$$

where $\alpha' \equiv^K \alpha$ and $\alpha'_K = \sigma$.

**Lemma 3.2.1** With the above notation, $g_{K,\alpha^K} \in A(\Omega_K)$ for each $\alpha \in \hat{\Omega}$, $g \in W_1$ and $K \in \mathfrak{K}$.

Let $W = \{g \in W_1 : (\forall K \in \mathfrak{K})(\forall \alpha \in \hat{\Omega})(g_{K,\alpha^K} \in G_K)\}; (W,\hat{\Omega})$ is called the Wreath Product of $\{(G_K,\Omega_K) : K \in \mathfrak{K}\}$ and is written $\text{Wr} \{(G_K,\Omega_K) : K \in \mathfrak{K}\}$. The elements of $W$ may be thought of as $\mathfrak{K} \times \hat{\Omega}$ matrices $(g_{K,\alpha})$ with $g_{K,\alpha} = g_{K,\beta}$ if $\alpha^K = \beta^K$.

**Lemma 3.2.2** Assume that each $(G_K,\Omega_K)$ is transitive. Then so is $(W,\hat{\Omega}) = \text{Wr} \{(G_K,\Omega_K) : K \in \mathfrak{K}\}.

Moreover, if $\Omega' \in \Omega$ is chosen as reference point and the resulting Wreath product is $(W',\Omega')$, then $(W,\Omega)$ and $(W',\Omega')$ are ($\ell$-)isomorphic.

The culmination of these considerations is:

**Theorem 3.2.3** [Holland & McCleary 1969] Let $(G,\Omega)$ be a transitive group of order-preserving permutations of a totally ordered set $\Omega$. Let $\mathfrak{R} = \mathfrak{R}(G,\Omega)$, an index set for the set of all covering pairs of convex congruences of $(G,\Omega)$ ordered in the natural way by the induced inclusions. Let $\{(G_K,\Omega_K) : K \in \mathfrak{R}\}$ be the set of all primitive components of $(G,\Omega)$ and

$$
(W,\hat{\Omega}) = \text{Wr} \{(G_K,\Omega_K) : K \in \mathfrak{R}\}.
$$
Then there are injections $\phi: \Omega \to \hat{\Omega}$ and $\psi: G \to W$ such that $\alpha\phi < \beta\phi$ if $\alpha < \beta$ and $g\psi < f\psi$ if $g < f$. Moreover, $(\alpha g)\phi = (\alpha\phi)(g\psi)$ for all $\alpha \in \Omega$, $g \in G$, and any finite suprema and infima that exist in $G$ are preserved by $\psi$.

3.3 An Example

Let $\Omega$ be the totally ordered set obtained from $\mathbb{R}$ by replacing each rational number by a copy of $\mathbb{Z}$ and each irrational number by a copy of $\mathbb{R}$. So

$$\Omega = \{(n,q) : n \in \mathbb{Z}, q \in \mathbb{Q}\} \cup \{(r,s) : r \in \mathbb{R}, s \in \mathbb{R} \setminus \mathbb{Q}\},$$

ordered by $(a,x) < (b,y)$ iff $(x < y$ in $\mathbb{R}$ or, $x = y$ & $a < b$ (in $\mathbb{Z}$ or $\mathbb{R}$)).

Let $G$ be the group of all “generalised translations” of $\Omega$; so if $g \in A(\Omega)$, then $g \in G$ iff there are $q \in \mathbb{Q}$, $f_1 : \mathbb{Q} \to \mathbb{Z}$ and $f_2 : \mathbb{R} \setminus \mathbb{Q} \to \mathbb{R}$ such that

$$(a,x)g = \begin{cases} 
(a + f_1(x), x + q) & \text{if } x \in \mathbb{Q} \\
(a + f_2(x), x + q) & \text{if } x \not\in \mathbb{Q}.
\end{cases}$$

Note that $G$ is normal-valued but not Abelian. There are two non-trivial natural partial $G$-congruences whose domains are not all of $\Omega$: $C_1$ has classes $C(q) = \{(n,q) : n \in \mathbb{Z}\} (q \in \mathbb{Q})$; $C_2$ has classes $C(s) = \{(r,s) : r \in \mathbb{R}\}$ $(s \in \mathbb{R} \setminus \mathbb{Q})$. In this case, $\mathcal{R}$ the associated root system of all covering pairs of partial natural $G$-congruences, is a three element root system with a single maximal element $K$ and two (unrelated) elements $C_1, C_2 < K$. The maximal totally ordered subsets of $\mathcal{R}$ are $\mathcal{C}_1 = \{C_1,K\}$ and $\mathcal{C}_2 = \{C_2,K\}$. Note that the points $(a,x)$ with $x \in \mathbb{Q}$ have no bearing on $C_2$, and the points $(a,x)$ with $x \in \mathbb{R} \setminus \mathbb{Q}$ have no bearing on $C_1$. We therefore do not wish to consider $W_1 = (\mathbb{Z},\mathbb{Z}) \text{ Wr } (\mathbb{Q},\mathbb{R})$ and $W_2 = (\mathbb{R},\mathbb{R}) \text{ Wr } (\mathbb{Q},\mathbb{R})$ but instead $W(\mathcal{C}_1) = (\mathbb{Z},\mathbb{Z}) \text{ Wr } (\mathbb{Q},\mathbb{Q})$ and $W(\mathcal{C}_2) = (\mathbb{R},\mathbb{R}) \text{ Wr } (\mathbb{Q},\mathbb{R} \setminus \mathbb{Q})$, and then sew these together.

So, in considering $W(\mathcal{C}_1)$, we delete from $\Omega/C_K$ those elements $xC_K$ for which $x \in \mathbb{R} \setminus \mathbb{Q}$; that is, we remove all classes whose points do not belong to $\text{dom}(C_1)$. Similarly, for $W(\mathcal{C}_2)$. So instead of taking $\Omega_K = \mathbb{R}$, we take $\Omega_K(\mathcal{C}_1) := \mathbb{Q}$ and $\Omega_K(\mathcal{C}_2) := \mathbb{R} \setminus \mathbb{Q}$.

Then $W(\mathcal{C}_j) = (G(C_j),\Omega(\mathcal{C}_j)) \text{ Wr } (\hat{G}(K),\Omega_K(\mathcal{C}_j))$ for $j = 1,2$.

In both cases we have a translation by a rational number in the “upstairs” part. Analogously to the rooted valuation product, we form the rooted Wreath product:

$$W(G) = \{(w_1, w_2) \in W(\mathcal{C}_1) \times W(\mathcal{C}_2) : (xC_1)w_1 = (xC_2)w_2\},$$

13
where we take the natural extensions of $w_j$ from $\Omega_K(C_j)$ to $\Omega_K$ ($j = 1, 2$). This is possible since both are translations of subgroups of $\mathbb{R}$.

### 3.4 Rooted Wreath products

We wish to generalise the Wreath product construction ([H], [HMc]) to give a universal representation for normal-valued permutation groups $(G, \Omega)$ which are not necessarily transitive.

Consider the normal-valued permutation group $(G, \Omega)$. As in the transitive case, let $K_0$ be the root system of all partial natural $G$-congruences on $\Omega$ and $K$ the associated root system of all covering pairs of partial natural $G$-congruences.

If $K \in K$, then $\text{dom}(C^K) = \text{dom}(C_K)$, and we will write $\text{dom}(K)$ as an abbreviation for this common domain.

For each $G$-orbit $O$ of $\text{dom}(K)$, choose exactly one point $x(K, O)$, and let $T(K)$ be the resulting set of points (a subset of $\text{dom}(K)$). We do this in such a way that $K < K'$ implies that $T(K') \subseteq T(K)$ (existence, op. cit.).

For each $K \in K$ and orbit $O$, let $X(K, O) = x(K, O)C^K/C_K$.

Let $M$ be the set of all maximal chains in $K$ and $C \in M$. For each $K \in C$, let

$$T(K, C) = \{ x(K, O) \in T(K) : x(K, O) \notin \bigcup \{ T(K') : K' \notin C \} \},$$

and

$$\Omega_K(C) = \{ X(K, O) : x(K, O) \in T(K, C) \},$$

the “$C$ restricted” domain of $K \in C$.

**Remark**: Since $(G, \Omega)$ is normal-valued, if the induced restriction of $g \in \hat{G}(K)$ to $\Omega_K(C)$ is the identity, then it is the identity on all of $\Omega_K$.

Let

$$\Omega(\mathcal{C}) := \bigcup_{C \in \mathcal{C}} \Omega_C(\mathcal{C}),$$

the union of the “restricted” domains of the members of $\mathcal{C}$.

Note that $\Omega(\mathcal{C}) g = \Omega(\mathcal{C})$ for all $g \in G$, $\mathcal{C} \in M$. Then, as in Section 3.2 (or op. cit.), we can form the Wreath product

$$(W(\mathcal{C}), \Omega(\mathcal{C})) = \text{Wr} \{ (\hat{G}(X(C, O)), X(C, O)) : x(C, O) \in T(C, \mathcal{C}), C \in \mathcal{C} \}.$$
Since \((G, \Omega)\) is a normal-valued permutation group, each \(\hat{G}(X(C, \mathcal{O}))\) is \((\ell, \Omega)\)-isomorphic to a subgroup \(R(x(C, \mathcal{O}))\) of \(\mathbb{R}\), and each \(X(C, \mathcal{O})\) is a collection of orbits of \(R(x(C, \mathcal{O}))\) on each of which its action is induced by the right regular action.

Let
\[
L(\mathcal{C}) = \bigcap \{ L(x(C, \mathcal{O})C^C, G) : x(C, \mathcal{O}) \in T(C, \mathcal{C}), C \in \mathcal{C} \},
\]
and \(G(\mathcal{C}) = G/L(\mathcal{C})\).

By the remark,
\[
L(\mathcal{C}) = \bigcap \{ L(x(C, \mathcal{O})C^C, G) : x(C, \mathcal{O}) \in T(C), C \in \mathcal{C} \}. \quad (**)
\]

As above, we get a pair of embeddings \((\phi_{\mathcal{C}}, \psi_{\mathcal{C}})\) of \((G(\mathcal{C}), \hat{\Omega}(\mathcal{C}))\) as in the transitive case (by the remark).

Now \(\bigcap \{ L(\mathcal{C}) : \mathcal{C} \in \mathcal{M} \} = \{ 1 \}\) (since if \(g \neq 1\), then \(xg \neq x\) for some \(x \in \Omega\); then \(g \notin val(xg, x)\) and so \(g \notin L(\mathcal{C})\) for any chain \(\mathcal{C}\) containing \(val(xg, x)\) by \((**))\). Thus we obtain an embedding \(\theta : G \to \prod_{\mathcal{C} \in \mathcal{M}} G/L(\mathcal{C})\) induced by the natural maps \(\nu_{\mathcal{C}} : g \mapsto L(\mathcal{C})g\) (\(\mathcal{C} \in \mathcal{M}\)). Thus we have an embedding of \(G\) into \(\prod_{\mathcal{C} \in \mathcal{M}} W(\mathcal{C})\) induced by \(\{ \nu_{\mathcal{C}} \psi_{\mathcal{C}} : \mathcal{C} \in \mathcal{M} \}\).

To complete the analysis, we need two further observations:

(1). Since \((G, \Omega)\) is normal-valued, we have that for each \(K \in \mathcal{R}\) and \(x(K, \mathcal{O}) \in T(K)\), either
\[
(yC_K)g = yC_K \quad \text{for all} \quad yC^K x(K, \mathcal{O})
\]
or
\[
(yC_K)g \neq yC_K \quad \text{for all} \quad yC^K x(K, \mathcal{O}).
\]
So, as in standard group theory, the right regular actions provide an index set and we do not need to resort to a full permutation representation approach as in [HMc].

(2). Suppose that \(\mathcal{C}_1, \mathcal{C}_2 \in \mathcal{M}\). Then for all \(g \in G\), \(y \in \text{dom}(C)\) and \(C \in \mathcal{C}_1 \cap \mathcal{C}_2\),
\[
(yC_C)(g\psi_{\mathcal{C}_1}) = (yC_C)(g\psi_{\mathcal{C}_2}).
\]
Since \(\mathcal{R}\) is a root system, it follows that if \(C \in \mathcal{C}_1 \cap \mathcal{C}_2\), and \(C < K \in \mathcal{R}\), then \(K \in \mathcal{C}_1 \cap \mathcal{C}_2\).

In analogy with the rooted valuation product, we need to consider compatibility conditions to get a tighter embedding.
First note that we may uniquely extend each element of \((G_K, \Omega_K(\mathcal{C}))\), and that if \(K \in \mathcal{C}_1 \cap \mathcal{C}_2\), then \(g \in \hat{G}(K)\) is the same translation of \(\Omega_K\) as that given by the extensions of the corresponding elements of each of \((G_K, \Omega_K(\mathcal{C}_1))\) and \((G_K, \Omega_K(\mathcal{C}_2))\).

We define the rooted Wreath product \(W(G)\) to comprise all

\[
w \in \prod_{\mathcal{C} \in \mathcal{M}} W(\mathcal{C})
\]

that satisfy

\[(\forall \mathcal{C}_1, \mathcal{C}_2 \in \mathcal{M})(\forall C \in \mathcal{C}_1 \cap \mathcal{C}_2)(\forall y \in \text{dom}(C))(y C_{C} w_{\mathcal{C}_1} = (y C_{C}) w_{\mathcal{C}_2})\ (\ast).\]

Thus if \(w \in W(G)\), then \(w_{\mathcal{C}_1}\) agrees with \(w_{\mathcal{C}_2}\) on the (possibly empty) upper segment of \(\mathcal{C}_1 \cap \mathcal{C}_2\).

By (1) and (2) we have an embedding \(\chi\) of \(G\) into \(W(G)\) that preserves the (pointwise) ordering on \(G\) and any finite suprema and infima that exist in \(G\). Consequently, we obtain the desired universal:

**Theorem 3.4.1** If \((G, \Omega)\) is any normal-valued (coherent) permutation group with natural primitive components \((G_K, \Omega_K)\) \((K \in \mathcal{K})\), then \((G, \Omega)\) can be embedded in the rooted Wreath product of \(\{(G_K, \Omega_K) : K \in \mathcal{K}\}\); this embedding preserves order (and any finite suprema and infima that exist in \(G\)).

### 3.5 Rooted valuation subgroups of rooted Wreath products

First note that if \(\Omega\) is a totally ordered set and \(G\) is a sublattice subgroup of \(A(\Omega)\), then the root system of covering pairs of partial natural \(G\)-congruences is a subset of the root system \(\Gamma(G)\): if \(g \in G_+,\) let \(\alpha \in \Omega\) be such that \(\alpha g \neq \alpha\). Let \(\text{val}(\alpha g, \alpha) = (\mathcal{C}, \mathcal{C}^*)\), and \(G(\alpha, g) = \{f \in G : (\forall \beta \in \alpha \mathcal{C}^*)(\beta f \mathcal{C} \beta)\}\), a convex sublattice subgroup of \(G\). Then \(G(\alpha, g)\) is a value of \(g\), and \(\bigcap\{G(\alpha, g) : g \in G_+, \ \alpha \in \Omega, \ \alpha g \neq \alpha\} = \{1\}\). Thus \(\Gamma'(G) = \{G(\alpha, g) : g \in G_+, \ \alpha \in \Omega, \ \alpha g \neq \alpha\} = \{1\}\) is a plenary subset of \(\Gamma(G)\).

Let \(G\) be a transitive totally ordered Abelian subgroup of \(A(\Omega)\). Then \(\mathcal{K} = \mathcal{K}(G, \Omega)\) is a totally ordered set (which is naturally order isomorphic to \(\Gamma(G)\)). Let \((W, \hat{\Omega})\) be the resulting Wreath product as in Theorem 3.2.3.
Observe that if \( \text{val}(\alpha, \alpha g) = (C_K, C_K^\alpha) \), then \( C_K \) is the value of \( g \). Hence, with the obvious identification \( \Gamma(G) = \mathcal{R} \), consider

\[
V = \{ g \in W : (\forall K \in \mathcal{R})(\forall \alpha, \beta \in \Omega)(g_{K,\alpha} = g_{K,\beta}) \}.
\]

Then \( V \) is naturally \( \ell \)-isomorphic to the valuation group \( V(G) \). Thus we have the (totally ordered) valuation group for \( G \) occurring as a natural subgroup of the Wreath product associated with \( G \).

Next, let \( G \) be a transitive normal-valued subgroup of \( A(\Omega) \). Then \( \mathcal{R} = \mathcal{R}(G, \Omega) \) is a totally ordered set. Let \((W, \hat{\Omega})\) be the resulting Wreath product as in Theorem 3.2.3. Consider

\[
V = \{ g \in W : (\forall K \in \mathcal{R})(\forall \alpha, \beta \in \Omega)(g_{K,\alpha} = g_{K,\beta}) \}.
\]

Then \( V \) is naturally \( \ell \)-isomorphic to the \( \mathcal{R} \)-valuation group. Thus we again have a valuation group occurring as a natural subgroup of the Wreath product associated with \( G \).

Now suppose that \( G \) is an Abelian sublattice subgroup of \( A(\Omega) \), and assume further that all components \( R_\gamma \) of \( G \) are divisible. Form the rooted Wreath product \( \mathcal{W}(G) \) as above. With the notation of the previous section, for each \( C \in \mathcal{M} \) let \( \mathcal{V}(C) \) comprise the elements \( w \in \mathcal{W}(C) \) that satisfy

\[
w_{C,\alpha} = w_{C,\beta} \quad \text{for all } \alpha, \beta \in \Omega(C) \text{ and } C \in \mathcal{C}.
\]

Let

\[
V = \{ w \in \mathcal{W}(G) : (\forall C \in \mathcal{M})(w_C \in \mathcal{V}(C)) \}.
\]

Since \( \mathcal{R} \) is a plenary subset of \( \Gamma(G) \), we get that \( V \) is \( (\ell) \)-isomorphic to the \( \mathcal{R} \)-valuation group with components \( \{R_\gamma : \gamma \in \mathcal{R}\} \). So again we have a rooted valuation group associated with \( G \) identified naturally in the rooted Wreath product associated with \( G \).

Finally, suppose that \( G \) is a normal-valued subgroup of \( A(\Omega) \). Form the rooted Wreath product \( \mathcal{W}(G) \) as above. With the notation of the previous section, for each \( C \in \mathcal{M} \) let \( \mathcal{V}(C) \) comprise the elements \( w \in \mathcal{W}(C) \) that satisfy

\[
w_{C,\alpha} = w_{C,\beta} \quad \text{for all } \alpha, \beta \in \Omega(C) \text{ and } C \in \mathcal{C}.
\]

Let

\[
V = \{ w \in \mathcal{W}(G) : (\forall C \in \mathcal{M})(w_C \in \mathcal{V}(C)) \}.
\]

Thus we again have a valuation group occurring as a natural subgroup of the rooted Wreath product associated with \( G \).
4 An application

4.1 Background

Throughout, if $X, Y \subseteq \mathbb{R}$, then we write $\overline{X}$ for the topological closure of $X$ in $\mathbb{R}$, $X^o$ for the interior, and $\delta X$ for the boundary of $X$.

Let $P_g = \{\Delta(g, y) : y \in \mathbb{R} \text{ and } \Delta(g, y) \neq \{y\}\}$. For any subgroup $G$ of $A(\mathbb{R})$, let

$$P_G = \{\Delta(g, y) : g \in G, y \in \mathbb{R}\} = \bigcup_{g \in G} P_g.$$ 

Most of the lemmata of Section 2 of [W] only required that the intervals $\Delta(g, y) (g \in G; y \in \mathbb{R})$ of a coherent subgroup $G$ be non-overlapping, and this is what Lemma 3.1.1 achieves. Specifically, we have

Lemma 4.1.1 (c.f., [W], Lemma 2.1.2) Let $G$ be a normal-valued subgroup of $A(\mathbb{R})$.

1. If $I = \Delta(g, y) = (a, b)$, $J = \Delta(f, y) = (c, d)$ and $I \subseteq J$ with $I \neq J$, then $c < a < b < d$.

2. $\Delta(f, x)$ is an extensive block of $G$ for each $f \in G$ and $x \in \mathbb{R}$.

3. Let $\mathcal{K}$ be a chain in $(P_G, \subseteq)$ and $J = (a, b) \in P_G$ with $\overline{\bigcap \mathcal{K}} \subseteq J$ for all $I \in \mathcal{K}$. Let $I_0 = (a_0, b_0)$ denote the union of all $I \in \mathcal{K}$. Then $a < a_0 < b_0 < b$, i.e. $\overline{I_0} \subset J$.

Proof: Only (2) is different, so we provide a proof for it.

Assume that $\Delta(f, x)g \cap \Delta(f, x) \neq \emptyset$ and $\Delta(f, x)g \neq \Delta(f, x)$. Then $\overline{\Delta(f, x)} \subseteq \Delta(f, x)g$ without loss of generality (by (1)). If $\Delta(f, x) = (a, b)$, then $ag < a < x < b < bg$. By the Intermediate Value Theorem, $g$ fixes some element of $(a, b)$ (which we may assume to be $x$ by rechristening). By taking appropriate powers of $f, g$ in place of $f, g$, we may further suppose that $xfg > b$ (and $zf > z$ for all $z \in \Delta(f, x)$). Then $x|g|f^2 = xgf^2 = xf^2 < b < xfg = x|f||g|$. This contradicts that $G$ is normal valued. \hfill \Box

So if $G$ is a normal-valued subgroup of $A(\mathbb{R})$ and $\mathcal{K} \neq \emptyset$ is a chain in $P_G$, then $\bigcap \mathcal{K}$ either has a least element $I \in P_G$, or $\bigcap \mathcal{K}$ is a closed interval by Lemma 4.1.1(1).

Let $P''_G$ be the set of all interiors $(\bigcap \mathcal{K})^o$ of non-singleton such $\bigcap \mathcal{K}$. Note that $\mathcal{K} = \{I\}$ implies that $I = \bigcap \mathcal{K}$; so $P_G \subseteq P''_G$. 18
If \( J \in P_G^\prime\prime \) and \( \mathcal{K} \) is a chain in \( P_G \) maximal with \( K \subseteq J \) for all \( K \in \mathcal{K} \), then \( \bigcup \mathcal{K} \) is called a component of \( J \). Clearly, if \( I \) and \( K \) are components of \( J \), then \( I = K \), \( I < K \) or \( K < I \). Let \( P_G^\prime \) denote the set of all elements of \( P_G^\prime\prime \) that have at least two components. Thus \( P_G \subseteq P_G^\prime \subseteq P_G^\prime\prime \).

The proofs of the following lemmata were derived in [W] from the assertions in our Lemma 3.1.1. The same derivations apply here.

**Lemma 4.1.2** (c.f., [W], Proposition 2.1.3) Let \( G \) be a normal-valued subgroup of \( A(\mathbb{R}) \).

1. If \( I, J \in P_G^\prime\prime \), then exactly one of the following five statements holds: \( I = J \), \( I < J \), \( J < I \), \( I \subseteq J \) or \( J \subseteq I \). Thus \( (P_G^\prime\prime, \subseteq) \) is a root system.

2. \( (P_G^\prime, \subseteq) \) is at most countable.

3. \( G \) acts on \( (P_G^\prime\prime, P_G^\prime, P_G, \subseteq, \leq) \), i.e. preserves \( P_G^\prime \), \( P_G \), \( \subseteq \) and \( \leq \), and all inclusions and orders.

*Proof:* The proof of (3) given in [W] relied heavily on \( G \) being Abelian, so we provide a word of explanation here. Note first that if \( f \in G \) and \( x \in \text{supp}(f) \), then \( \Delta(f, x) \) is an extensive block of \( G \) by Lemma 4.1.1(2), and the \( G \)-congruence \( \mathcal{C} \) on \( \Delta(f, x)^2 \) whose blocks are \( \Delta(f, x)g \) \((g \in G)\) is extensive. Let \( \mathcal{D} \subseteq \mathcal{C} \) be the union of all natural \( G \)-congruences on \( \Delta(f, x)^2 \) in which \( x \) and \( f(x) \) belong to different classes. Then \( \mathcal{C} \) covers \( \mathcal{D} \) and the induced primitive action of \( G \) on \( \Delta(f, x)^2 \) is non-trivial and Abelian (Lemma 3.1.1) with \( zDg = zD \) if \( yDg = yD \) and \( zCg \). It follows that \( \Delta(f, x)g = xCg = xgC = \Delta(f, xg) \). Thus \( G \) maps \( P_G \) to itself and preserves \( \subseteq \) and \( \leq \). The rest of the proof follows from this by definition. //

**Lemma 4.1.3** (c.f., [W], Lemma 2.1.4) Let \( G \) be a normal-valued subgroup of \( A(\mathbb{R}) \) and \( g \in G \). Then \( g \) is the identity on \( P_G \) iff \( (\Delta(g, x) \) is minimal in \( (P_G, \subseteq) \) whenever \( xy \neq x \).

Let \( H(G) \) denote the set of those \( g \in G \) satisfying either of the equivalent conditions of Lemma 4.1.3. Then \( H(G) \) is a normal convex subgroup of \( G \) and the quotient \( G' = G/H(G) \) acts on \( (P_G^\prime, \subseteq, \leq) \) in the natural way.

We write \( (G', P_G^\prime, \subseteq, \leq) \) for this action, which also preserves \( P_G \) and \( P_G^\prime \).
Lemma 4.1.4 (c.f., [W], Proposition 2.1.5) Let $G$ be a normal-valued subgroup of $A(\mathbb{R})$. Then the actions $(G', P_G, \subseteq, \leq)$, $(G', P'_G, \subseteq, \leq)$ and $(G', P''_G, \subseteq, \leq)$ are faithful, i.e. only the identity in $G'$ acts as the identity map.

The situation described in Lemma 4.1.2(1) distinguishes different types of points $x \in \mathbb{R}$. Let $K(x) = \{\Delta(g, x) : g \in G\}$. If $K(x)$ contains an element $I$ that is minimal in $(P_G, \subseteq)$, then $x$ is called $G$-regular of the first type and $I$ is called the $G$-interval of $x$. If $\bigcap K(x) = \{x\}$, we say that $x$ is $G$-regular of the second type. In all other cases $\bigcap K(x)$ is a closed convex subset of $\mathbb{R}$ containing more than one point and we say that $x$ is $G$-singular.

If $I = \Delta(f, x) = (a, b)$, let $G_{(I)} = \{g \in G_{\{I\}} : \text{Fix}(g) \cap I \neq \emptyset\}$ and $\text{Fix}_I = I \cap \bigcap \{\text{Fix}(g) : g \in G_{(I)}\}$ be the set of common fixed points of all $g \in G_{(I)}$ in $I$.

Lemma 4.1.5 (c.f., [W], Proposition 2.1.6) Let $G$ be a normal-valued subgroup of $A(\mathbb{R})$. Then a point $x \in \mathbb{R}$ is $G$-singular if and only if $x \in \text{Fix}_J$ or $x \in \delta J$ for some $J \in P''_G$.

As shown in Lemma 4.1.2(2), $P'_G$ is countable. This places a restriction on the size of $P''_G$.

Lemma 4.1.6 (c.f., [W], Proposition 2.1.7) Let $G$ be a normal-valued subgroup of $A(\mathbb{R})$. Then the set of all boundary points $a_J < b_J$ of $J = (a_J, b_J) \in P''_G \setminus P_G$ is meagre.

4.2 The local action

Lemma 4.2.1 (c.f., [W], Proposition 2.2.1) Let $G$ be a normal-valued subgroup of $A(\mathbb{R})$.

1. Fix$_I \subseteq I$ is closed in $I$ and is either empty or infinite with $a, b$ accumulation points of Fix$_I$.

2. $G_{(I)}$ is a subgroup of $G_{\{I\}}$.

3. $Q(I) = G_{\{I\}} / G_{(I)}$ is an Archimedean ordered group, and so isomorphic to a subgroup $U(I)$ of $(\mathbb{R}, +, \leq)$.
Proof: In [W], the proof of (1) used the Abelian property. We can use Lemma 4.1.1(2) instead to get that the action is “locally Abelian”, and this is all that is necessary for the original proof. //

The set $I \setminus \text{Fix}_I$ splits into open connected components $C$. If $x \in C$, we write $C = C(x)$ or $C_I(x)$ (if we want to emphasise the dependence on $I \in P_G$) for the connected component containing $x$. Let $S_I$ denote this set of components. We will impose conditions to ensure that $\text{Fix}_I$ is nowhere dense, whence most (in a topological sense) $x$ are contained in a unique $C(x)$.

Lemma 4.2.2 (c.f., [W], Proposition 2.2.2) Let $G$ be a normal-valued subgroup of $A(\mathbb{R})$. Let $I = (a, b) \in P_G$ with $\text{Fix}_I \neq \emptyset$. Let $x \in I \setminus \text{Fix}_I$ and $C(x) = (a_x, b_x)$.

1. $a_x = \sup(\text{Fix}_I \cap (a, x))$ and $b_x = \inf(\text{Fix}_I \cap (x, b))$.
2. $a < a_x < b_x < b$.
3. $C(x) = \bigcup\{\Delta(g, x) : g \in G_{\langle I \rangle}\}$.
4. $Q = Q(I)$ acts as an ordered group on $S_I$ so that, for each $C \in S_I$, $f(C) = C$ iff $f \in G_{\langle I \rangle}$.

We can now use these lemmata to obtain a rooted Wreath product representation for any (coherent) normal-valued subgroup of $A(\mathbb{R})$, analogously to the Abelian case in [W]. The proof is identical (see [W]).

4.3 The Abelian case

If $G$ is Abelian and $\Delta(f, x)g \neq \Delta(f, x)$, then for each $y \in \Delta(f, x)g$, we have $yg^{-1} \in \Delta(f, x)$. Hence $yf = yg^{-1}fg$; that is, the action of $f$ on all its supporting intervals is completely determined by, and obtainable from, its action on one representative interval of $f$ for each orbit of intervals under $G$. By Lemma 4.1.1(2), this corresponds to a representative for each orbit of the corresponding extensive $G$-congruence. Thus the subgroup of the rooted Wreath product $\mathcal{W}(G)$ that is sufficient for the embedding of $G$ is actually the rooted valuation group $\mathcal{V}(G) = V(\Gamma(G), \leq \mathbb{R})$ (given in [W], Section 3.1, and Section 2.2 above), where $\leq \mathbb{R}$ is either $\mathbb{R}$ or $\{0\}$ as in [W]. Thus the group $\mathcal{V}(G)$ given in [W] occurs naturally as a subgroup of a corresponding rooted Wreath product $\mathcal{W}(G)$ which in turn arises naturally.
from (the natural partial $G$-congruences of) the normal-valued permutation subgroup $G$ of $A(\mathbb{R})$.

The conditions for admissibility of quintuples in the sense of [W] (certain objects which are complete invariants for the classification of maximal Abelian actions on $(\mathbb{R}, <)$ up to conjugacy) will be considered in future research for normal-valued permutation subgroups of $A(\mathbb{R})$.

### 4.4 An Example

We conclude with an easy example to show that even for maximal Abelian subgroups of $A(\mathbb{R})$, it is not sufficient to consider transitive Wreath products: rooted Wreath products and rooted valuation groups are necessary.

For each $a \in \mathbb{R}^+$, let $xf_a = ax$ if $x \geq 0$ and $xf_a = x$ if $x \leq 0$; and $xg_a = ax$ if $x \leq 0$ and $xg_a = x$ if $x \geq 0$. Let $F = \{f_a : a \in \mathbb{R}^+\}$ and $G = \{g_a : a \in \mathbb{R}^+\}$. Then $F, G$ are Abelian subgroups of $A(\mathbb{R})$ and generate $H = F \times G$; so $0h = 0$ for all $h \in H$. Now $F$ and $G$ are totally ordered (under the pointwise ordering) so $H$ is a sublattice subgroup of $A(\mathbb{R})$, whence $H$ is coherent. Note that if $x \in \mathbb{R}$, then the orbit of $x$ under $H$ is either $(-\infty, 0)$, $\{0\}$ or $(0, \infty)$. Hence any transitive subgroup $K$ of $A(\mathbb{R})$ that contains $H$ has no natural convex $H$-blocks except $\mathbb{R}$ and singletons. Thus $(K, \mathbb{R})$ is primitive and coherent. But if $k \in K$ with $0k > 0$, let $x \in (0k^{-1}, 0)$ and $a \neq 1$; so $ax \neq x$. Then $xkg_a = xk \neq (ax)k = xg_a k$. Hence $K$ is non-Abelian and therefore not normal-valued by Lemma 3.1.1. Consequently, the Abelian sublattice subgroup $H$ of $A(\mathbb{R})$ is not contained in any normal-valued transitive permutation subgroup of $A(\mathbb{R})$ and it is necessary to consider intransitive permutation subgroups of $A(\mathbb{R})$ even for maximal Abelian subgroups of $A(\mathbb{R})$. In this example, the set of natural covering congruences is the four point root system with a single maximal element and three incomparable elements below it. The corresponding permutation groups are $\{(1), \{-, 0, +\}\}$, $(G, \mathbb{R}_-)$, $(\{1\}, \{0\})$ and $(F, \mathbb{R}_+)$. If we ignore the trivial actions we get $G \times F$ for both $\mathcal{W}(H)$ and $\mathcal{V}(H)$.

### References


Authors’ addresses:
amwg@dpmms.cam.ac.uk
Department of Pure Mathematics and Mathematical Statistics,
Centre for Mathematical Sciences,
Wilberforce Rd.,
Cambridge CB3 0WB,
England

reinhard.winkler@tuwien.ac.at
Institut für Algebra und Computermathematik
Technische Universität Wien,
Wiedner Hauptstraße 8-10,
Wien A-1040,
Austria.