Poly-weakly-abelian ordered groups.

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To Nikolai Ya. Medvedev — in memoriam.

Abstract

Theorem *The variety of lattice-ordered groups that is generated by all o-groups that are poly-weakly-abelian is strictly contained in the variety of all residually ordered lattice-ordered groups.*

The proof uses the techniques introduced in [8] and [6].

In August 2004, Nikolai Medvedev presented some theorems at an Algebra and Logic Conference in Irkutsk organised by Vasily Bludov. Nikolai explained his proof [8] to me in detail afterwards. From this discussion, we were able to considerably generalise his result [6]. The present work uses Nikolai's ideas to obtain the new result. Sadly, he is unable to continue the work himself. This paper is a small tribute to Nikolai Ya. Medvedev, the mathematician. On a more personal note, I have many happy memories of thirty years of our correspondence but all too rare meetings in England, Russia and the U.S.A.

1 Introduction

In [8], Medvedev proved

Theorem A *The variety of lattice-ordered groups that is generated by all o-groups that are soluble as groups is strictly contained in the variety of all residually ordered lattice-ordered groups.*

In [6], we extended the result.

F. Point [9] called a group word $u(x, y)$ a *Milnor word* if it can be written in the form

$$y^m(x^{m_0})^{y^{n_0}} \dots (x^{m_k})^{y^{n_k}} \cdot v,$$

where $m, m_j, n_j \in \mathbb{Z}$ $(j = 0, \dots, k)$, $\gcd\{m_0, \dots, m_k\} = 1$, $n_0 < \dots < n_k$, and v belongs to the derived group of the subgroup generated by the conjugates of x by integer powers of y. In [6] we called any such word a *unilateral word* if $m = 0 = n_0$, $m_0 \in \{\pm 1\}$ and the conjugates of x appearing in v are all by non-negative integer powers of y. [This last condition was inadvertantly omitted from [6]. I am most grateful to Vasily Bludov for pointing out this typographical error.] If $m_j \neq 0$ for some $j \in \{1, \dots, k\}$, we called the unilateral word *hegemonic*. We extend the definition and call a Milnor word *hegemonic* if $m = n_0 = 0$, $m_0 \neq 0$, $m_j \neq 0$ for some $j \in \{1, \dots, k\}$, and the conjugates of x appearing in v are all by non-negative integer powers of y.

We call a class of groups $\mathcal{U}$ *unilateral* if for each $G \in \mathcal{U}$ and $g, h \in G$, there is unilateral word $u(x, y)$ (dependent on g, h, G) such that $u(g, h) = 1$. So the abelian law $x^{-1}x^y = 1$ defines the hegemonic variety of abelian groups.

Following P. Hall and A. Mal′cev, let $\mathcal{V}$ be a class of groups and $\mathrm{P}\mathcal{V}$ be the class of poly-$\mathcal{V}$ groups; that is, $\mathrm{P}\mathcal{V} := \bigcup_{k \in \mathbb{Z}_+} \mathcal{V}^k$. As usual, we call poly-abelian groups *soluble*.

Our main result in [6] was:

Theorem B *Let $\mathcal{V} \supseteq \mathcal{A}$ be a variety of groups defined by a single hegemonic group law. Then the variety of lattice-ordered groups that is generated by all o-groups that are poly-$\mathcal{V}$ as groups is strictly contained in the variety of all residually ordered lattice-ordered groups.*

Vasily Bludov has kindly pointed out

Corollary 1.1 *Let $\mathcal{V} \supseteq \mathcal{A}$ be a variety of groups defined by a single unilateral group law. Then the variety of lattice-ordered groups that is generated by all o-groups that are poly-$\mathcal{V}$ as groups is strictly contained in the variety of all residually ordered lattice-ordered groups.*

Proof: Let $u = u(x, y)$ be a unilateral word defining $\mathcal{V}$. Then $u_0 := u^{-1}u^y$ is equal to a hegemonic word $u_1(x, y)$ and defines a variety $\mathcal{V}_1$. Clearly, $\mathcal{V} \subseteq \mathcal{V}_1$. The corollary follows at once from Theorem B. //

The purpose of this note is to extend this result to an important subclass of the variety of all lattice-ordered groups that are residually ordered. This is the class of all *weakly abelian lattice-ordered groups* which was introduced in [7] and which we now describe.

Let $v_0(x,y) := x^{-2}x^y[x^y, x^{-2}]$ $(= x^y x^{-2} = y^{-1}xyx^{-2})$, a hegemonic Milnor word, and $v(x,y) := v_0(|x|, y) \vee 1$. The variety $\mathcal{W}$ of all weakly abelian lattice-ordered groups is precisely the variety of lattice-ordered groups defined by the law $v(x,y) = 1$.

As is standard, we write G is an *o-group* if the order on G is total, and *ℓ-group* as a shorthand for lattice-ordered group. We let $\mathcal{O}$ be the class of all o-groups and $\mathrm{R}_\ell\mathcal{O}$ be the class of all residually ordered ℓ-groups, the residual operator being in the class of ℓ-groups. Then $\mathcal{W} \subseteq \mathrm{R}_\ell\mathcal{O}$ (see [7] or [5], Section 6.4); moreover, $\mathcal{W}$ properly contains the variety of ℓ-groups generated by all ℓ-groups that are nilpotent as groups [1]. Weakly abelian ℓ-groups are subdirect products of *centally ordered groups* (see [2] and [3] for the definition and for more group-theoretic properties).

In analogy with the P operator on classes of groups, we can form the P_ℓ operator on classes of ℓ-groups. For any class $\mathcal{V}$ of ℓ-groups, let $G \in \mathcal{V}^2$ iff there is a convex normal ℓ-subgroup N of G such that N and G/N both belong to $\mathcal{V}$. We can similarly define $\mathcal{V}^m$ for any $m \in \mathbb{Z}_+$ and let $\mathrm{P}_\ell\mathcal{V}$ be the union $\bigcup\{\mathcal{V}^m : m \in \mathbb{Z}_+\}$. We call this the class of *poly-$\mathcal{V}$* ℓ-groups. The proof of Theorem B relied on the technical Corollary 1.4 whose proof used the "unilateral" property. For P_ℓ, this is unnecessary and the proof in [6] is easily seen (see below) to give

Theorem C *Let $\mathcal{V} \supseteq \mathcal{A}$ be a variety of lattice-ordered groups defined by a single hegemonic Milnor group law. Then the variety of lattice-ordered groups that is generated by all o-groups that are poly-$\mathcal{V}$ as lattice-ordered groups is strictly contained in the variety of all residually ordered lattice-ordered groups.*

If $\mathcal{A}$ is the variety of all abelian ℓ-groups, then every residually ordered ℓ-group belongs to $\mathrm{P}_\ell\mathcal{A}$. Thus $\mathrm{P}_\ell\mathcal{A} \cap \mathrm{R}_\ell\mathcal{O} = \mathrm{R}_\ell\mathcal{O}$. This contrasts with Medvedev's Theorem A which shows that the variety of ℓ-groups generated by $\mathrm{P}_\mathcal{O}\mathcal{A} := \bigcup\{\mathcal{A}^m \cap \mathcal{O} : m \in \mathbb{Z}_+\}$ is properly contained in $\mathrm{R}_\ell\mathcal{O}$.

In trying to extend Theorem B to ℓ-group terms, the weakly abelian word $v(x,y)$ obtained from the hegemonic Milnor word $v_0(x,y)$ is a natural starting place. Since $\mathcal{W}$ is a much larger variety of ℓ-groups than $\mathcal{A}$ and $\mathcal{W} \cap \mathcal{O}$ is not contained in $\mathrm{P}_\mathcal{O}\mathcal{A}$, it is natural to ask if $\mathrm{P}_\mathcal{O}\mathcal{W}$ generates the variety $\mathrm{R}_\ell\mathcal{O}$. The answer is given in the following theorem.

Theorem D *The variety of lattice-ordered groups that is generated by all o-groups that are poly-weakly-abelian is strictly contained in the variety of all residually ordered lattice-ordered groups.*

The proof uses a very slight variant of the ideas and techniques introduced in [8] and [6].

2 Proof of Theorem D

Let $v_0(x,y)$ be the hegemonic Milnor word equal to $x^y x^{-2}$ and $v(x,y) := v_0(|x|,y) \vee 1$, as above. Let F be the free group on free generators x, y and F' be the commutator subgroup of F.

Let $\Theta(x,y)$ be the identity obtained from the inequality

$$\bigvee_{z\in\{x^{\pm 1},y^{\pm 1}\}} |v([x,y],[x,y]^z)| \leq |[x,y]|^2,$$

and for $w(x,y) \in F$, let $\Psi_w(x,y) = \Psi(w(x,y),x,y)$ be the identity given by

$$|[x,y]|^{w(x,y)} \leq |[x,y]| \vee \bigvee_{z\in\{x^{\pm 1},y^{\pm 1}\}} |[x,y]|^z.$$

Note that in any o-group, $|a| \wedge |b| = 1$ iff $a = 1$ or $b = 1$.

To prove Theorem D, we will show that for any $w(x,y) \in F'$,

$$|\Theta(x,y)| \wedge |\Psi_w(x,y)| = 1$$

holds in every $\mathrm{P}_\ell\mathcal{W}$ o-group; but it fails in Chehata's Example for some $w(x,y) \in F'$.

Lemma 2.1 *Let $G \in \mathrm{P}_\ell\mathcal{W}$ be an o-group. Then for all $w(x,y) \in F'$,*

$$|\Theta(x,y)| \wedge |\Psi_w(x,y)| = 1 \quad \textit{holds in } G.$$

Proof: Suppose that G were a counterexample to the lemma. Let $a, b \in G$ witness this for some $w \in F'$. So $|\Theta(a,b)| > 1$ and $|\Psi_w(a,b)| > 1$. Hence the o-subgroup $\langle a,b\rangle$ of G generated by a and b would be a counterexample to the lemma. So we may assume that $G = \langle a,b\rangle$. Let L be the convex normal subgroup of G generated by $[a,b]$; so $L \supseteq G'$ and G/L is Abelian. Let D be the convex normal subgroup of L generated by $[a,b]$ and C be the union of the chain of all convex normal subgroups of L that do not contain $[a, b]$. Since $D \lhd L$ and $w \in G' \subseteq L$, we have $[a, b]^w \in D$. But $\Psi_w(a, b) \neq 1$ in G, so $|[a, b]|^z < |[a, b]|^w$ (whence $[a, b]^z \in D$) for all $z \in \{a^{\pm 1}, b^{\pm 1}\}$. It follows that $D \lhd G$ (whence $D = L$). Thus $C \lhd G$. Since $L/C \in \mathrm{P}_\ell \mathcal{W}$ is an o-group and contains no proper convex normal subgroup, it must belong to $\mathcal{W}$ by definition. Since $\Theta(a, b) \neq 1$ in G, we must have $|v([a, b], [a, b]^t)| > |[a, b]|^2$ for some $t \in \{a^{\pm 1},\ b^{\pm 1}\}$. Thus, for any such t, we have $[a, b], [a, b]^t \in L \setminus C$ and $v([a, b], [a, b]^t) \in L \setminus C$. Hence $L/C \notin \mathcal{W}$, a contradiction. //

The above lemma applies equally for any hegemonic Milnor group word.

Now let G be Chehata's simple o-group, namely the group of piecewise linear automorphisms of $(\mathbb{R}, \leq)$ that have bounded support (see [4], Chapter 6). For each $g \in G$ with $g \neq e$, let $\sigma = sup\{supp(g)\} \in \mathbb{R}$ and let $\alpha < \sigma$ be such that g is linear on (α, σ). Define $g \in G_+$ iff $\beta g > \beta$ for all (some) $\beta \in (\alpha, \sigma)$. Then G is an o-group. Let $a, b \in G_+$ each have two non-identity linear pieces. Let a have support $(0, 2)$ and change linear definition at $(1, 1a)$; let b have support $(1, 3)$ and change linear definition at $(2, 2b)$.

Let $d = |[a, b]|$. An easy computation shows that $[2, 2b) \subset supp(d)$ and $sup\{supp(d)\} = 2b$. Hence $sup\{supp(d^b)\} = 2b^2 > 2b$. It follows immediately that $d, d^{a^{\pm 1}}, d^{b^{-1}} \ll d^b$. Moreover, $sup\{supp(v(d, d^b))\} = 2db > 2b$, whence $\Theta(a, b) \neq 1$ in G.

Let $c = d^b$. Then $supp(cc^b) \supseteq [2b, 2b^3)$; so some power of cc^b maps $2b$ above $2b^2$. Let w be cc^b raised to this power. Then $w \in G'$ and $d^w \gg d,\ d^{a^{\pm 1}},\ d^{b^{\pm 1}}$. Consequently, $\Psi_w(a, b) \neq 1$ in G for this w.

Theorems C and D follow immediately. //

In view of Theorems C and D, Corollary 1.1 and the results in [8] and [6], one might consider:

> *If $\mathcal{V}$ is a variety of ℓ-groups properly contained in $\mathrm{R}_\ell\mathcal{O}$, must the variety of ℓ-groups generated by all poly-$\mathcal{V}$ o-groups be properly contained in $\mathrm{R}_\ell\mathcal{O}$?*

This interesting question should be pursued.

References

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