Right orders and amalgamation for lattice-ordered groups

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Dedicated to W. Charles Holland on his 75th birthday, with gratitude for his research.

Abstract

Let H_i be a sublattice subgroup of a lattice-ordered group G_i ($i = 1, 2$). Suppose that H_1 and H_2 are isomorphic as lattice-ordered groups, say by φ . In general, there is no lattice-ordered group in which G_1 and G_2 can be embedded (as lattice-ordered groups) so that the embeddings agree on the images of H_1 and H_1^φ . In this article we prove that the group free product of G_1 and G_2 amalgamating H_1 and H_1^φ is right orderable and so embeddable (as a group) in a lattice-orderable group. To obtain this, we use our necessary and sufficient conditions for the free product of right-ordered groups with amalgamated subgroup to be right orderable [1]. We also provide new limiting examples to show that amalgamation can fail in the category of lattice-ordered groups even when the amalgamating sublattice subgroups are convex and normal (ℓ -ideals) and solve Problem 1.42 of [6].

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1 Introduction

A *partially ordered group* (or *p.o. group*, for short) is a group with a partial order $\leq$ such that $axb \leq ayb$ whenever $x \leq y$. A special case is provided when the partial order is a lattice. A *lattice-ordered group* G is a group and a lattice such that

$$a(x \vee y)b = axb \vee ayb \quad \text{and} \quad a(x \wedge y)b = axb \wedge ayb$$

for all $a, b, x, y \in G$. We will frequently use the abbreviation ℓ -group for lattice-ordered group. We will write ℓ -subgroup as a shorthand for a sublattice subgroup, ℓ -homomorphism for a map that is both a group and a lattice homomorphism, *etc.* Lattice operations $\vee$ and $\wedge$ on an ℓ -group G induce a partial order $\leq$ on G : $g \leq h$ if and only if $g \vee h = h$ (or equivalently $g \wedge h = g$). We say that g_1 and g_2 are *orthogonal* if $g_1 \wedge g_2 = 1$. A normal convex ℓ -subgroup of an ℓ -group is called an ℓ -ideal. An ℓ -ideal N of an ℓ -group G is called *prime* if the ℓ -group G/N (with the naturally induced lattice order) is totally ordered. Thus an ℓ -ideal N of an ℓ -group G is prime if and only if for any pair g_1, g_2 of orthogonal elements of G , at least one of g_1 and g_2 belongs to N .

There are two especially important ways to form new ℓ -groups from old. If G_1 and G_2 are ℓ -groups, we can partially order their direct product $G_1 \times G_2$ by: $(g_1, g_2) \geq 1$ if and only if $g_i \geq 1$ ($i = 1, 2$). This is called *the cardinal product of G_1 and G_2* ; it is an ℓ -group. Note that if $f_i \in G_i$ with $f_i \geq 1$ ($i = 1, 2$), then the images of f_1 and f_2 in the cardinal product of G_1 and G_2 are orthogonal. Let K be an *o-group* (i.e., an ℓ -group with the order total). If G is an ℓ -group and K acts on G as a group of ℓ -automorphisms, then the splitting extension of G by K is an ℓ -group, where $1 \leq (g, k)$ if and only if $k > 1$ or both $k = 1$ and $g \geq 1$. We write $G \ltimes K$ for this ℓ -group and call it the *lexicographic split extension of G by K* .

A *right-ordered group* G is a group with a total order $\leq$ such that $xg \leq yg$ whenever $x \leq y$ in G . Every right-ordered group can be (group) embedded in a lattice-ordered group; and every lattice-ordering on a group is the intersection of all the right orders on the group that extend the original lattice ordering (see [2], Examples 1.3.19 and 1.3.20). This close relationship between right orders and lattice orders is at the heart of this article.

Given groups G_1 and G_2 with isomorphic subgroups H_1 and H_2 respectively (say $\varphi : H_1 \cong H_2$), the free product of G_1 and G_2 with H_1 and H_2 amalgamated by φ is the quotient of $G_1 * G_2$ by the normal subgroup generated by $\{h_1^\varphi h_1^{-1} \mid h_1 \in H_1\}$. We denote it by $G_1 * G_2 (H_1 \stackrel{\varphi}{\cong} H_2)$. The main result is that the group $G_1 * G_2 (H_1 \stackrel{\varphi}{\cong} H_2)$ with natural maps from G_1 and G_2 is an amalgam; *i.e.*, it is the “free-est” such generated by the union of the images of G_1 and G_2 where the natural maps are embeddings.

In [1], we determined necessary and sufficient conditions for a free product of groups with amalgamated subgroups to be right orderable. Let G be a right orderable group and $\mathcal{R}$ be a non-empty family of right orders on G . $\mathcal{R}$ is said to be a *normal family* if $\leq \in \mathcal{R}$ and $g \in G$ imply that $\leq^g \in \mathcal{R}$, where $a \leq^g b$ if and only if $ga \leq gb$. Let G_i be a right orderable group and $\mathcal{R}_i$ be a family of right orders on G_i ($i = 1, 2$). If H_i is a subgroup of G_i ($i = 1, 2$) with H_1 and H_2 isomorphic (say $\varphi : H_1 \cong H_2$), then we say that $(\mathcal{R}_1, \mathcal{R}_2)$ is *compatible for φ* if

$$(\forall \leq_1 \in \mathcal{R}_1)(\exists \leq_2 \in \mathcal{R}_2)(\forall h_1 \in H_1)(1 \leq_1 h_1 \implies 1 \leq_2 h_1^\varphi), \quad (1)$$

and

$$(\forall \leq_2 \in \mathcal{R}_2)(\exists \leq_1 \in \mathcal{R}_1)(\forall h_2 \in H_2)(1 \leq_2 h_2 \implies 1 \leq_1 h_2^{\varphi^{-1}}). \quad (2)$$

We proved

Theorem 1 ([1], Theorem A). *Let H_i be a subgroup of a right-orderable group G_i ($i = 1, 2$). Assume that $\varphi : H_1 \cong H_2$ is an isomorphism. Then $L := G_1 * G_2 (H_1 \stackrel{\varphi}{\cong} H_2)$ is right orderable if and only if there is a normal family $\mathcal{R}_i$ of right orders on G_i ($i = 1, 2$) such that $(\mathcal{R}_1, \mathcal{R}_2)$ is compatible for φ . Moreover, the right orders $\leq_i$ on G_i ($i = 1, 2$) extend to a right order on L if and only if $(\leq_1, \leq_2)$ is compatible with φ and $\leq_i$ is contained in a normal family $\mathcal{R}_i$ of right orders on G_i ($i = 1, 2$) such that $(\mathcal{R}_1, \mathcal{R}_2)$ is compatible for φ .*

We use this result to prove

Theorem A *Let H_i be a sublattice subgroup of a lattice-ordered group G_i ($i = 1, 2$). If $\varphi : H_1 \cong H_2$ is an ℓ -isomorphism, then $G_1 * G_2 (H_1 \stackrel{\varphi}{\cong} H_2)$ is right orderable.*

We can also obtain lattice orderable conditions equivalent to the group $G_1 * G_2 (H_1 \stackrel{\varphi}{\cong} H_2)$ being right orderable.

Theorem B *Let G_1, G_2 be right-orderable groups with isomorphic subgroups $H_1 \stackrel{\varphi}{\cong} H_2$ respectively. Then $G_1 * G_2 (H_1 \stackrel{\varphi}{\cong} H_2)$ is right orderable if and only if there are lattice-ordered groups $\hat{G}_i$, group embeddings $\varepsilon_i : G_i \rightarrow \hat{G}_i$ ($i = 1, 2$) and an ℓ -isomorphism $\hat{\varphi}$ between the sublattice subgroups generated by $H_1^{\varepsilon_1}$ and $H_2^{\varepsilon_2}$ such that*

$$h_1^{\varphi\varepsilon_2} = h_1^{\varepsilon_1\hat{\varphi}}$$

for all $h_1 \in H_1$.

Since every right-orderable group can be embedded in a lattice-orderable group, we obtain

Corollary 1.1 *Let H_i be a sublattice subgroup of a lattice-ordered group G_i ($i = 1, 2$). If $\varphi : H_1 \cong H_2$ is an ℓ -isomorphism, then there is a lattice-ordered group L and group embeddings $\varepsilon_i : G_i \rightarrow L$ ($i = 1, 2$) such that $g_1^{\varepsilon_1} = g_2^{\varepsilon_2}$ if and only if $g_1 \in H_1$ and $g_2 = g_1^{\varphi}$.*

This is in sharp contrast to the lattice-ordered case which is false in general even when the phrase “and only if” is removed (see [5], or [2], Theorem 7.C). We strengthen this to show that the corollary fails if we replace “group embeddings” by “ ℓ -embeddings” even when both amalgamated subgroups are ℓ -ideals and one of them is prime (Example 5.2). However, such ℓ -embeddings are possible in some partial cases that we study in Section 4 (see Propositions 4.1 – 4.3). This situation (when group embeddings are possible and the right orders extend but no ℓ -embeddings are possible) allows us to construct a p.o. group giving a negative answer for Problem 1.42 listed in [6] (see Example 6.1).

2 Notation and Preliminaries

If H is a subset of a right-ordered or lattice-ordered group G , let $\text{Conv}_G(H)$ denote the least convex subgroup of G containing H . It is the intersection of all convex subgroups of G containing H . If H is an ℓ -subgroup of an ℓ -group G , then $\text{Conv}_G(H) = \{g \in G \mid (\exists h \in H)(|g| \leq |h|)\}$, where

$|x| := x \vee x^{-1}$. Note that $|x| \geq 1$ with $|x| = 1$ if and only if $x = 1$ (see [2], Lemma 2.3.8).

If H is subset of an ℓ -group G , we write $\langle H \rangle_\ell$ for the least ℓ -subgroup of G containing H .

By Zorn's Lemma, if G is an ℓ -group and $g \in G \setminus \{1\}$, there is a convex ℓ -subgroup C_g of G that is maximal with respect to not containing g . It is called a *value* of g . The set $R(C_g)$ of all right cosets of C_g in G inherits a total order from G in the natural way (see [2], Corollary 3.3.7).

Let $(\Omega, \leq)$ be a totally ordered set. Then $\text{Aut}(\Omega, \leq)$ is an ℓ -group under the lattice ordering given by $\omega(g \vee h) := \max\{\omega g, \omega h\}$ and $\omega(g \wedge h) := \min\{\omega g, \omega h\}$ for all $\omega \in \Omega$.

One of the main tools in the theory of ℓ -groups is in the same spirit as Cayley's permutation representation, though the proof is far less straightforward.

Theorem 2 (Holland [3]). *Every lattice-ordered group G is ℓ -isomorphic to a sublattice subgroup of $\text{Aut}(\Omega, \leq)$ for some totally ordered set $(\Omega, \leq)$.*

We may therefore regard any ℓ -group G as an ℓ -subgroup of $A(\Lambda) := \text{Aut}(\Lambda, \leq)$ for a suitable totally ordered set $\Lambda = \Lambda(G)$.

We first recall some standard notation. For $g \in G \leq A(\Lambda)$, let

$$\text{fix}(g) = \{\lambda \in \Lambda \mid \lambda g = \lambda\}, \quad \text{supp}(g) = \{\lambda \in \Lambda \mid \lambda g \neq \lambda\}.$$

So for all $g \in G$,

$$\text{fix}(g) = \text{fix}(g^{-1}), \quad \text{fix}(g^h) = \text{fix}(g) \cdot h,$$

$$\text{supp}(g) = \text{supp}(g^{-1}), \quad \text{supp}(g^h) = \text{supp}(g) \cdot h,$$

and

$$\Lambda = \text{fix}(g) \cup \text{supp}(g).$$

We also use the standard notation

$$g^+ = g \vee 1, \quad g^- = g^{-1} \vee 1.$$

Indeed (see [2], Lemma 2.3.8),

$$g = g^+(g^-)^{-1}, \quad |g| = g^+ \vee g^-, \quad g^+ \wedge g^- = 1.$$

For $g \in G$, let

$$S_+(g) := \text{supp}(g^+), \quad S_-(g) := \text{supp}(g^-).$$

So

$$S_+(g) = \{\lambda \in \Lambda \mid \lambda g > \lambda\} = S_-(g^{-1}), \quad S_+(g^h) = S_+(g) \cdot h$$

and

$$S_-(g) = \{\lambda \in \Lambda \mid \lambda g < \lambda\} = S_+(g^{-1}), \quad S_-(g^h) = S_-(g) \cdot h.$$

In any ℓ -group G ,

$$S_+(g) \cap S_+(h) = \emptyset \quad \Longleftrightarrow \quad g^+ \wedge h^+ = 1;$$

i.e., if and only if g^+ and h^+ are orthogonal.

Clearly, for each $g \in G$,

$$\{S_+(g), \text{fix}(g), S_-(g)\} \text{ is a partition of } \Lambda. \quad (3)$$

We use a corollary that is easily deduced from the proof of Theorem 2.

Corollary 2.1 *For any lattice-ordered group G , there is a totally ordered set Λ and ℓ -embedding of G into $A(\Lambda)$ satisfying*

- (I) *if L is a proper convex ℓ -subgroup of G , then $\text{supp}(g) \cap \text{fix}(h) \neq \emptyset$ for all $g \in G \setminus L$, $h \in L$, and*
- (II) *$\text{fix}(g) \neq \emptyset$ for all $g \in G$.*

Proof. The proof is a minor modification of Holland's original embedding (see [3] or [2], Theorem 7.A); we use *all* values of all non-identity elements of G instead of just one value for each non-identity element. Let $\Gamma(G)$ denote the set of *all* values of all nontrivial elements g from G . Order $\Gamma(G)$ by some total order $\preceq$. Totally order $\Omega := \bigcup \{R(C) \mid C \in \Gamma(G)\}$ by: $C_1 g \leq C_2 f$ if and only if $C_1 \prec C_2$ or both $C_1 = C_2$ and $C_1 g \leq C_1 f$ in $R(C_1)$. We can ℓ -embed G in $A(\Omega)$ by $(Cf)g = Cfg$ ($f, g \in G$). We adjoin to Ω an extra point at plus infinity, ν ; *i.e.*, $\Lambda = \Omega \cup \{\nu\}$, where $\omega < \nu$ for all $\omega \in \Omega$, and $\nu g = \nu$ for all $g \in G$. Thus property (II) holds. Property (I) also holds because L is contained in some value C_g of $g \in G \setminus L$ and C_g is a point in Λ with the

desired property: $C_g g \neq C_g$ and $C_g h = C_g$ for all $h \in L \subseteq C_g$. Thus $C_g \in \text{supp}(g) \cap \text{fix}(h) \neq \emptyset$. //

Recall that an ultrafilter on Λ is a maximal filter on Λ . A family of subsets of Λ forms an *ultrafilter* $\mathfrak{u}$ on Λ if and only if $\mathfrak{u}$ satisfies the properties:

- (U1) $\emptyset \notin \mathfrak{u}$;
- (U2) $S, T \in \mathfrak{u}$ imply $S \cap T \in \mathfrak{u}$;
- (U3) if $S \in \mathfrak{u}$ and $S \subseteq T$, then $T \in \mathfrak{u}$; and
- (U4) either $S \in \mathfrak{u}$ or $\Lambda \setminus S \in \mathfrak{u}$ for all $S \subseteq \Lambda$.

Recall that a family $\mathfrak{F}$ of subsets of Λ satisfying properties (U1) – (U3) is called a *filter* on Λ , and a family $\mathfrak{B}$ of subsets of Λ satisfying properties (U1) – (U2) is called a *filter base*. Each filter base generates a filter and each filter can be extended to an ultrafilter.

Remark 2.2 It follows from (U1) – (U4) that if $\mathfrak{u}$ is an ultrafilter on Λ and $\{\Lambda_1, \dots, \Lambda_n\}$ is a partition of Λ , then $\Lambda_i \in \mathfrak{u}$ for a unique $i \in \{1, \dots, n\}$. In particular, by (3) and (U4), if $\mathfrak{u}$ is an ultrafilter on Λ and $g \in A(\Lambda)$, then exactly one of $S_+(g)$, $\text{fix}(g)$ and $S_-(g)$ belongs to $\mathfrak{u}$.

Let $\mathcal{U} = \{\mathfrak{u}_\alpha \mid \alpha \in A\}$ denote the set of *all* ultrafilters on Λ and let $\preceq$ be a well ordering of $\mathcal{U}$. We can define a right order on an ℓ -group $G \leq A(\Lambda)$ (associated with $(\mathcal{U}, \preceq)$) as follows.

Let $G \leq A(\Lambda)$ and $g \in G$. If $g \neq 1$, then $\text{supp}(g) \neq \emptyset$. So there is an ultrafilter $\mathfrak{u}_0$ on Λ containing $\text{supp}(g)$; thus $\text{fix}(g) \notin \mathfrak{u}_0$ and either $S_+(g) \in \mathfrak{u}_0$ or $S_-(g) \in \mathfrak{u}_0$ (and not both). Thus there is an ultrafilter $\mathfrak{u}_0$ such that $\text{fix}(g) \in \mathfrak{u}$ for all $\mathfrak{u} \prec \mathfrak{u}_0$ and $\text{supp}(g) \in \mathfrak{u}_0$. Define $g > 1$ if $S_+(g) \in \mathfrak{u}_0$ and $g < 1$ if $S_-(g) \in \mathfrak{u}_0$. So $g > 1$ or $g < 1$ (and not both) if $g \neq 1$. Let $1 < g, h \in G$. So there are ultrafilters $\mathfrak{u}_1$ and $\mathfrak{u}_2$ on Λ such that $S_+(g) \in \mathfrak{u}_1$ and $\text{fix}(g) \in \mathfrak{u}$ for all $\mathfrak{u} \prec \mathfrak{u}_1$, and $S_+(h) \in \mathfrak{u}_2$ and $\text{fix}(h) \in \mathfrak{u}$ for all $\mathfrak{u} \prec \mathfrak{u}_2$. Since $\text{fix}(gh) \supseteq \text{fix}(g) \cap \text{fix}(h)$, $\text{fix}(gh) \in \mathfrak{u}$ for all $\mathfrak{u} \prec \min_{\preceq} \{\mathfrak{u}_1, \mathfrak{u}_2\}$. If $\mathfrak{u}_1 = \mathfrak{u}_2$, then $S_+(gh) \supseteq S_+(g) \cap S_+(h) \in \mathfrak{u}_1$, so $gh > 1$. On the other hand, if $\mathfrak{u}_1 \prec \mathfrak{u}_2$ (without loss of generality), then $S_+(g), \text{fix}(h) \in \mathfrak{u}_1$. Since $S_+(gh), S_+(hg) \supseteq S_+(g) \cap \text{fix}(h) \in \mathfrak{u}_1$, we have $gh > 1$ and $hg > 1$. This therefore gives a right order $\leq$ on G which we call *the right order associated with $(\mathcal{U}, \preceq)$* .

Remark 2.3 If $\mathfrak{u} \in \mathcal{U}$ and $g \in G$, then $\mathfrak{u}g = \{Sg \subseteq \Lambda \mid S \in \mathfrak{u}\} \in \mathcal{U}$. So we can define a conjugate $\preceq^g$ ($g \in G$) of $\preceq$ by

$$\mathfrak{u}_\alpha \preceq^g \mathfrak{u}_\beta \iff \mathfrak{u}_\alpha g^{-1} \preceq \mathfrak{u}_\beta g^{-1}.$$

We also see that if $\preceq$ is a right order associated with $(\mathcal{U}, \preceq)$, then its conjugate right order $\preceq^g$ (given by $f \preceq^g h$ if and only if $gf \preceq gh$) is the right order associated with $(\mathcal{U}, \preceq^g)$.

Let G_i be a group with subgroup H_i ($i = 1, 2$). Suppose that H_1 and H_2 are isomorphic, say $\varphi : H_1 \cong H_2$. An *amalgam of groups* G_1, G_2 (with isomorphic subgroups H_1 and H_2 amalgamated by φ) is a group L with embeddings $\varepsilon_i : G_i \rightarrow L$ ($i = 1, 2$) such that $h_1^{\varepsilon_1} = h_1^{\varphi\varepsilon_2}$ for all $h_1 \in H_1$. Amalgams of two groups always exist in the category of groups. When groups G_1, G_2 carry an additional structure (e.g., they are right-ordered, ℓ -groups or o-groups), we require that their amalgam L has the same structure and that the embeddings are also order or lattice embeddings. For (right- or totally) orderable groups, we only require that an amalgam is (right or totally) orderable.

An amalgam L of ℓ -groups G_1, G_2 with embeddings $\varepsilon_i : G_i \rightarrow L$ ($i = 1, 2$) is called an ℓ -*amalgam* if each ε_i is an ℓ -embedding ($i = 1, 2$) (so $G_i^{\varepsilon_i}$ are ℓ -subgroups of the ℓ -group L ($i = 1, 2$)).

We distinguish between two versions of amalgams for lattice-orderable groups G_1 and G_2 . We call the lattice-orderable group L a g -*amalgam* if ε_i are group embeddings. And if there are lattice orders on L, G_1, G_2 such that L becomes an ℓ -amalgam of ℓ -groups L, G_1, G_2 , then we call L an $\mathcal{L}$ -*amalgam*. Clearly, any $\mathcal{L}$ -amalgam is a g -amalgam.

3 Proofs of Theorems

Proof of Theorem A: Let $G_i \leq A(\Lambda_i)$ be an ℓ -group and Λ_i satisfy (I) and (II) of Corollary 2.1 ($i = 1, 2$). Let $H_1 \stackrel{\varphi}{\cong} H_2$ be ℓ -isomorphic ℓ -subgroups of G_1 and G_2 , respectively.

For $i \in \{1, 2\}$, let $\mathcal{U}_i$ be the set of *all* ultrafilters on Λ_i and $\{\preceq_\kappa \mid \kappa \in K_i\}$ be the set of *all* well orderings of $\mathcal{U}_i$. Let $\mathcal{R}_i$ be set of all right orders on G_i associated with $\{\preceq_\kappa \mid \kappa \in K_i\}$. As observed in Remark 2.3, $\mathcal{R}_i$ is

closed under conjugate orders and so is a normal family of right orders on G_i ($i = 1, 2$).

By Theorem A of [1] (stated in Section 1 above), it remains to show that $(\mathcal{R}_1, \mathcal{R}_2)$ is compatible for φ ; *i.e.*, we must verify conditions (1) and (2). To achieve this, we first establish a lemma.

Lemma 3.1 *Let $\mathfrak{u}_\alpha \in \mathcal{U}_1$ and*

$$\text{Fix}(\alpha) = \{h \in H_1 \mid \text{fix}(h) \in \mathfrak{u}_\alpha\}.$$

Then $\text{Fix}(\alpha)$ is an ℓ -subgroup of G_1 and $\text{Conv}_{G_1}(\text{Fix}(\alpha)) \cap H_1 = \text{Fix}(\alpha)$.

Proof. Clearly, $\text{Fix}(\alpha)$ is a subgroup of H_1 . Moreover, as $\text{fix}(h \vee 1) \supseteq \text{fix}(h)$, we have $h \vee 1 \in \text{Fix}(\alpha)$ whenever $h \in \text{Fix}(\alpha)$. Hence $\text{Fix}(\alpha)$ is an ℓ -subgroup of H_1 . But $\text{fix}(|h|) \subseteq \text{fix}(|g|) \subseteq \text{fix}(g)$ if $h \in H_1$, $g \in G_1$ and $|g| \leq |h|$. Thus $g \in \text{Conv}_{G_1}(\text{Fix}(\alpha))$ implies that $\text{fix}(g) \in \mathfrak{u}_\alpha$. The lemma follows. //

Since φ is an ℓ -isomorphism between H_1 and H_2 , we immediately deduce

Corollary 3.2 *If $\mathfrak{u}_\alpha$ is an ultrafilter on Λ_1 , then $\text{Conv}_{G_2}(\text{Fix}(\alpha)^\varphi) \cap H_2 = \text{Fix}(\alpha)^\varphi$.*

Let $\leq_1 \in \mathcal{R}_1$. By construction, $\leq_1$ is associated with a well ordering $\preceq_1$ of $\mathcal{U}_1$. We seek a right order $\leq_2 \in \mathcal{R}_2$ compatible with φ . We need to use $\preceq_1$ to define a “compatible” well order $\preceq_2$ on $\mathcal{U}_2$.

First, we define a function $\psi : \mathfrak{u}_\alpha \mapsto \mathfrak{B}_\alpha$ which associates with each ultrafilter $\mathfrak{u}_\alpha \in \mathcal{U}_1$ a filter base $\mathfrak{B}_\alpha$ in Λ_2 . For this, let $\mathfrak{u}_\alpha$ be an ultrafilter on Λ_1 . Let

$$S_+(\alpha) := \{h_1 \in H_1 \mid S_+(h_1) \in \mathfrak{u}_\alpha\}, \quad S_-(\alpha) := \{h_1 \in H_1 \mid S_-(h_1) \in \mathfrak{u}_\alpha\}.$$

By Remark 2.2, $\{S_+(\alpha), \text{Fix}(\alpha), S_-(\alpha)\}$ is a partition of H_1 .

We now define a filter base $\mathfrak{B}_\alpha$ on Λ_2 associated with the ultrafilter $\mathfrak{u}_\alpha$ on Λ_1 . Since φ is an ℓ -isomorphism, $\{S_-(\alpha)^\varphi, \text{Fix}(\alpha)^\varphi, S_+(\alpha)^\varphi\}$ is a partition of H_2 . Let $\mathfrak{B}_\alpha$ be the set of finite intersections of sets belonging to

$$\{S_+(h_2) \mid h_2 \in S_+(\alpha)^\varphi\} \cup \{\text{fix}(h_2) \mid h_2 \in \text{Fix}(\alpha)^\varphi\} \cup \{S_-(h_2) \mid h_2 \in S_-(\alpha)^\varphi\}.$$

We show that $\emptyset \notin \mathfrak{B}_\alpha$.

Now $h_2 \in S_k(\alpha)^\varphi$ if and only if $S_k(h_2^{\varphi^{-1}}) \in \mathfrak{U}_\alpha$ ($k \in \{\pm\}$). Since $\emptyset \notin \mathfrak{U}_\alpha$ and φ is an ℓ -isomorphism, $S_k(h_2) \neq \emptyset$ if $S_k(h_2) \in \mathfrak{B}_\alpha$ ($k \in \{\pm\}$). Also, $\text{fix}(h_2) \neq \emptyset$ for all $h_2 \in H_2$ by Corollary 2.1(I). So the empty set is not a member of the union used to form $\mathfrak{B}_\alpha$.

By Corollary 2.1(I), $\bigcap_{j=1}^n \text{fix}(f_j) \neq \emptyset$ for all $f_1, \dots, f_n \in H_2$.

Assume that $S_+(f_1), \dots, S_+(f_n) \in \mathfrak{B}_\alpha$. Then $f_1, \dots, f_n \in S_+(\alpha)^\varphi$ and so $f_1^{\varphi^{-1}}, \dots, f_n^{\varphi^{-1}} \in S_+(\alpha)$. Hence $S_+(f_1^{\varphi^{-1}}), \dots, S_+(f_n^{\varphi^{-1}}) \in \mathfrak{U}_\alpha$, whence $\bigcap_{j=1}^n S_+(f_j^{\varphi^{-1}}) \neq \emptyset$. Thus $(f_1^{\varphi^{-1}})^+ \wedge \dots \wedge (f_n^{\varphi^{-1}})^+ \neq 1$. Therefore, $f_1^+ \wedge \dots \wedge f_n^+ \neq 1$ in G_2 . Consequently, $\bigcap_{j=1}^n S_+(f_j) \neq \emptyset$.

By replacing $f_1, \dots, f_n$ by their inverses, the same argument applies if $S_-(f_1), \dots, S_-(f_n) \in \mathfrak{B}_\alpha$. The analogous argument applies when $S_+(f_1), \dots, S_+(f_m), S_-(f_{m+1}), \dots, S_-(f_n) \in \mathfrak{B}_\alpha$.

By the above technique, to complete the proof that $\mathfrak{B}_\alpha$ is a filter base, it suffices to show that $\text{fix}(h_0) \cap S_+(h_2) \in \mathfrak{B}_\alpha$ whenever $\text{fix}(h_0), S_+(h_2) \in \mathfrak{B}_\alpha$. Let $h_0, h_2 \in H_2$ be such that $\text{fix}(h_0), S_+(h_2) \in \mathfrak{B}_\alpha$. By the construction, $h_0^{\varphi^{-1}} \in \text{Fix}(\alpha)$, $h_2^{\varphi^{-1}} \in S_+(\alpha)$; so $\text{fix}(h_0^{\varphi^{-1}}), S_+(h_2^{\varphi^{-1}}) \in \mathfrak{U}_\alpha$. By Lemma 3.1, $\text{Fix}(\alpha)$ is the ℓ -subgroup of G_1 and $\text{Conv}_{G_1}(\text{Fix}(\alpha)) \cap H_1 = \text{Fix}(\alpha)$. By Corollary 3.2, the analogous result is true for $\text{Fix}(\alpha)^\varphi$ in G_2 . But $(h_2^{\varphi^{-1}})^+ \notin \text{Fix}(\alpha)$ and so, by Lemma 3.1, $(h_2^{\varphi^{-1}})^+ \not\leq |h|$ for any $h \in \text{Fix}(\alpha)$, i.e., $(h_2^{\varphi^{-1}})^+ \vee |h| \neq |h|$ for all $h \in \text{Fix}(\alpha)$. Applying the ℓ -isomorphism φ , we obtain $h_2^+ \vee |h| \neq |h|$ for all $h \in \text{Fix}(\alpha)^\varphi$. Thus $h_2^+ \notin \text{Conv}_{G_2}(\text{Fix}(\alpha)^\varphi)$ and by Corollary 2.1(I), $S_+(h_2) \cap \text{fix}(h_0) \neq \emptyset$, as desired. This completes the proof that $\mathfrak{B}_\alpha$ is a filter base on Λ_2 .

Therefore, the map $\psi : \mathfrak{U}_\alpha \mapsto \mathfrak{B}_\alpha$ ($\alpha \in A$) is defined. However, in general, ψ need not be a one-to-one correspondence. We therefore take the subset A_1 ,

$$A_1 := \{\alpha \in A \mid \alpha = \min_{\preceq} \{\beta \in A \mid \mathfrak{U}_\beta^\psi = \mathfrak{U}_\alpha^\psi\}\}$$

of A so that the restriction of ψ to $\{\mathfrak{U}_\alpha \mid \alpha \in A_1\}$ is a one-to-one correspondence and $\{\mathfrak{U}_\alpha^\psi \mid \alpha \in A_1\} = \{\mathfrak{U}_\alpha^\psi \mid \alpha \in A\}$.

For each $\alpha \in A_1$, we extend $\mathfrak{B}_\alpha$ to an ultrafilter $\mathfrak{V}_\alpha$ on Λ_2 and transfer the well-ordering $\preceq_1$ from $\{\mathfrak{U}_\alpha \mid \alpha \in A_1\}$ to $\mathcal{V}_1 = \{\mathfrak{V}_\alpha \mid \alpha \in A_1\}$:

$$\mathfrak{V}_\alpha \preceq_2 \mathfrak{V}_\beta \iff \mathfrak{U}_\alpha \preceq_1 \mathfrak{U}_\beta.$$

We enumerate $\mathcal{U}_2$, the set of all ultrafilters on Λ_2 , say $\mathcal{U}_2 := \{\mathfrak{U}_\beta \mid \beta \in B\}$, where $B \supseteq A_1$. If $\mathcal{U}_2 \neq \mathcal{V}_1$, let $\preceq_2$ be an arbitrary well-ordering of $\mathcal{V}_2 := \mathcal{U}_2 \setminus \mathcal{V}_1$. Finally, extend the well-order $\preceq_2$ to all of $\mathcal{U}_2$ by putting

$$\mathfrak{U}_\alpha \prec_2 \mathfrak{U}_\beta \quad \text{for all } \mathfrak{U}_\alpha \in \mathcal{V}_1, \mathfrak{U}_\beta \in \mathcal{V}_2.$$

The right orders $\leq_1$ on G_1 associated with $(\mathcal{U}_1, \preceq_1)$ and $\leq_2$ on G_2 associated with $(\mathcal{U}_2, \preceq_2)$ satisfy the compatibility conditions (1) as we now show. Let $h_1 \in H_1$. If $1 <_1 h_1$, then by the definition of $\leq_1$ there is an ultrafilter $\mathfrak{U}_\alpha \in \mathcal{U}_1$ such that $S_+(h_1) \in \mathfrak{U}_\alpha$ and $\text{fix}(h_1) \in \mathfrak{U}_\beta$ for all $\mathfrak{U}_\beta \prec_1 \mathfrak{U}_\alpha$. By construction, $S_+(h_1^\varphi) \in \mathfrak{U}_\alpha$ and $\text{fix}(h_1^\varphi) \in \mathfrak{U}_\beta$ for all $\mathfrak{U}_\beta = \mathfrak{U}_\beta^\psi$ when $\mathfrak{U}_\beta \prec_1 \mathfrak{U}_\alpha$. It follows that $\alpha = \min_{\preceq} \{\gamma \in A \mid \mathfrak{U}_\gamma^\psi = \mathfrak{U}_\alpha^\psi\}$. Hence $\alpha \in A_1$ and $\mathfrak{U}_\alpha \subseteq \mathfrak{U}_\alpha \in \mathcal{V}_1$. Thus $S_+(h_1^\varphi) \in \mathfrak{U}_\alpha \in \mathcal{V}_1$ and $\text{fix}(h_1^\varphi) \in \mathfrak{U}_\beta$ for all $\mathfrak{U}_\beta \prec_2 \mathfrak{U}_\alpha$. By the definition of the right order $<_2$ associated with $(\mathcal{U}_2, \preceq_2)$, we get $h_1^\varphi >_2 1$. By symmetry, condition (2) also holds and the proof of the theorem is complete. //

Proof of Theorem B: If $G_1 * G_2 (H_1 \overset{\varphi}{\cong} H_2)$ is right orderable, then there is a group embedding of $G_1 * G_2 (H_1 \overset{\varphi}{\cong} H_2)$ into an ℓ -group L . So there are group embeddings $\varepsilon_i : G_i \rightarrow L$ ($i = 1, 2$) such that $h_1^{\varphi\varepsilon_2} = h_1^{\varepsilon_1}$ for all $h_1 \in H_1$. Since ℓ -subgroups $\langle H_1^{\varepsilon_1} \rangle_\ell$ and $\langle H_2^{\varepsilon_2} \rangle_\ell$ of L generated by $H_1^{\varepsilon_1}$ and $H_2^{\varepsilon_2}$ coincide, we can take $\hat{\varphi}$ to be the identity mapping on L .

Conversely, suppose that there are embeddings ε_i of G_i in ℓ -groups $\hat{G}_i$ ($i = 1, 2$) and an ℓ -isomorphism $\hat{\varphi}$ between $\hat{H}_1 := \langle H_1^{\varepsilon_1} \rangle_\ell$ and $\hat{H}_2 := \langle H_2^{\varepsilon_2} \rangle_\ell$ such that $h_1^{\varphi\varepsilon_2} = h_1^{\varepsilon_1\hat{\varphi}}$ for all $h_1 \in H_1$. Then the group $\hat{G}_1 * \hat{G}_2 (\hat{H}_1 \overset{\hat{\varphi}}{\cong} \hat{H}_2)$ is right orderable by Theorem A and contains an isomorphic copy of $G_1 * G_2 (H_1 \overset{\varphi}{\cong} H_2)$ by construction. //

4 Amalgamations and prime ℓ -ideals

Proposition 4.1 *Let G_i be a lattice-ordered group and H_i be a prime ℓ -ideal of G_i ($i = 1, 2$). Suppose that $\varphi : H_1 \cong H_2$ is an ℓ -isomorphism. Then $L = G_1 * G_2 (H_1 \overset{\varphi}{\cong} H_2)$ is lattice-orderable $\mathcal{L}$ -amalgam of groups G_1, G_2 with amalgamated subgroups H_1, H_2 .*

Proof: Let $i \in \{1, 2\}$. Denote by H_i^+ the set of positive elements of H_i under the original lattice order on G_i and denote by $\bar{G}_i^+$ the set of positive elements of G_i/H_i under the order induced on G_i/H_i by the original lattice orders on G_i . This induces a lexicographic lattice order on G_i whose set of positive elements is

$$P_i = \{g \in G_i \mid H_i g_i \in \bar{G}_i^+ \setminus \{H_i\} \text{ or } g \in H_i^+\} \quad (4)$$

By (4), H_i is also a prime ℓ -ideal of G_i under this new lattice ordering. Moreover, $P_i \cap H_i = H_i^+$ and the new order on H_i coincides with the old one. Thus φ is also an ℓ -isomorphism with respect to the new orders on H_1 and H_2 . We can extend the total orders from G_1/H_1 and G_2/H_2 to the free product $G_1/H_1 * G_2/H_2$ (see [7]). Since $L/H_i^{\varepsilon_i} \cong G_1/H_1 * G_2/H_2$ and the order on $H_i^{\varepsilon_i}$ is invariant under conjugation by elements from L (since L is generated by $G_1^{\varepsilon_1} \cup G_2^{\varepsilon_2}$), we can obtain a lexicographic lattice order on L . Let P be the set of positive elements of L under this ordering. Now $P \cap G_i = P_i$ ($i = 1, 2$). Let $i \in \{1, 2\}$ and $g \in G_i^{\varepsilon_i}$. If $g \in H_i^{\varepsilon_i}$, then $g \vee 1 \in H_i^{\varepsilon_i}$ since $H_i^{\varepsilon_i}$ is an ℓ -ideal of L . If $g \in G_i^{\varepsilon_i} \setminus H_i^{\varepsilon_i}$, either g is strictly positive or strictly negative in L ; thus $g \vee 1 \in G_i^{\varepsilon_i}$. Consequently, $G_i^{\varepsilon_i}$ is an ℓ -subgroup of ℓ -group L and the proposition is proved. //

A proof of the existence of an ℓ -amalgam is possible in the special case when the ℓ -groups are splitting extensions of the prime ℓ -ideals; considering it leads to Example 5.1.

Let $G_i = H_i \overleftarrow{\rtimes} K_i$ for K_i an o-group and H_i an ℓ -group ($i = 1, 2$) — so G_i is an ℓ -group ($i = 1, 2$). Suppose that $\varphi : H_1 \cong H_2$ is an ℓ -isomorphism. We say that the extensions K_1 and K_2 are *compatible for φ* if for all $k_i \in K_i$ ($i = 1, 2$) and all $h_1 \in H_1$,

$$[((h_1^{k_1})^\varphi)^{k_2}]^{\varphi^{-1}} = [((h_1^\varphi)^{k_2})^{\varphi^{-1}}]^{k_1}.$$

Proposition 4.2 *Let $G_i = H_i \overleftarrow{\rtimes} K_i$ for K_i an o-group and H_i a lattice-ordered group ($i = 1, 2$). Suppose that $\varphi : H_1 \cong H_2$ is an ℓ -isomorphism and K_1 and K_2 are compatible for φ . Then there is a lattice-ordered group L and ℓ -embeddings $\varepsilon_i : G_i \rightarrow L$ ($i = 1, 2$) such that $g_1^{\varepsilon_1} = g_2^{\varepsilon_2}$ if and only if $g_1 \in H_1$ and $g_2 = g_1^\varphi$ ($g_i \in G_i$, $i = 1, 2$).*

Proof: Let $L := (H_2 \overleftarrow{\rtimes} K_2) \overleftarrow{\rtimes} K_1$, where the action of $k_1 \in K_1$ on (h_2, k_2) is given by

$$(h_2, k_2)^{k_1} := (((h_2^{\varphi^{-1}})^{k_1})^\varphi, k_2) \quad (h_2 \in H_2, k_2 \in K_2).$$

This is well-defined as the extensions are compatible for φ . Now L is an ℓ -group and $\varepsilon_2 : G_2 \rightarrow L$ given by $(h_2, k_2) \mapsto (h_2, k_2, 1)$ is an ℓ -embedding. Let $\varepsilon_1 : G_1 \rightarrow L$ be given by $(h_1, k_1) \mapsto (h_1^\varphi, 1, k_1)$. Then ε_1 is a well-defined ℓ -embedding by compatibility and the definition of the splitting extension of $G_2 = H_2 \overleftarrow{\rtimes} K_2$ by K_1 . Moreover, if $g_i = (h_i, k_i)$ ($i = 1, 2$), then $g_1^{\varepsilon_1} = g_2^{\varepsilon_2}$ if and only if $(h_1^\varphi, 1, k_1) = (h_2, k_2, 1)$ if and only if $(k_1 = 1 = k_2 \text{ and } h_2 = h_1^\varphi)$. //

Note that if, say, K_1 is a direct extension of H_1 , then the compatibility condition automatically holds for any split extension of H_2 by K_2 and so the lemma applies. However, we can do better in this case; G_2 can be an arbitrary ℓ -group with H_2 not necessarily convex or normal or prime in G_2 .

Proposition 4.3 *Let G_i be a lattice-ordered group with ℓ -subgroup H_i ($i = 1, 2$). If $\varphi : H_1 \cong H_2$ is an ℓ -isomorphism and $G_1 = H_1 \overleftarrow{\rtimes} K_1$ for some o -group K_1 , then there is an ℓ -group L and ℓ -embeddings $\varepsilon_i : G_i \rightarrow L$ ($i = 1, 2$) such that $g_1^{\varepsilon_1} = g_2^{\varepsilon_2}$ if and only if $g_1 \in H_1$ and $g_2 = g_1^\varphi$ ($g_i \in G_i$, $i = 1, 2$).*

Proof: Let $L := G_2 \overleftarrow{\rtimes} K_1$, an ℓ -group, and $\varepsilon_2 : G_2 \rightarrow L$ be given by $g_2 \mapsto (g_2, 1) \in L$. Clearly, ε_2 is an ℓ -embedding. Let $\varepsilon_1 : G_1 \rightarrow L$ be given by $(h_1, k_1) \mapsto (h_1^\varphi, k_1)$. Then ε_1 is also an ℓ -embedding. Now $g_1^{\varepsilon_1} = g_2^{\varepsilon_2}$ if and only if $g_1 = (h_1, 1)$ for some $h_1 \in H_1$ and $g_2 = h_1^\varphi$; i.e., g_1^φ is defined and equal to g_2 . //

Remark 4.4 Pierce proved in [5] that ℓ -amalgams always exist when the amalgamated ℓ -subgroups are cyclic and generated by a positive element. If we take an element h_i unrelated to the identity in G_i , the ℓ -subgroup generated by h_i is generated by the disjoint elements h_i^+ and h_i^- ($i = 1, 2$). Pierce's proof (*ibid*) can be adapted to give an ℓ -group L containing G_1 and G_2 as ℓ -subgroups such that one can simultaneously conjugate h_1^+ to h_2^+ and h_1^- to h_2^- in L ; so the ℓ -subgroups generated by h_1 and h_2 can be ℓ -amalgamated in this case, too. In contrast, Pierce's example (*ibid*) or Example 5.1 below shows that an ℓ -amalgam need not exist if the amalgamated ℓ -subgroups are abelian and generated by two elements.

5 Limiting examples

In [5], K. R. Pierce gave an example to show that, in general, ℓ -amalgams do not exist. In this section, we show that such failure can occur even when the amalgamated ℓ -subgroups are ℓ -ideals. In the second example, we show that this failure can occur for all lattice orders on the constituent ℓ -groups.

Example 5.1 *There are lattice-ordered groups G_i with prime ℓ -ideals H_i ($i = 1, 2$) such that $\varphi : H_1 \cong H_2$ is an ℓ -isomorphism, but there is no ℓ -amalgam for these ℓ -groups.*

The idea is to use splitting extensions that are incompatible for φ .

Let $A_{23} = \langle a_2 \rangle \times \langle a_3 \rangle$ be the cardinal direct product of two cyclic groups $\langle a_2 \rangle$ and $\langle a_3 \rangle$ with $a_2, a_3 > 1$, and $B_1 = A_{23} \overleftarrow{\rtimes} \langle b_1 \rangle$ be the lexicographic split extension of A_{23} by the cyclic group $\langle b_1 \rangle$ given by the automorphism $a_2^{b_1} = a_3, a_3^{b_1} = a_2$. Then B_1 is an ℓ -group with $b_1 > a$ for all $a \in A_{23}$. Let $G_1 = B_1 \times \langle a_1 \rangle$ be the cardinal direct product of B_1 and the cyclic group $\langle a_1 \rangle$ with $a_1 > 1$. Similarly, let $A_{13} = \langle a_1 \rangle \times \langle a_3 \rangle$ be the cardinal product and $B_2 = A_{13} \overleftarrow{\rtimes} \langle b_2 \rangle$ be the splitting extension of A_{13} given by the automorphism $a_1^{b_2} = a_3, a_3^{b_2} = a_1$ with $b_2 > a$ for all $a \in A_{13}$. Let $G_2 = B_2 \times \langle a_2 \rangle$ be the cardinal product. Then $H_1 = A_{23} \times \langle a_1 \rangle$ and $H_2 = A_{13} \times \langle a_2 \rangle$ are prime ℓ -ideals of the ℓ -groups G_1 and G_2 respectively. The identity map $\varphi : a_i \mapsto a_i$ ($i = 1, 2, 3$) is an ℓ -isomorphism between H_1 and H_2 . We claim that there is no ℓ -group L admitting ℓ -embeddings $\varepsilon_i : G_i \rightarrow L$ ($i = 1, 2$) such that $h_1^{\varepsilon_1} = h_1^{\varphi \varepsilon_2}$ for all $h_1 \in H_1$.

First note that a_1, a_2, a_3 are pairwise orthogonal and that a_i is orthogonal to b_i ($i = 1, 2$):

$$a_1 \wedge a_2 = a_1 \wedge a_3 = a_2 \wedge a_3 = 1, \quad a_1 \wedge b_1 = a_2 \wedge b_2 = 1.$$

For *reductio ad absurdum*, suppose that an ℓ -amalgam L were to exist. By the Holland Embedding Theorem (Theorem 2), represent L by order-preserving permutations of a totally ordered set $(\Lambda, \leq)$. Since $a_2^{b_1^{\pm 1}} = a_3$ and orthogonal elements have disjoint supports, we have

$$\text{supp}(a_2) \cup \text{supp}(a_3) \subseteq \text{supp}(b_1) \quad \text{and} \quad \text{supp}(a_1) \subseteq \text{fix}(b_1).$$

Similarly,

$$\text{supp}(a_1) \cup \text{supp}(a_3) \subseteq \text{supp}(b_2) \quad \text{and} \quad \text{supp}(a_2) \subseteq \text{fix}(b_2).$$

It follows that

$$\text{supp}(a_1)b_2^{\pm 1} = \text{supp}(a_3) \quad \text{and} \quad \text{supp}(a_2)b_1^{\pm 1} = \text{supp}(a_3). \quad (5)$$

Take $\lambda_1 \in \text{supp}(a_1)$ and let $\lambda_3 := \lambda_1 b_2$, $\lambda_2 := \lambda_3 b_1^{-1}$. Then $\lambda_1, \lambda_2 < \lambda_3$ (since $b_1, b_2 > 1$) and $\lambda_1 < \lambda_2$ (since $\lambda_1 \in \text{fix}(b_1)$). Hence $\lambda_1 < \lambda_2 < \lambda_3$. By (5) $\lambda_2 \in \text{supp}(a_2) \subseteq \text{fix}(b_2)$; so $\lambda_3 = \lambda_1 b_2 < \lambda_2$ — the desired contradiction. //

By Proposition 4.1, there is an $\mathcal{L}$ -amalgam for the ℓ -groups considered in Example 5.1. If we want to build an example of lattice-orderable groups which have no $\mathcal{L}$ -amalgams, we must weaken the hypotheses of Proposition 4.1. As we now show, this is possible even when both amalgamated subgroups are ℓ -ideals and one of them is prime (see Example 5.2 below).

Example 5.2 *There are lattice-ordered groups G_i with ℓ -ideals H_i ($i = 1, 2$) such that $\varphi : H_1 \cong H_2$ is an ℓ -isomorphism, but there is no $\mathcal{L}$ -amalgam for these lattice-orderable groups.*

Corollary 1.1 guarantees the existence of lattice-orderable g -amalgams. But for any lattice order on such an amalgam L in this example, at least one of images of G_1 and G_2 cannot be a sublattice of the ℓ -group L .

To prove the result, we recall a lemma.

Lemma 5.3 (Holland and Scrimger [4]). *If $g_1 \wedge g_2 = 1$, $h_1 \wedge h_2 = 1$ for elements g_1, g_2, h_1, h_2 in an ℓ -group G , then $[h_1, g_2^{-1} h_2 g_1 h_2^{-1} g_2 h_2 g_1^{-1}] = 1$.*

We now construct the required ℓ -groups and their ℓ -ideals.

Let A be the cardinal direct product of sixty eight cyclic ℓ -groups

$$A_{i,j} := \langle a_{i,j} \mid a_{i,j} \vee 1 = a_{i,j} \rangle, \quad 1 \leq i \leq 4, \quad 0 \leq j \leq 16.$$

So all ℓ -groups $A_{i,j}$ are cyclic totally ordered groups with positive generators $a_{i,j}$ and $a_{1,0}^{n_{1,0}} \dots a_{4,15}^{n_{4,15}} > 1$ in A if and only if all the exponents $n_{i,j}$ are non-negative and at least one $n_{i,j}$ is not equal to 0.

Let A_1 be the ℓ -subgroup of A generated by the elements

$$a_{1,j} \quad (j = 0, 1, 8, 9, 16), \quad a_{2,j} \quad (j = 2, 3, 4, 5, 6, 7, 10, 11, 12, 13, 14, 15)$$

$$a_{3,j} \quad (j = 0, 1, 2, 5, 6, 11, 12, 15, 16), \quad a_{4,j} \quad (j = 3, 4, 7, 8, 9, 10, 13, 14).$$

Let A_2 be the ℓ -subgroup generated by the remaining generators. Thus $A = A_1 \times A_2$, the cardinal product of A_1 and A_2 .

We define ℓ -automorphisms β_k ($0 \leq k \leq 15$), γ , δ on A and b_m on A_m ($m = 1, 2$). It suffices to define the requisite ℓ -automorphisms on the generators $a_{i,j}$. Let

$$\beta_k : a_{i,k} \mapsto a_{i,k+1} \mapsto a_{i,k}, \quad a_{i,j} \mapsto a_{i,j} \quad (j \neq k, k+1);$$

$$\gamma : a_{1,j} \mapsto a_{2,j} \mapsto a_{1,j}, \quad a_{3,j} \mapsto a_{4,j} \mapsto a_{3,j};$$

$$\delta : a_{1,j} \mapsto a_{3,j} \mapsto a_{1,j}, \quad a_{2,j} \mapsto a_{4,j} \mapsto a_{2,j};$$

$$\begin{aligned} b_1 : \quad & a_{1,j} \mapsto a_{1,j}^{\beta_0\beta_8}, \quad a_{2,j} \mapsto a_{2,j}^{\beta_2\beta_4\beta_6\beta_{10}\beta_{12}\beta_{14}}, \\ & a_{3,j} \mapsto a_{3,j}^{\beta_1\beta_5\beta_{11}\beta_{15}}, \quad a_{4,j} \mapsto a_{4,j}^{\beta_3\beta_7\beta_9\beta_{13}}; \end{aligned} \tag{6}$$

$$b_2 : a_{i,j} \mapsto a_{i,j}^{\gamma b_1 \gamma}. \tag{7}$$

By the definitions,

$$A_2 = A_1^\gamma, \quad A_1^{b_1} = A_1, \quad A_2^{b_2} = A_2.$$

We build the lexicographic split extensions

$$H_1 := A_1 \overleftarrow{\rtimes} \langle b_1 \rangle \quad \text{and} \quad H_2 := A_2 \overleftarrow{\rtimes} \langle b_2 \rangle$$

prescribed by the automorphisms (6) and (7), respectively. Let

$$H := H_1 \times H_2,$$

the cardinal product of H_1 and H_2 , and build one more lexicographic split extension

$$G_1 := H \overleftarrow{\rtimes} \langle c \rangle$$

given by the automorphism

$$c : a_{i,j} \mapsto a_{i,j}^\gamma, \quad b_1 \mapsto b_2 \mapsto b_1. \tag{8}$$

By (6) — (8)

$$a_{i,j}^{b_1^2} = a_{i,j}^{b_2^2} = a_{i,j}^{c^2} = a_{i,j} \quad (1 \leq i \leq 4, 0 \leq j \leq 16). \quad (9)$$

By construction, G_1 is an ℓ -group with $b_1 \wedge b_2 = 1$. Further, A is an ℓ -ideal in G_1 and each element of G_1 can be written uniquely in the form

$$g = ub_1^n b_2^m c^k \quad \text{for some } u \in A, n, m, k \in \mathbb{Z}. \quad (10)$$

Lemma 5.4 *For any lattice ordering $\vee, \wedge$ on G_1 , there are orthogonal elements $h_1, h_2 \in G_1$ such that*

$$w^{h_1} = w^{b_1} \quad \text{and} \quad w^{h_2} = w^{b_2} \quad \text{for all } w \in A.$$

Proof. Consider $b = b_1 b_2^{-1}$. We have

$$(b^+)^c = (b \vee 1)^c = b^{-1} \vee 1 = b^-.$$

So $b = b^+ (b^+)^{-c}$. Write b^+ in the normal form described in (10):

$$b^+ = ub_1^n b_2^m c^k.$$

Since $b_1 b^+ = b^+ b_1$, we must have that k is even; say

$$b^+ = ub_1^n b_2^m c^{2s}.$$

Further,

$$(b^+)^c = u^c b_1^m b_2^n c^{2s}$$

and

$$(b^+)^c = b^- = b^{-1} b^+ = u_1 b_1^{n-1} b_2^{m+1} c^{2s}.$$

So $m = n - 1$ and

$$b^+ = ub_1^n b_2^{n-1} c^{2s}$$

with $u \in A$.

If n is odd, say $n = 2r + 1$, let

$$h_1 = b^+ = ub_1^{2r} b_2^{2r} c^{2s} b_1, \quad h_2 = b^- = u^c b_1^{2r} b_2^{2r} c^{2s} b_2.$$

If n is even, say $n = 2r$, let

$$h_2 = b^+ = ub_1^{2r} b_2^{2r-2} c^{2s} b_2, \quad h_1 = b^- = u^c b_1^{2r-2} b_2^{2r} c^{2s} b_1.$$

Thus $h_1 \wedge h_2 = b^+ \wedge b^- = 1$. By (9),

$$w^{h_1} = w^{b_1} \quad w^{h_2} = w^{b_2}$$

for all $w \in A$. //

We next construct the group G_2 as a lexicographic split extension of A by an infinite cyclic group $\langle d \rangle$;

$$G_2 := A \overleftarrow{\rtimes} \langle d \rangle,$$

where the action of d on A is defined by

$$d : a_{i,j} \mapsto a_{i,j}^\delta. \quad (11)$$

By construction, A is a prime ℓ -ideal of G_2 .

Suppose that there are lattice orders on the groups G_1, G_2 such that there is an ℓ -group L with ℓ -embeddings $\varepsilon_1 : G_1 \rightarrow L$, $\varepsilon_2 : G_2 \rightarrow L$ coinciding on A (i.e. $a^{\varepsilon_1} = a^{\varepsilon_2}$ in L for all $a \in A$). By Lemma 5.4, there are orthogonal elements $h_1, h_2 \in G_1^{\varepsilon_1} < L$ such that

$$a_{i,j}^{\varepsilon_1 h_1} = a_{i,j}^{b_1 \varepsilon_1}, \quad a_{i,j}^{\varepsilon_1 h_2} = a_{i,j}^{b_2 \varepsilon_1}$$

for all $1 \leq i \leq 4$, $0 \leq j \leq 16$. Let $g_1 = h_1^{d^{\varepsilon_2}}$ and $g_2 = h_2^{d^{\varepsilon_2}}$. Then g_1, g_2 are also orthogonal. By Lemma 5.3,

$$f = [h_1, g_2^{-1} h_2 g_1 h_2^{-1} g_2 h_2 g_1^{-1}] = 1. \quad (12)$$

We calculate $a_{1,0}^{\varepsilon_1 f}$ using (6), (7), and (11).

$$\begin{aligned} a_{1,0}^{\varepsilon_1 f} &= a_{1,0}^{\varepsilon_1 h_1^{-1} g_1 h_2^{-1} g_2^{-1} h_2 g_1^{-1} h_2^{-1} g_2 h_1 g_2^{-1} h_2 g_1 h_2^{-1} g_2 h_2 g_1^{-1}} = \\ &= a_{1,1}^{\varepsilon_1 g_1 h_2^{-1} g_2^{-1} h_2 g_1^{-1} h_2^{-1} g_2 h_1 g_2^{-1} h_2 g_1 h_2^{-1} g_2 h_2 g_1^{-1}} = \\ &= a_{1,2}^{\varepsilon_2 h_2^{-1} g_2^{-1} h_2 g_1^{-1} h_2^{-1} g_2 h_1 g_2^{-1} h_2 g_1 h_2^{-1} g_2 h_2 g_1^{-1}} = \\ &= a_{1,3}^{\varepsilon_2 g_2^{-1} h_2 g_1^{-1} h_2^{-1} g_2 h_1 g_2^{-1} h_2 g_1 h_2^{-1} g_2 h_2 g_1^{-1}} = \dots = a_{1,16}^{\varepsilon_1} \neq a_{1,0}^{\varepsilon_1}. \end{aligned}$$

Consequently, $f \neq 1$ which contradicts (12). This establishes the claim of this example. //

6 Intersections of right orders

By an analysis of ℓ -groups which have an $\mathcal{L}$ -amalgam but no ℓ -amalgam, we are able to give a negative answer to Problem 1.42 of [6]. This question was the original motivation for the paper.

If G is a p.o. group which is the intersection of right orders, then can G be order-embedded in an ℓ -group with the embedding preserving any existing suprema and infima?

For our goal, we adapt a Pierce's example [5].

Example 6.1 Let $\Omega_i = (\mathbb{Z} \times \mathbb{Z})_i$ ($i = 1, 2$) ordered lexicographically from the right; i.e., $(n, m) \leq (r, s)$ if either $m < s$ or both $m = s$ and $n \leq r$. Let $\Omega = \Omega_1 \cup \Omega_2$ with $\Omega_1 < \Omega_2$. Consider the subgroup $G \leq A(\Omega)$ generated by the permutations a, b, c, h where:

$$a : \begin{aligned} (n, 2m+1)_1 &\mapsto (n+1, 2m+1)_1, & (n, 2m+1)_2 &\mapsto (n, 2m+1)_2, \\ (n, 2m)_1 &\mapsto (n, 2m)_1, & (n, 2m)_2 &\mapsto (n+1, 2m)_2; \end{aligned}$$

$$b : \begin{aligned} (n, 2m+1)_1 &\mapsto (n, 2m+1)_1, & (n, 2m+1)_2 &\mapsto (n+1, 2m+1)_2, \\ (n, 2m)_1 &\mapsto (n+1, 2m)_1, & (n, 2m)_2 &\mapsto (n, 2m)_2; \end{aligned}$$

$$c : (n, m)_i \mapsto (n, m+1)_i \quad (i = 1, 2);$$

$$h : (n, m)_1 \mapsto (n, m+2)_1, \quad (n, m)_2 \mapsto (n, m)_2;$$

$$(n, m)_i \in (\mathbb{Z} \times \mathbb{Z})_i \quad (i = 1, 2).$$

It is easy to see that the subgroup $A = \langle a, b, h \rangle$ is a free abelian group of rank three and that G is the split extension $A \rtimes \langle c \rangle$ of A under the cyclic subgroup with generator the automorphism c given by:

$$a^c = b, \quad b^c = a, \quad h^c = h. \tag{13}$$

So G is metabelian; it has 6 defining relations, namely (13) and

$$[a, b] = [a, h] = [b, h] = 1. \tag{14}$$

So the elements $g \in G$ have unique canonical form:

$$g = a^{t_1} b^{t_2} h^{t_3} c^{t_4} \quad (t_1, \dots, t_4 \in \mathbb{Z}).$$

Let $\mathcal{F} = \{\prec_\lambda \mid \lambda \in \Lambda\}$ be the family of all well-orderings of Ω and $\lambda \in \Lambda$. Let $\leq_\lambda$ be the right order on G associated with $\prec_\lambda \in \mathcal{F}$; *i.e.*, $1 \leq_\lambda g$ if and only if $(n, m)_i \leq (n, m)_i g$ in Ω , where $(n, m)_i$ is the least element of $\text{supp}(g)$ under $\prec_\lambda$. Let $\leq_0$ be the intersection of all $\leq_\lambda$ ($\lambda \in \Lambda$) and their conjugates.

We claim that G equipped with the partial order $\leq_0$ cannot be order-embedded in any ℓ -group with the embedding preserving the three infima $c^2 \wedge a$, $c^2 h^{-1} \wedge a$ and $h \wedge a$.

We first establish that $g = a^{t_1} b^{t_2} h^{t_3} c^{t_4} \geq_0 1$ if and only if one of three conditions is fulfilled:

- (a) $t_4 > 0$ and $t_4 + 2t_3 > 0$;
- (b) $t_4 > 0$, $t_4 + 2t_3 = 0$ and $t_1, t_2 \geq 0$;
- (c) $t_4 = 0$, $t_3 \geq 0$ and $t_1, t_2 \geq 0$.

If $t_4 > 0$ and $t_4 + 2t_3 > 0$, then $(n, m)_1 g = (n', m + t_4 + 2t_3)_1 > (n, m)_1$ and $(n, m)_2 g = (n', m + t_4)_2 > (n, m)_2$; so $\alpha g > \alpha$ for all $\alpha \in \Omega$. This gives $1 <_\lambda g$ for all $\lambda \in \Lambda$. Since conjugates of g have the same exponents t_3, t_4 as g , we get $1 <_\lambda g^f$ for all $f \in G$ and $\lambda \in \Lambda$. Hence $1 <_0 g$.

If $t_4 > 0$, $t_4 + 2t_3 = 0$ and $t_1, t_2 \geq 0$, then

$$\begin{aligned} (n, 2m + 1)_1 g &= (n + t_1, 2m + 1)_1 \geq (n, 2m + 1)_1, \\ (n, 2m + 1)_2 g &= (n + t_2, 2m + 1 + t_4)_2 > (n, 2m + 1)_2, \\ (n, 2m)_1 g &= (n + t_2, 2m + 1)_1 \geq (n, 2m + 1)_1, \\ (n, 2m)_2 g &= (n + t_1, 2m + 1 + t_4)_2 > (n, 2m + 1)_2. \end{aligned}$$

So $\alpha g \geq \alpha$ for all $\alpha \in \Omega$ and $(0, 1)_2 g > (0, 1)_2$. This gives $1 <_\lambda g$ for all $\lambda \in \Lambda$. Now t_4 is even and the conjugates of g are equal to g or to $a^{t_2} b^{t_1} h^{t_3} c^{t_4}$, and therefore satisfy condition (b).

If $t_4 = 0$, $t_3 \geq 0$ and $t_1, t_2 \geq 0$, then

$$\begin{aligned} (n, 2m + 1)_1 g &= (n + t_1, 2m + 1 + 2t_3)_1 \geq (n, 2m + 1)_1, \\ (n, 2m + 1)_2 g &= (n + t_2, 2m + 1)_2 \geq (n, 2m + 1)_2, \\ (n, 2m)_1 g &= (n + t_2, 2m + 1 + 2t_3)_1 \geq (n, 2m + 1)_1, \\ (n, 2m)_2 g &= (n + t_1, 2m + 1)_2 \geq (n, 2m + 1)_2. \end{aligned}$$

Now $\alpha g \geq \alpha$ for all $\alpha \in \Omega$ and if $g \neq 1$, then $\alpha g > \alpha$ for some $\alpha \in \Omega$. So t_4 is also even and, as above, the conjugates of g are g and $a^{t_2}b^{t_1}h^{t_3}c^{t_4}$ and so satisfy condition (c).

In all cases $1 \leq_0 g$.

Assume now that at least one of the conditions (a) – (c) fails for some $g \in G$. For such an element g , we must have either $t_4 < 0$, or $t_4 + 2t_3 < 0$, or $t_4 = t_3 = 0$ and one or both of t_1, t_2 is strictly negative. If $t_4 < 0$, then $(0, 0)_2 g < (0, 0)_2$ and $g \prec_\lambda 1$ for the well-ordering $\preceq_\lambda \in \mathcal{F}$ with the least point $(0, 0)_2$. So $1 \not\leq_0 g$. If $t_4 + 2t_3 < 0$, then $(0, 0)_1 g < (0, 0)_1$, and $1 \not\leq_0 g$ similarly. If $t_4 = t_3 = 0$ and $t_2 < 0$ then $(0, 0)_1 g < (0, 0)_1$; and if $t_4 = t_3 = 0$ and $t_1 < 0$ then $(0, 0)_2 g < (0, 0)_2$. In all cases, $1 \not\leq_0 g$.

We next calculate $c^2 \wedge a$, $c^2 h^{-1} \wedge a$ and $h \wedge a$. Using, respectively, conditions (a), (b), (c) and (c), we see that $c^2 >_0 1$, $c^2 h^{-1} >_0 1$, $h >_0 1$, and $a >_0 1$. So all $c^2 \wedge a$, $c^2 h^{-1} \wedge a$, $h \wedge a$ are positive under $\leq_0$. Further, $c^2 a^{-1} = a^{-1} c^2 >_0 1$ by (a) and so $c^2 \wedge a = a$. Suppose $g = a^{t_1} b^{t_2} h^{t_3} c^{t_4} \in G$ satisfies

$$1 \leq_0 g, \quad (15)$$

$$g \leq_0 c^2 h^{-1}, \quad (16)$$

$$g \leq_0 a \quad (17)$$

Now (15) and (a) – (c) give $t_4 \geq 0$ and $t_4 + 2t_3 \geq 0$, but (17) and (a) – (c) give $t_4 \leq 0$ and $t_4 + 2t_3 \leq 0$. Thus $t_4 = t_3 = 0$ and $g = a^{t_1} b^{t_2}$. Further, (15) and (c) give $t_1, t_2 \geq 0$; but (16) and (c) give $t_1, t_2 \leq 0$. So $g = 1$ and consequently $c^2 h^{-1} \wedge a = 1$. Similarly, $h \wedge a = 1$.

In any ℓ -group, $c^2 h^{-1} \wedge a = 1$ and $h \wedge a = 1$ imply $c^2 \wedge a = 1$. But we have $c^2 \wedge a = a$ in the p.o. group G . This completes the negative solution of Problem 1.42. //

References

- [1] V. V. Bludov and A. M. W. Glass, *Word problems, embeddings, and free products of right-ordered groups with amalgamated subgroup*, Proc. London Math. Soc. 2009; doi: 10.1112/plms/pdp008

- [2] A. M. W. Glass, *Partially Ordered Groups*, Series in Algebra **7**, World Scientific Pub. Co., Singapore, 1999.
- [3] W. C. Holland, *The lattice-ordered group of automorphisms of an ordered set*, Michigan Math. J., **10** (1963), 399–408.
- [4] W. C. Holland and E. B. Scrimger, *Free products of lattice-ordered groups*, Algebra Universalis **2** (1972), 247–254.
- [5] K. R. Pierce, *Amalgamations of lattice-ordered groups*, Trans. Amer. Math. Soc. **172** (1972), 249–260.
- [6] V. M. Kopytov and N. Ya. Medvedev, *Ordered groups*, Selected Problems in Algebra. Collection of works dedicated to the memory of N. Ya. Medvedev, Altai State University, Barnaul, 2007, 15–112 (in Russian).
- [7] A. A. Vinogradov, *On the free product of ordered groups*, Math. Sbornik **25** (1949), 163–168 (in Russian).

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