# Free products and Higman-Neumann-Neumann type extensions of lattice-ordered groups.

A. M. W. Glass

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To Akbar H. Rhemtulla with thanks for years of enjoyable collaboration.

**Abstract**

We prove the lattice-ordered group analogues of two easy results from group theory.

**Theorem A** *Let $G, H$ be lattice-ordered groups with soluble word problem. Then the free product of $G$ and $H$ (in the category of lattice-ordered groups) has soluble word problem.*

**Theorem B** *Let $G$ be a lattice-ordered group and $H$ a convex sublattice subgroup of $G$. Then $G$ can be $\ell$-embedded in $L$, where $L$ has presentation $\langle G, t : t^{-1}ht = h\ (h \in H)\rangle$ in the category of lattice-ordered groups. If $g \in G$, then in $L$, $[t, g] = 1$ iff $g \in H$, and if $\mathbf{f}$, $\mathbf{g}$ are finite subsets of $G$ (which may overlap), then $w(\mathbf{f}, \mathbf{g}) \neq 1$ in $G$ implies $w(t^{-1}\mathbf{f}t, \mathbf{g}) \neq 1$ in $L$.*

The proofs use permutation groups, a technique of Holland and McCleary, and the ideas used to prove the lattice-ordered group analogue of the Boone-Higman Theorem.

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# 1 Introduction

Since the category of lattice-ordered groups is equationally defined, free lattice-ordered groups on any set of free generators exists. So does the free product (in this category) of any set of lattice-ordered groups; moreover, each lattice-ordered group in this set is naturally embeddable in this free product and we will identify it with its image. If $G_1, G_2$ are lattice-ordered groups, then their free product $G_1 *_{\mathcal{L}} G_2$ in this category is defined (to within isomorphism between lattice-ordered groups) by the standard property:

if $L$ is a lattice-ordered group and $\varphi_j : G_j \to L$ $(j = 1, 2)$ are $\ell$-homomorphisms (in this category), then there is a unique $\ell$-homomorphism $\varphi : G_1 *_{\mathcal{L}} G_2 \to L$ whose restriction to $G_j$ is $\varphi_j$ $(j = 1, 2)$

**Cautions:** (1) The subgroup of $G *_{\mathcal{L}} H$ generated by $G \cup H$ is <u>not</u> in general the *group* free product of $G$ and $H$.

(2) If a lattice-ordered group $A$ is $\ell$-embeddable in $G_1$ and $G_2$, there is not usually a lattice-ordered group $L$ in which $G_1$ and $G_2$ can be $\ell$-embedded to make the diagram commute; thus $HNN$-extensions do not in general exist in the category of lattice-ordered groups.

For more details, see the next section.

Despite these cautions, we can prove very special analogues of the corresponding group theoretic results.

**Theorem A** *Let $G, H$ be lattice-ordered groups with soluble word problem. Then the free product of $G$ and $H$ (in the category of lattice-ordered groups) has soluble word problem.*

**Theorem B** *Let $G$ be a lattice-ordered group and $H$ a convex sublattice subgroup of $G$. Then $G$ can be $\ell$-embedded in $L$, where $L$ has presentation $\langle G, t : t^{-1}ht = h \ (h \in H)\rangle$ in the category of lattice-ordered groups. If $g \in G$, then in $L$, $[t, g] = 1$ iff $g \in H$, and if $\mathbf{f}$, $\mathbf{g}$ are finite subsets of $G$ (which may overlap), then $w(\mathbf{f}, \mathbf{g}) \neq 1$ in $G$ implies $w(t^{-1}\mathbf{f}t, \mathbf{g}) \neq 1$ in $L$.*

More general results should be true.

# 2 Background and notation

Throughout we will use $\mathbb{N}$ for the set of non-negative integers, $\mathbb{Z}_+$ for the set of positive integers, $\mathbb{Q}$ for the set of rational numbers and $\mathbb{R}$ for the set of real numbers. The only order on $\mathbb{Q}$ and $\mathbb{R}$ that we will consider will be the usual one.

If $X$ and $Y$ are totally ordered sets, let $X \overleftarrow{\times} Y$ be the set $X \times Y$ totally ordered by: $(x, y) < (x', y')$ if either ($y < y'$ in $Y$) or ($y = y'$ in $Y$ & $x < x'$ in $X$).

We assume that the reader has a minimal knowledge of recursive function theory (see [16]).

In any group $G$ we write $[f, g]$ for $f^{-1}g^{-1}fg$. If $H$ is a subgroup of $G$, we write $[g, H]$ for $\{[g, h] : h \in H\}$.

A *lattice-ordered group* is a group which is also a lattice that satisfies the identities $x(y \wedge z)t = xyt \wedge xzt$ and $x(y \vee z)t = xyt \vee xzt$. Throughout we write $x \leq y$ as a shorthand for $x \vee y = y$ or $x \wedge y = x$, and *$\ell$-group* as an abbreviation for lattice-ordered group. A sublattice subgroup of an $\ell$-group is called an *$\ell$-subgroup*. An $\ell$-group that is totally ordered is called an o-*group*.

Lattice-ordered groups are torsion-free and $f \vee g = (f^{-1} \wedge g^{-1})^{-1}$. Moreover, as lattices, they are distributive ([5], Lemma 2.3.5). Each element of $G$ can be written in the form $fg^{-1}$ where $f, g \in G^+ = \{h \in G : h \geq 1\}$ — see, *e.g.*, [5], Corollary 2.1.3, Lemma 2.3.2 & Lemma 2.1.8. For each $g \in G$, let $|g| = g \vee g^{-1}$. Then $|g| \in G_+$ iff $g \neq 1$, where $G_+ = G^+ \setminus \{1\}$. Therefore, $(w_1 = 1 \;\&\; \ldots \;\&\; w_n = 1)$ iff $|w_1| \vee \ldots \vee |w_n| = 1$ [*ibid*, Lemma 2.3.8 & Corollary 2.3.9]. Consequently, in the language of lattice-ordered groups (and in sharp contrast to group theory) any finite number of equalities can be replaced by a single equality.

We will write $f \perp g$ as a shorthand for $|f| \wedge |g| = 1$ and say that $f$ and $g$ are *orthogonal*. As is well-known and easy to prove, $f \perp g$ implies $[f, g] = 1$.

We will write $\mathcal{L}$ for the category of all lattice-ordered groups. Its morphisms, called *$\ell$-homomorphisms*, are group and a lattice homomorphisms. Kernels are precisely the normal $\ell$-subgroups that are convex (if $k_1, k_2$ belong to the kernel and $k_1 \leq g \leq k_2$, then $g$ belongs to the kernel). They are called *$\ell$-ideals*.

The free $\ell$-group on any set of generators exists by universal algebra. The free $\ell$-group on a single generator is $\mathbb{Z} \oplus \mathbb{Z}$ ordered by: $(m_1, m_2) \geq (0, 0)$ iff $m_1, m_2 \geq 0$; $(1, -1)$ is a generator since $(1, -1) \vee (0, 0) = (1, 0)$.

We will write

$$\langle Y \;:\; w_i(Y) = 1 \; (i \in I) \rangle$$

for the quotient $F/K$ where $F$ is the free $\ell$-group on the generating set $Y$ and $K$ is the $\ell$-ideal generated (as an $\ell$-ideal) by $\{w_i(Y) : i \in I\}$.

If $G = \langle Y \ : \ w_i(Y) = 1 \ (i \in I)\rangle$ as above and $t$ is a new symbol, let $F(Y,t)$ be the free $\ell$-group on the free generators $Y \cup \{t\}$. If $\{u_j(Y,t) : j \in J\}$ is a set of $\ell$-group words in $Y \cup \{t\}$, then we write

$$\langle G, t : u_j(Y,t) = 1 \ (j \in J)\rangle \quad \text{for} \quad F(Y,t)/K_0,$$

where $K_0$ is the $\ell$-ideal of $F(Y,t)$ generated by $\{w_i(Y), u_j(Y,t) : i \in I, j \in J\}$.

If $G_1$ and $G_2$ are $\ell$-groups, then we can analogously write $\langle G_1, G_2\rangle$ for $G_1 *_{\mathcal{L}} G_2$. In sharp contrast to groups, it is far from clear, *ab initio*, what "mixed" $\ell$-group expressions in the generators $Y_1 \cup Y_2$ are the identity in $\langle G_1, G_2\rangle$.

**Lemma 2.1** (Holland and Scrimger [11]) *Let $G, H$ be $\ell$-groups with $g_1 \wedge g_2 = 1$ in $G$ and $h_1 \wedge h_2 = 1$ in $H$. Then*

$$[h_1, g_2^{-1}h_2g_1h_2^{-1}g_2h_2g_1^{-1}]$$

*is a non-identity reduced word of length* 16 *in the group free product of $G$ and $H$ but is equal to the identity in $G *_{\mathcal{L}} H$. Hence the subgroup of $G *_{\mathcal{L}} H$ generated by $G \cup H$ is not the group free product if $G$ and $H$ are not o-groups.*

This also appears as Lemma 1.11.5 in [3] in a slightly different context.

In contrast to groups, the amalgamation property fails for $\mathcal{L}$: there are $\ell$-groups $G, H_1, H_2$ with $\ell$-embeddings $\sigma_j : G \to H_j$ $(j = 1, 2)$ such that there is no $\ell$-group $L$ such that $H_j$ can be $\ell$-embedded in $L$ $(j = 1, 2)$ so that the resulting diagram commutes (see [15] or [5], Theorem 7.C). Hence $HNN$-extension techniques fail (see [2]). That is, there is an $\ell$-group $G$ with $\ell$-isomorphic $\ell$-subgroups $A, B$ (via $\varphi$) such that $G$ cannot be $\ell$-embedded in $\langle G, t : t^{-1}at = a\varphi \ (a \in A)\rangle$.

Let $\{G_x : x \in X\}$ be a family of $\ell$-groups. Then the full Cartesian product $C := \prod\{G_x : x \in X\}$ is an $\ell$-group under the ordering

$$(g_x)_{x \in X} \in C^+ \quad \text{iff} \quad g_x \in G_x^+ \quad \text{for all} \quad x \in X.$$

We call $C$ the *cardinal product* of $\{G_x : x \in X\}$.

If $X$ is finite, say $X = \{1, \ldots, m\}$, we write $G_1 \oplus \ldots \oplus G_m$ for this $\ell$-group.

Let $(\Omega, \leq)$ be a totally ordered set. Then $Aut(\Omega, \leq)$ is an $\ell$-group when the group operation is composition and the lattice operations are just the pointwise supremum and infimum ($\alpha(f \vee g) = max\{\alpha f, \alpha g\}$, *etc.*) There is an analogue of Cayley's Theorem for groups, namely the Cayley-Holland Theorem ([5], Theorem 7.A):

**Theorem C** (Holland [9]) *Every lattice-ordered group can be $\ell$-embedded in $Aut(\Omega, \leq)$ for some totally ordered set $(\Omega, \leq)$; every countable lattice-ordered group can be be $\ell$-embedded in $Aut(\mathbb{Q}, \leq)$ and hence in $Aut(\mathbb{R}, \leq)$.*

We will write $A(\Omega)$ as a shorthand for $Aut(\Omega, \leq)$ when the total order on $\Omega$ is clear.

Since we will need the ideas in the proof of the Cayley-Holland Theorem to prove Theorem B, we give an outline of its proof here. For more details, see [*op. cit.*].

*Outline of Proof:* Given an $\ell$-group $G$ and a convex $\ell$-subgroup $H$ of $G$, let $g \in G \setminus H$. By Zorn's Lemma there is a convex $\ell$-subgroup $C_g \supseteq H$ that is maximal with respect to not containing $g$; it is called a *value* of $g$ in $G$. The set $\Omega_g$ of right cosets of $C_g$ in $G$ inherits a partial order from $G$; with respect to this partial order, $\Omega_g$ is actually a totally ordered set and the right representation of $G$ on $\Omega_g$ gives an $\ell$-homomorphism (in $\mathcal{L}$) of $G$ into $A(\Omega_g)$. By letting $H = \{1\}$ and $g$ range over all non-identity elements of $G$, we get an $\ell$-embedding of $G$ into $\prod\{A(\Omega_g) : \ g \in G \setminus \{1\}\}$. Let $\prec$ be a total order on $G \setminus \{1\}$ (not necessarily compatible with the group structure) and $(\Omega, \leq)$ be $\bigcup\{(\Omega_g, \leq) : \ g \in G \setminus \{1\}\}$ ordered by: let $\alpha \in \Omega_f$ and $\beta \in \Omega_g$; then $\alpha < \beta$ if $f \prec g$ or both $f = g$ and $\alpha < \beta$ in $(\Omega_g, \leq)$. Then the $\ell$-homomorphisms induce an $\ell$-embedding of $G$ in $A(\Omega)$. //

If $G$ is a subgroup of $A(\Omega)$ and $n \in \mathbb{Z}_+$, then $G$ is said to be *order-$n$-transitive* if for all $\alpha_1 < \ldots < \alpha_n$ and $\beta_1 < \ldots < \beta_n$ in $\Omega$, there is $g \in G$ with $\alpha_j g = \beta_j$ $(j = 1, \ldots, n)$. If $G$ is order-$n$-transitive for all $n \in \mathbb{Z}_+$, then $G$ (acting on $\Omega$) is said to be *highly order-transitive.* If $\Omega$ is infinite, then any order-2-transitive $\ell$-subgroup of $A(\Omega)$ is higly order-transitive (see [3], Lemma 1.10.1). If $F$ is any ordered field, then $A(F)$ is order-2-transitive and so highly order-transitive.

If $g \in A(\Omega)$, then the *support* of $g$, *supp*$(g)$, is the set $\{\beta \in \Omega : \beta g \neq \beta\}$.

Since each real interval $(\alpha, \beta)$ is order-isomorphic to $(\mathbb{R}, \leq)$, we obtain:

**Corollary 2.2** *Let $\alpha, \beta \in \mathbb{R}$ with $\alpha < \beta$. Then every countable $\ell$-group $G$ can be $\ell$-embedded in $A(\mathbb{R})$ so that supp$(g) \subseteq (\alpha, \beta)$ for all $g \in G$.*

If $g \in A(\Omega)$ and $\alpha \in supp(g)$, then the convexification of the $g$-orbit of $\alpha$ is called the *interval of support of $g$ containing $\alpha$*; *i.e.*, the supporting interval of $g$ containing $\alpha$ is $\Delta(g, \alpha) := \{\beta \in \Omega : \ (\exists m, n \in \mathbb{Z})(\alpha g^n \leq \beta \leq \alpha g^m)\}$. So the support of an element is the disjoint union of its non-singleton supporting intervals. The restriction of $g$ to one of its non-singleton intervals of support is called a *bump* of $g$. We will also call an element of $A(\Omega)$ a *bump* if it has just one bump. If $g$ is a bump, we write $\Delta_g$ for its unique non-singleton supporting interval.

By considering intervals of support, it is easy to establish the well-known fact:

**Proposition 2.3** *For all $f, g \in A(\Omega)$, $supp(g^{-1}fg) = supp(f)g$. Hence if $g^{-1}fg \perp f$ and $g \geq 1$, then $|f|^n \leq g$ for all $n \in \mathbb{Z}_+$.*

We complete this section with an application of the Cayley-Holland Theorem that we will need in the proof of Theorem A.

V. M. Kopytov and S. H. McCleary independently proved that the free lattice-ordered group on a finite number of generators has a faithful highly order-transitive representation ([12], [14] or [5], Theorem 8.D). This result was extended in [4] (or see [5], Theorem 8.F) to show that the free product $G_1 *_{\mathcal{L}} G_2$ of countable $\ell$-groups $G_1, G_2$ has a highly order-transitive representation in $A(\mathbb{Q})$. For countable free $\ell$-groups, more is true.

**Proposition 2.4** [10]. *Given any order-preserving bijections $z_j$ with domain and range finite subsets of $\mathbb{Q}$ $(j = 1, \ldots, n)$, these maps can be extended to elements $y_j \in A(\mathbb{Q})$ $(j = 1, \ldots, n)$ so that the $\ell$-subgroup of $A(\mathbb{Q})$ generated by $\{y_1, \ldots, y_n\}$ is the free $\ell$-group $F$ on $\{y_1, \ldots, y_n\}$.*

Holland and McCleary applied Proposition 2.4 to prove (*op. cit.*)

**Theorem D** [10] *For any positive integer $n$, the free lattice-ordered group on $n$ free generators has soluble word problem.*

We provide a summary of the proof as we will need it later.

*Summary of proof:* First consider a single group term $w(y_1, \ldots, y_n)$, say $w := y_{j_1}^{\epsilon_1} \ldots y_{j_k}^{\epsilon_k}$, where $j_1, \ldots, j_k \in \{1, \ldots, n\}$ and $\epsilon_1, \ldots, \epsilon_k \in \{\pm 1\}$. We draw two diagrams, one with $0y_{j_1} > 0$, the other with $0y_{j_1} < 0$.

From each of these diagrams we construct three new diagrams if $j_2 \neq j_1$. For the first diagram ($0y_{j_1} > 0$), we make the following modification. If $\epsilon_1 = 1$, we draw three diagrams, the first with

$$0y_{j_1}^{\epsilon_1} y_{j_2}^{\epsilon_2} > 0y_{j_1}^{\epsilon_1} > 0,$$

the second with

$$0y_{j_1}^{\epsilon_1} > 0y_{j_1}^{\epsilon_1} y_{j_2}^{\epsilon_2} > 0,$$

and the third with

$$0y_{j_1}^{\epsilon_1} > 0 > 0y_{j_1}^{\epsilon_1}y_{j_2}^{\epsilon_2};$$

on the other hand, if $\epsilon_1 = -1$, we construct three diagrams: in the first, we have

$$0y_{j_1}^{\epsilon_1}y_{j_2}^{\epsilon_2} > 0 > 0y_{j,1}^{\epsilon_1},$$

in the second

$$0 > 0y_{j_1}^{\epsilon_1}y_{j_2}^{\epsilon_2} > 0y_{j,1}^{\epsilon_1},$$

and in the third

$$0 > 0y_{j_1}^{\epsilon_1} > 0y_{j_1}^{\epsilon_1}y_{j_2}^{\epsilon_2}.$$

If $0y_{j_1} > 0$ and $j_1 = j_2$, then we construct a single diagram with

$$0y_{j_1}^{\epsilon_1}y_{j_2}^{\epsilon_2} > 0y_{j_1}^{\epsilon_1} > 0 \quad \text{if } \epsilon_1 = \epsilon_2 = 1;$$

a single diagram with

$$0y_{j_1}^{\epsilon_1}y_{j_2}^{\epsilon_2} < 0y_{j_1}^{\epsilon_1} < 0 \quad \text{if } \epsilon_1 = \epsilon_2 = -1;$$

a single diagram with

$$0y_{j_1}^{\epsilon_1}y_{j_2}^{\epsilon_2} = 0 < 0y_{j_1}^{\epsilon_1} \quad \text{if } \epsilon_1 = 1 \text{ and } \epsilon_2 = -1;$$

and a single diagram with

$$0y_{j_1}^{\epsilon_1}y_{j_2}^{\epsilon_2} = 0 > 0y_{j_1}^{\epsilon_1} \quad \text{if } \epsilon_1 = -1 \text{ and } \epsilon_2 = 1.$$

Similarly, we construct diagrams from the second case ($0y_{j_1} < 0$). We proceed with the spelling ensuring only that when we consider $y_{j_i}^{\epsilon_i}$, the element $y_{j_i}$ and its inverse respect all the inequalities declared previously involving $y_\ell$ where $j_i = \ell$.

By Proposition 2.4, if in *all* possible resulting legitimate diagrams we have $0w = 0$, then $w = 1$ in $F$; if in *some* resulting legitimate diagram we get $0w \neq 0$, then $w \neq 1$ in $F$ by the same proposition.

This completes the solubility of the group word problem in $F$.

For a general $\ell$-group word $w(y_1, \ldots, y_n)$, enumerate the group words used to constitute

$$w := \bigvee_{i=1}^{k} \bigwedge_{j=1}^{r_i} w_{i,j};$$

*i.e.*, $w_{1,1}, \ldots, w_{1,r_1}, w_{2,1}, \ldots, w_{k,r_k}$. Form all possible legitimate diagrams as above for $w_{1,1}$. For each of these diagrams, do the same for $w_{1,2}$ subject only that all inequalities that occurred in that diagram for $w_{1,1}$ are respected in the diagrams for $0w_{1,2}$. For each of the resulting diagrams, do the same for $w_{1,3}$, *etc.* Then $w = 1$ in $F$ if $0w = 0$ in *all* resulting diagrams; and $w \neq 1$ in $F$ if $0w \neq 0$ in *some* resulting diagram. //

The proof trivially extends to the solubility of the word problem for the free $\ell$-group on a countably infinite set of free generators. Theorem D was a crucial ingredient in the proof of the full $\ell$-group analogue of Higman's Embedding Theorem (see [6]) which was the topic of my talk at the Ischia Conference. The proof of Theorem 2.4 was generalised in [7] to prove the full $\ell$-group analogue of the Boone-Higman Theorem; one step of the proof in [7] is developed here to prove Theorem A.

# 3 The proof of Theorem A.

The key to the proof is to build sufficiently many (not necessarily faithful) representations of $G *_{\mathcal{L}} H$ in $A(\mathbb{Q}) \subseteq A(\mathbb{R})$ to determine algorithmically whether or not an arbitrary $\ell$-group expression in the alphabet of $G \cup H$ is the identity of $G *_{\mathcal{L}} H$. I have not been able to use the faithful highly order-transitive representation of $G *_{\mathcal{L}} H$ in $A(\mathbb{Q})$, so have had to proceed more circuitously.

**Lemma 3.1** *Let $G$ be a countable $\ell$-group and $\Lambda$ be any set of pairwise disjoint non-empty open intervals in $\mathbb{Q}$. Then there is an $\ell$-embedding of $G$ into $A(\mathbb{Q}) \subseteq A(\mathbb{R})$ such that $supp(g) \subseteq \Lambda$ for all $g \in G$.*

*Proof:* Let $\Delta \in \Lambda$. By Corollary 2.2, there is an $\ell$-embedding $\varphi_\Delta$ of $G$ into $A(\mathbb{Q})$ such that $supp(g\varphi_\Delta) \subseteq \Delta$ for all $g \in G$. Define $\varphi : G \to A(\mathbb{Q})$ by:
if $\alpha \in \Delta \in \Lambda$, let $\alpha(g\varphi) = \alpha(g\varphi_\Delta)$; and if $\alpha \notin \bigcup_{\Delta \in \Lambda} \Delta$, let $\alpha(g\varphi) = \alpha$.
Then $\varphi$ is the desired $\ell$-embedding.//

We will often take $\Lambda$ to be a subset of $\{(n, n+1) : n \in \mathbb{Z}\}$, or a set of subintervals of these; *e.g.*, take a dense set of open intervals of $(0, 1)$ without greatest or least interval and their translates by $2n$ for all $n \in \mathbb{Z}$.

**Lemma 3.2** *Let $G$ be a countable $\ell$-group. If $\alpha_1 < \ldots < \alpha_n$ in $\mathbb{Q}$ and $g_1, \ldots, g_n \in G_+$, then there is an $\ell$-embedding $\psi : G \to A(\mathbb{Q})$ such that $\alpha_j < \alpha_j(g_j\psi)$ $(j = 1, \ldots, n)$.*

*Proof:* Let $\Lambda = \{(2m, 2m+1) : n \in \mathbb{Z}\}$. By Lemma 3.1, there is an $\ell$-embedding $\varphi : G \to A(\mathbb{Q})$ and $\beta_j \in (2j, 2j+1)$ such that $\beta_j < \beta_j(g_j\varphi)$

$(j = 1, \ldots, n)$. Let $f \in A(\mathbb{Q})$ be such that $\beta_j f = \alpha_j$ $(j = 1, \ldots, n)$. Define $\psi : G \to A(\mathbb{Q})$ by $g\psi = f^{-1}(g\varphi)f$ $(g \in G)$. Then $\psi$ is the desired $\ell$-embedding.//

We now modify the Holland-McCleary algorithm (outlined in the previous section) to obtain Theorem A.

*Proof of Theorem A:* First note that since $G$ and $H$ are recursively generated and are defined by recusively enumerable sets of relations, the same is true of $G *_{\mathcal{L}} H$; the generators and relations are the unions of the respective sets. Hence there is an algorithm to enumerate all $\ell$-group expressions in the alphabet of $G \cup H$ that are the identity in $G *_{\mathcal{L}} H$. Thus we can determine (for any $\ell$-group expression $w$ in this alphabet) if $w = 1$ in $G *_{\mathcal{L}} H$.

We now provide an algorithm that demonstrates that $w \neq 1$ in $G *_{\mathcal{L}} H$ for any $\ell$-group expression $w$ that is not the identity in $G *_{\mathcal{L}} H$.

We first consider a special case. Let $w$ be a group expression in alternately elements of $G$ and elements of $H$; say $w = g_1 h_1 \ldots g_m h_m$. Assume that $w \neq 1$ in $G *_{\mathcal{L}} H$. Using the solubility of the word problem for $G$, we may determine whether or not $g_1 = 1$ in $G$. Remove $g_1$ if it is equal to 1 in $G$. We can likewise test if $h_m$ is or is not 1 in $H$ and remove $h_m$ if it is equal to 1 in $H$. We now consider the new word. We can determine whether or not any $g_i$ $(i = 2, \ldots, m)$ is equal to the identity of $G$. If it is, replace $h_{i-1} g_i h_i$ by $h_{i-1} h_i$ regarded as a single symbol in $H$. Do the same with $H$ instead of $G$: for any $h_j$ that is equal to 1 in $H$, replace $g_j h_j g_{j+1}$ by $g_j g_{j+1}$ regarded as a single symbol in $G$ $(j = 1, \ldots, m-1)$. Continue with the rechristened new-formed expression until this is no longer possible. The entire procedure is completely algorithmic. If the new word $w'$ begins with an element $h \in H$, then we may consider the conjugate $h^{-1} w' h$ instead of $w'$ $(h^{-1} w' h = 1$ iff $w' = 1)$. Similarly, if $w'$ ends with an element of $G$. Thus we can reduce to examining the special case that $w = g_1 h_1 \ldots g_m h_m$ with $g_1, \ldots, g_m \neq 1$ in $G$ and $h_1, \ldots, h_m \neq 1$ in $H$.

By the Cayley-Holland Theorem, there is a (faithful) representation of $G *_{\mathcal{L}} H$ in $A(\mathbb{R})$ such that $0w \neq 0$. We wish to construct every legitimate *finite* diagram for $w$ (to within equivalence). That is, all consistent orderings of the migratory points with attached arrows.

The *migratory points for $w$* are

$$0, \;\; 0g_1, \;\; 0g_1h_1, \;\; 0g_1h_1g_2, \;\; \ldots, \;\; 0g_1h_1 \ldots g_m h_m = 0w.$$

These points may or may not be distinct.

Let $v_0 := 0$, $u_{j+1} := v_j g_{j+1}$ and $v_{j+1} := u_{j+1} h_{j+1}$ $(j = 0, \ldots, m-1)$.

Place a $g_{j+1}$-*arrow* from $v_j$ to $u_{j+1}$ and an $h_{j+1}$-*arrow* from $u_{j+1}$ to $v_{j+1}$ $(j = 0, \ldots, m-1)$. We also place a $g_{j+1}^{-1}$-arrow from $u_{j+1}$ to $v_j$ and an $h_{j+1}^{-1}$-arrow from $v_{j+1}$ to $u_{j+1}$ $(j = 0, \ldots, m-1)$. Any $g$-arrow $(g \in G)$ will be called a $G$-*arrow* and any $h$-arrow $(h \in H)$ will be called an $H$-*arrow*. We extend the definition of $G$-arrows as follows. If there is a subsequence of migratory points $\alpha_1, \ldots, \alpha_k$ with $G$-arrows between each $\alpha_j$ and $\alpha_{j+1}$ $(j = 1, \ldots, k)$, then we will regard the composition of the arrows as providing a $G$-arrow between $\alpha_1$ and $\alpha_k$. *Mutatis mutandis* for $H$-arrows. These are the only arrows we will consider.

We will regard two diagrams for $w$ as *equivalent* if there is an order-preserving bijection between their migratory points with corresponding points having the same arrows to within equality in $G$ or $H$.

Let $\alpha$ be a migratory point for $w$ and $\{i_1, \ldots, i_r\} \subseteq \{1, \ldots, m\}$. Let $\epsilon_1, \ldots, \epsilon_r \in \{\pm 1\}$. We say that $\alpha$ is *forced up by* $g := g_{i_1}^{\epsilon_1} \ldots g_{i_r}^{\epsilon_r}$ in a diagram if there are migratory points $\beta, \gamma$ with $\beta < \alpha < \gamma$ such that $\beta g = \gamma$. We may similarly define such an element $g \in G$ as forcing a migratory point down. In either case, we will say that $\alpha$ is *surrounded by* a $g$-arrow. Note that if $g' := g_{j_1}^{\eta_1} \ldots g_{j_s}^{\eta_s}$ is another such group word of $G$ $(\eta_1, \ldots, \eta_s \in \{\pm 1\})$ and $g' \perp g$, then $\alpha g' = \alpha$ for any migratory point $\alpha$ surrounded by a $g$-arrow. Moreover, if $\beta < \alpha < \gamma$ with $\beta g = \gamma$, then $\delta g' < \beta$ if $\delta < \beta$ and $\delta g' > \gamma$ if $\delta > \gamma$. *Mutatis mutandis*, for a group expression in $H$ associated with $w$. These cause the only modifications needed to the original Holland-McCleary method.

By uniform continuity and Lemma 3.2, if $w \neq 1$ in $G *_{\mathcal{L}} H$, there must be a consistent diagram in which these $2m+1$ migratory points are distinct with the following exceptions. Let $g \perp g'$ with $g, g'$ as above. For any migratory point $\alpha$, we must have $\alpha g = \alpha$ or $\alpha g' = \alpha$. Indeed, we will construct sets of consistent diagrams one in which a migratory point $\alpha$ is fixed by $g$, another in which $\alpha$ is fixed by $g'$. If a migratory point $\beta$ is surrounded by a $g$-arrow, then $\beta g' = \beta$; and if $\beta$ is surrounded by a $g'$-arrow, then $\beta g = \beta$. And, of course, the diagram must be consistent. *Mutatis mutandis* with $G$ replaced by $H$.

To achieve this, we consider all subwords $g$ as above and list all pairs of them that are pairwise orthogonal. For any such pair $(g, g')$ appearing by stage $i$, we will extend each consistent diagram so far constructed as follows: for each migratory point so far appearing, we will impose either $\alpha g = \alpha$ or $\alpha g' = \alpha$ and take all consistent possibilities.

**Aside:** In constructing a diagram for $w$, we may have to apply $f \in c\ell s(g_i)$, the convex $\ell$-subgroup of $G$ generated by $g_i \in G$. If for some migratory point

$\alpha \in \mathbb{R}$, we have $\alpha g_i = \alpha$, then $\alpha f = \alpha$. Moreover, if $\gamma < \alpha < \delta$, then we must have $\gamma f < \alpha < \delta f$. Although $G$ has soluble word problem, there is no algorithm in general to determine whether or not an element $f \in G$ belongs to $c\ell s(g_i)$: we can test if $|f| \leq |g_i|$, $|f| \leq |g_i|^2, \ldots$ , and so will be able to determine if $f \in c\ell s(g_i)$; but since there is no *a priori* bound on the power of $|g_i|$, we have no algorithm to determine if $f \notin c\ell s(g_i)$. So we seem to have no way of knowing that we might have restrictions on $f$. However, the only such constraints on $g_i$ arose because $g_i$ was orthogonal to some $g_{j_1}^{\eta_1} \ldots g_{j_r}^{\eta_r}$ with $\eta_1, \ldots, \eta_r \in \{\pm 1\}$; that is all that is needed for $f \in G$. If $f$ is not orthogonal to any element of $G$ with an arrow moving a point below $\alpha$ to one above, then we may assume that $\alpha f \neq \alpha$. On the other hand, if there is a $G$-arrow surrounding $\alpha$ with label, say $g$, orthogonal to $f$, then $\alpha f = \alpha$ and points above (below) the surrounding interval must be mapped by $f$ to the same interval. So it is quite unnecessary to determine if $f \in c\ell s(g_i)$. Of course, if there is a $g$-arrow from $\mu$ to $\nu$ and $f \leq g$, then $\mu f \leq \nu$. But we can test if $f \leq g$ for any of the finitely many $g \in G$ that are usable as labels for arrows. So the difficulty is avoided.

We now give the algorithm explicitly to demonstrate that $w \neq 1$ in $G *_{\mathcal{L}} H$.

Use the solubility of the word problem for $G$ to determine if $g_1 > 1$ (*i.e.*, $g_1 \wedge 1 = 1 \neq g_1$), $g_1 < 1$ (*i.e.*, $g_1 \vee 1 = 1 \neq g_1$) or neither.

**Case 1.1**. $g_1 > 1$.
The initial consistent diagram is $0g_1 > 0$ with the obvious arrows attached.

**Case 1.2**. $g_1 < 1$.
The initial consistent diagram is $0g_1 < 0$ with the obvious arrows attached.

**Case 1.3**. $g_1 \not> 1$ and $g_1 \not< 1$.
In this case, there are two initial consistent diagrams, in the first $0g_1 > 0$ and in the second $0g_1 < 0$, each with the obvious arrows attached.

Next (using the solubility of the word problem for $G$), determine if $g_1$ is orthogonal to any $g_{j_1}^{\epsilon_1} \ldots g_{j_r}^{\epsilon_r}$ with $\epsilon_1, \ldots, \epsilon_r \in \{\pm 1\}$. If so, in each of Cases 1.1–1.3, add a new initial consistent diagram with $0g_1 = 0$ and $g_1$ and $g_1^{-1}$ arrows from 0 to 0.

Now use the solubility of the word problem for $H$ to determine if $h_1 > 1$ (*i.e.*, $h_1 \wedge 1 = 1 \neq h_1$), $h_1 < 1$ (*i.e.*, $h_1 \vee 1 = 1 \neq h_1$) or neither.

**Case 2.1**. $h_1 > 1$.

If $g_1 > 1$, extend the consistent diagram from Case 1.1 to $0g_1h_1 > 0g_1 > 0$ with the obvious arrows attached. If we were required to add an extra diagram with $0g_1 = 0$, then extend it to $0 = 0g_1 < 0g_1h_1$ with the obvious arrows attached.

If $g_1 < 1$, we need two extensions of the consistent diagram from Case 1.2; in the first $0g_1h_1 > 0 > 0g_1$ and in the second $0 > 0g_1h_1 > 0g_1$. In each case, we attach the obvious arrows. If we were required to add an extra diagram with $0g_1 = 0$, then extend it to $0 = 0g_1 > 0g_1h_1$ with the obvious arrows attached.

If $g_1$ fell into Case 1.3, then we extend each of the consistent diagrams to get all cases covered by the two preceeding paragraphs. Likewise if we were required to add an extra diagram with $0g_1 = 0$.

**Case 2.2**. $h_1 < 1$.
This is analogous to Case 2.1.

**Case 2.3**. $h_1 \not\geq 1$ and $h_1 \not\leq 1$.
In this case, there are two sets of consistent extensions, in the first $0g_1h_1 > 0g_1$ and in the second $0g_1h_1 < 0g_1$ with all valid possibilities for the order relationship between $0g_1h_1$ and 0 as in Cases 2.1 and 2.2, respectively.

Finally we use the solubility of the word problem for $H$ to see if $h_1$ is orthogonal to any $h_{j_1}^{\eta_1} \dots h_{j_r}^{\eta_r}$ with $\eta_1, \dots, \eta_r \in \{\pm 1\}$. If so, we also add a new extension diagram with $0g_1h_1 = 0g_1$ (with the appropriate arrows) in any of the (possibly also modified) Cases 1.1–1.3.

Now suppose the set of consistent diagrams has been constructed for the placement of $v_0,\ u_1,\ v_1,\ \dots,\ u_i,\ v_i$ with the appropriate arrows. We first use the solubility of the word problem for $G$ to determine which of the three cases ($g_{i+1} > 1$, $g_{i+1} < 1$ or neither) pertains. We also determine if $g_{i+1}$ is orthogonal to any of the words $g := g_{j_1}^{\epsilon_1} \dots g_{j_r}^{\epsilon_r}$ with $\epsilon_1, \dots, \epsilon_r \in \{\pm 1\}$. If, say $g_{i+1} > 1$, then we need to determine (in each diagram) if there are any $G$-arrows moving a migratory point below $v_i$ to a migratory point above $v_i$. If there are, we need to determine how $g_{i+1}$ compares with these elements of $G$. If, for example, $\alpha < v_i < \alpha g$ and $g_i \geq g$ in $G$ and a $g$-arrow appears on the considered diagram at stage $i$, then we must have $u_{i+1} = v_i g_{i+1} > \alpha g$. And if $g_{i+1} \perp g$ for some $g$, then $u_{i+1} = v_i g_{i+1} = v_i$. Also, if some $G$-arrow $g$ moves a migratory point $\beta > v_i$ on a diagram at stage $i$ and $g \geq g_{i+1}$, then we must have $u_{i+1} = v_i g_{i+1} < \beta g$. And if $\delta > v_i$ and $\delta$ is fixed by some $g$-arrow already appearing in the diagram at stage $i$, then $u_{i+1} = v_i g_{i+1} < \delta$ if $g_{i+1} \perp g$. We also check the pairs of subwords associated with $g_1 \dots g_{i+1}$ and do likewise for these subject to consistency. Since the number of points and possibilities is finite and known, we know exactly how many tests to do, and then extend each diagram in all consistent ways according to the answers. Do the analogous procedure for $h_{i+1}$.

We therefore arrive at a set of diagrams which are *all* consitent diagrams for $w$ to within equivalence. Then $w \neq 1$ in $G *_{\mathcal{L}} H$ iff $0w \neq 0$ in *at least one* of these consistent diagrams.

Thus $G *_{\mathcal{L}} H$ has soluble *group* word problem.

More generally, any $\ell$-group expression in the alphabet of $G \cup H$ can be mechanically reduced to one of the form $\bigvee_{i=1}^{r} \bigwedge_{j=1}^{s_i} w_{i,j}$ where each $w_{i,j}$ is a group expression in this alphabet. We can reduce each $w_{i,j}$ to one in the form $g_1 h_1 \ldots g_m h_m$, where $g_1, \ldots, g_m \in G$, $h_1, \ldots, h_m \in H$ with $g_2, \ldots, g_m \neq 1$ and $h_1, \ldots, h_{m-1} \neq 1$. Enumerate $\{w_{i,j} : \ 1 \leq j \leq s_i, \ 1 \leq i \leq r\}$ in all possible ways.

Fix an enumeration. Construct all possible diagrams for the first group word in the enumeration as described for the group case. This is the first step. For the $(k+1)^{th}$ step, consider the $(k+1)^{th}$ group word in the enumeration. Extend each diagram in the $k^{th}$ step by adjoining all possible diagrams for the $(k+1)^{th}$ group word as explained above, but also ensuring that if some string of elements of $G$ or $H$ occurring in the $(k+1)^{th}$ group word is orthogonal, less than, *etc.*, to a string of elements from $G$ or $H$ occurring in one of the first $k$ group words in the enumeration, then we extend that diagram consistently as explained in the group case. Again, the amount of checking is determined and presents no algorithmic difficulties.

Thus we obtain a finite set of consistent diagrams for each enumeration of the group words. We therefore obtain the set of *all* consistent diagrams for the $\ell$-group word to within equivalence (possibly with repetitions). If the $\ell$-group expression $w$ is not equal to 1 in $G *_{\mathcal{L}} H$, then $0w \neq 0$ in *at least one* of the constructed diagrams; and $0w = 0$ in *all* constructed diagrams if $w = 1$ in $G *_{\mathcal{L}} H$. Consequently, we have solved the word problem for $G *_{\mathcal{L}} H$. //

**Example**

If $g_1 \wedge g_2 = 1$ in $G$ and $h_1 \wedge h_2 = 1$ in $H$, then the method shows that

$$g_1 h_2 g_1^{-1} \wedge g_1 h_1 \wedge g_2 = 1 \quad \text{in } G *_{\mathcal{L}} H.$$

# 4 Proof of Theorem B and generalisations

For groups, if $A, B$ are isomorphic subgroups of a group $G$ via isomorphism $\varphi$, then $G$ can be embedded in the group $(G, t : t^{-1}at = a\varphi (a \in A))$ (see [8] or [13]). In the very special case that $A$ and $B$ are equal convex $\ell$-subgroups of an $\ell$-group $G$ and $\varphi$ is the identity, then we can obtain a very limited Higman-Neumann-Neumann Theorem. This is given by Theorem B.

**Explanation**: $L$ is obtained from $G$ by adjoining $t$ as a new generator and requiring that it conjugates the elements of $H$ as the identity did, the resulting $L$ being as "free" as possible. The $\ell$-subgroup of $L$ generated by $G$ and $t^{-1}Gt$ *may* be their free product in $\mathcal{L}$ with amalgamated convex $\ell$-subgroup $H$; since there is no known test for whether an $\ell$-group is a free product in $\mathcal{L}$, I know of no way to determine this. Also, I suspect that there are counterexamples to Theorem B if the $\ell$-automorphism of $H$ is not the identity.

The proof again relies on the permuation group techniques outlined in the proof of Theorem C.

We begin with the proof of Theorem B, and then generalise the result to get a result which yields

**Corollary 4.1** *Let $G$ be an o-group and $H$ a subgroup of $G$. Then $G$ can be $\ell$-embedded in $L = \langle G, t: \ [t, H] = 1\rangle$. If $g \in G$, then in $L$, $[t, g] = 1$ iff $g \in H$, and if $\mathbf{f}$, $\mathbf{g}$ are finite subsets of $G$ (which may overlap), then $w(\mathbf{f}, \ \mathbf{g}) \neq 1$ in $G$ implies $w(t^{-1}\mathbf{f}t, \mathbf{g}) \neq 1$ in $L$.*

*Proof of Theorem B:* We first consider the case that $H$ is an $\ell$-ideal.

The proof proceeds in three parts. The first ensures that the $\ell$-homomorphism of $G$ into the resulting $\ell$-group is injective; the second ensures that if $\mathbf{f}$, $\mathbf{g}$ are finite subsets of $G$ (which may overlap), then $w(\mathbf{f}, \ \mathbf{g}) \neq 1$ in $G$ implies $w(t^{-1}\mathbf{f}t, \mathbf{g}) \neq 1$ in $L$; and the third ensures that $[t, g] \neq 1$ if $g \in G \setminus H$.

For each $g \in G \setminus \{1\}$, let $C_g$ be a value of $g$ (containing $H$ if $g \notin H$). Let $\Omega_g$ be the totally ordered set of right cosets of $C_g$ in $G$ (see the outline of the proof of Theorem C in Section 2). Let $\varphi_g : G \to A(\Omega_g)$ be the $\ell$-homomorphism given by:

$$(C_g f_1)(f\varphi_g) = C_g f_1 f \quad \text{for all} \quad f, f_1 \in G,$$

and let $K_g = ker(\varphi_g)$. Let $t_g$ be the identity of $A(\Omega_g)$ and

$$L_1 = \prod \{A(\Omega_g) : \ g \in G \setminus \{1\}\}.$$

Let $\varphi : G \to L_1$ be the $\ell$-homomorphism of $G$ in $L_1$ induced by $\{\varphi_g : \ g \in G \setminus \{1\}\}$. Then $\varphi$ is injective. Let $t_\flat$ be the automorphism of $L_1$ induced by $\{t_g : g \in G \setminus \{1\}\}$; *i.e.*, $t_\flat$ is the identity. This achieves the first and second goals; since $L_1$ is an $\ell$-homomorphic image of $L$, $w(\mathbf{f}, \mathbf{g}) \neq 1$ in $G$ implies

$w(t_\flat^{-1}\mathbf{f}t_\flat, \mathbf{g}) \neq 1$ in $L_1$. Since the diagram of $\ell$-homomorphisms commutes, the corresponding results are true in $L$.

If $g \in G \setminus H$, we extend $\Omega_g$ to an ordered field, $\Lambda_g$, such that the extension provides an $\ell$-embedding (which we will take to be the identity) of $A(\Omega_g)$ in $A(\Lambda_g)$ (see [3], Section 2.5). Hence the free product (in $\mathcal{L}$) of $G/K_g$ and the free $\ell$-group on one new generator, $a_g$, is contained in $A(\Lambda_g)$. By [15] or [3], Theorem 10.B, there is an $\ell$-group $L_{2,g}$ containing $A(\Lambda_g)$ and an element $t_{2,g} \in L_{2,g}$ such that $t_{2,g}^{-1}K_g|g|t_{2,g} = |a_g|$. Let $\psi_g : G \to L_{2,g}$ be the $\ell$-homomorphism arising. Let $L_2 = \prod\{L_{2,g} : \ g \in G \setminus H\}$, $(t_\#)_g = t_{2,g}$ and $\psi$ be the product of the $\psi_g$ $(g \in G \setminus H)$. Since $H$ is an $\ell$-ideal of $G$, the stabiliser of each right $C_g$ coset contains $H$. Thus $H \subseteq K_g = ker(\psi_g)$ (whence $H$ is the identity in $L_{2,g}$) for all $g \in G \setminus H$. So $[t_{2,g}, H\psi_g] = 1$ and $[t_{2,g}, g\psi_g] \neq 1$ for all $g \in G \setminus H$.

Let $L_0 = L_1 \oplus L_2$ with the cardinal ordering and $(\varphi, \psi)$ the induced $\ell$-homomorphism. Then $G$ is $\ell$-embedded in $L_0$ (since $\varphi$ is injective) and satisfies the desired properties by construction with $t = (t_\flat, t_\#)$. Since $L_0$ is an $\ell$-homomorphic image of $L$, the theorem follows when $H$ is an $\ell$-ideal of $G$.

If $H$ is a not-necessarily-normal convex $\ell$-subgroup, then we need to modify the argument in the last step:

By the Cayley-Holland Theorem, there is a totally ordered set $\Sigma_g$ and an $\ell$-embedding of $L_{2,g}$ into $A(\Sigma_g)$. We identify $L_{2,g}$ with its image and let $T_g = \mathbb{R} \overleftarrow{\times} \Sigma_g$. We embed $A(\Sigma_g)$ in $A(T_g)$ by $\bar{\psi}_g$ where

$$(r, \sigma)(f\bar{\psi}_g) = (r, \sigma f) \qquad (r \in \mathbb{R},\ \sigma \in \Sigma_g,\ f \in A(\Sigma_g)).$$

Let $\Delta = \mathbb{R} \overleftarrow{\times} \{C_g\}$ and $t_{2,g} \in A(T_g)$ be defined by:

$$(r, \sigma)t_{2,g} = \begin{cases} (r+1, \sigma) & \text{if } \sigma = C_g \\ (r, \sigma) & \text{otherwise.} \end{cases}$$

Since $C_g \supseteq H$, we have $t_{2,g} \neq 1$. Further, $supp(t_g) \subseteq \Delta$. Thus $[t_{2,g}, h] = 1$ for all $h \in H$ since $t_{2,g}$ and the elements of $H$ have disjoint supports. However, if $\rho \in supp(t_{2,g})$, then $\rho[t_{2,g}, g\psi_g] \neq \rho$. So we can proceed as in the $\ell$-ideal case with $t = (t_\flat, t_\#)$, where $(t_\#)_g$ is the new $t_{2,g}$.

This completes the proof of Theorem B. //

The extension of the above proof from the $\ell$-ideal case to a general convex $\ell$-subgroup permits further generalisation. The key is that for each $g \in G \setminus H$, we found an interval on which an $\ell$-homomorphic image of $H$ is the identity but the image of $g$ is not.

Let $H$ be an $\ell$-subgroup of an $\ell$-group $G$ and $g \in G$. We say that $g$ is *distinguishable from* $H$ if there is a totally ordered set $(\Omega, \leq)$ and an $\ell$-homomorphism $\theta : G \to A(\Omega)$ such that $\alpha(g\theta) \notin \alpha(H\theta)$ for some $\alpha \in \Omega$.

Let $\hat{H}$ be the set of all elements of $G$ that are *indistinguishable* from $H$. It is easy to see that $\hat{H}$ is an $\ell$-subgroup of $G$ containing $H$. Moreover, if $c\ell s(H)$ is the convex $\ell$-subgroup of $G$ generated by $H$ and $g \in G \setminus c\ell s(H)$, let $C_g$ be any value of $g$ containing $c\ell s(H)$. If $\Omega_g$ is the totally ordered set of right cosets of $C_g$ in $G$ and $\alpha_g = C_g \in \Omega_g$, then $\alpha_g H = \alpha_g$ and $\alpha_g g \neq \alpha_g$. Thus if we take $\theta_g : G \to A(\mathbb{R} \overleftarrow{\times} \Omega_g)$ given by:

$$(r, C_g f_1)(f\theta_g) = (r, C_g f_1 f) \quad (r \in \mathbb{R} : \ f, f_1 \in G),$$

then we see that $g$ is distinguishable from $H$. Thus $H \subseteq \hat{H} \subseteq c\ell s(H)$.

We now generalise Theorem B to

**Theorem E** *Let $G$ be a lattice-ordered group and $H$ an $\ell$-subgroup of $G$. Then $G$ can be $\ell$-embedded in $L = \langle G, t : \ [t, H] = 1 \rangle$. If $g \in G$, then in $L$, $[t, g] \neq 1$ if $g \notin \hat{H}$, and if $\mathbf{f}$, $\mathbf{g}$ are finite subsets of $G$ (which may overlap), then $w(\mathbf{f}, \mathbf{g}) \neq 1$ in $G$ implies $w(t^{-1}\mathbf{f}t, \mathbf{g}) \neq 1$ in $L$.*

*Proof:* Let $H_0 = c\ell s(H)$ and $L_1, \ L_2$ be as in the proof of Theorem B with $H_0$ in place of $H$. This achieves the last part and also gives $[t, g] \neq 1$ if $g \in G \setminus H_0$.

If $g \in H_0 \setminus \hat{H}$, then (by the definition of distinguishable) there is a totally ordered set $\Sigma_g$, a map $\chi_g : G \to A(\Sigma_g)$ and $\alpha_g \in \Sigma_g$ such that $\alpha_g(g\chi_g) \notin \alpha(H\chi_g)$. Let $T_g = \mathbb{R} \overleftarrow{\times} \Sigma_g$ and $\theta_g : G \to A(T_g)$ be given by:

$$(r, \sigma)(f\theta_g) = (r, \sigma(f\chi_g)) \quad (r \in \mathbb{R}; \ \sigma \in \Sigma_g; \ f \in G).$$

Let $s_g \in A(T_g)$ be defined by:

$$(r, \sigma)s_g = \begin{cases} (r+1, \sigma) & \text{if } \sigma \in \alpha_g(H\chi_g) \\ (r, \sigma) & \text{otherwise.} \end{cases}$$

A simple computation shows that $[s_g, H\theta_g] = 1$ but $[s_g, g\theta_g]$ moves $(0, \alpha_g)$.

Let $L_3 = \prod\{A(T_g) : \ g \in H_0 \setminus \hat{H}\}$. Then (under $\theta$ and $t \mapsto s$) $L_3$ is an $\ell$-homomorphic image of $L$ in which $[s, g] \neq 1$ for all $g \in H_0 \setminus \hat{H}$. Hence the same is true of $L$. Let $L_0$ be the cardinal product of $L_1, \ L_2, \ L_3$. Then $L_0$ is an $\ell$-homomorphic image of $L$ and has the required properties. The same is therefore true of $L$. This completes the proof of the theorem. //

As noted above, $\hat{H} = H$ if $H$ is a convex $\ell$-subgroup of $G$. Hence Theorem E does generalise Theorem B.

Further observe that if $G$ is an o-group, then $\hat{H} = H$ for every subgroup $H$ (take the (faithful) right regular representation $\theta : G \to A(G)$ and $\alpha$ the identity of $G$). Thus the corollary follows from Theorem E.

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Author's addresses:

Queens' College,
Cambridge CB3 9ET,
England,

and

Department of Pure Mathematics and Mathematical Statistics,
Centre for Mathematical Sciences,
Wilberforce Rd.,
Cambridge CB3 0WB,
England

amwg@dpmms.cam.ac.uk