Sublattice subgroups of finitely presented lattice-ordered groups.

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Abstract

Graham Higman proved that a finitely generated group can be embedded in a finitely presented group iff it has a recursively enumerable set of defining relations. We consider the analogue for lattice-ordered groups. We prove a result that implies:

Theorem. A finitely generated lattice-ordered group that is defined by group words can be \( \ell \)-embedded in a finitely presented lattice-ordered group iff it has a recursively enumerable set of such defining relations.

and

Theorem. A finitely generated totally ordered group can be \( \ell \)-embedded in a finitely presented lattice-ordered group iff it has a recursively enumerable set of defining relations.

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1 Introduction


**Theorem A** The finitely generated subgroups of finitely presented groups are precisely those defined by a recursively enumerable set of relations.

The natural analogue for lattice-ordered groups was listed in [13], Problem 11, [10], 12.12, and in similar form in [2], Chapter 11 Question 12.

As a first step, we proved in [5]

**Theorem B** The finite rank Abelian sublattice subgroups of finitely presented lattice-ordered groups are precisely those defined by a recursively enumerable set of relations.

In both of these theorems, the only difficulty is to show that every finitely generated (lattice-ordered) group (of the kind specified) that is defined by a recursively enumerable set of relations actually occurs as a (sublattice) subgroup of some finitely presented (lattice-ordered) group.

The purpose of this article is to prove the difficult half for the class of finitely generated lattice-ordered groups that are defined by “left strings” (definition below):

**Theorem C** Given a finitely generated lattice-ordered group $G$ that is defined by a recursively enumerable set of left string relations, there is an $\ell$-homomorphic image $G_\flat$ of $G$ that occurs as a sublattice subgroup of a finitely presented lattice-ordered group (and such that the set of left strings that are the identity in $G$ is precisely the same as that in $G_\flat$).

The converse is trivial.

Since every group word is a left string, we deduce:

**Corollary 1.1** Every finitely generated lattice-ordered group that is defined by a recursively enumerable set of group words occurs as a sublattice subgroup of a finitely presented lattice-ordered group.

By Theorem B of [5], in every finite rank Abelian lattice-ordered group with a recursively enumerable set of defining relations, there is an algorithm which converts every word into an equal group word. Thus Corollary 1.1 implies Theorem B above.

Also, for any group word $w$, terms of the form $w \land 1$ and $w \lor 1$ are left strings. Hence we obtain
Corollary 1.2  The finitely generated totally ordered groups that are sublattice subgroups of finitely presented lattice-ordered groups are precisely those defined by a recursively enumerable set of relations.

We will actually prove Corollary 1.1 first (since the coding for group terms is easier) and then extend the ideas to prove Theorem C. The proof relies heavily on the ideas and construction in [4] as modified in [5]. The reader is recommended to consult these where necessary. The diagram on page 130 of the former may be intuitively helpful.

2 Background and notation

Throughout we will use \(\mathbb{N}\) for the set of non-negative integers, \(\mathbb{Z}_+\) for the set of positive integers, and \(\mathbb{R}\) for the set of real numbers. The only order on \(\mathbb{R}\) that we will consider will be the usual one.

We assume that the reader has a minimal knowledge of recursive function theory (see [12]).

In any group \(G\) we write \(f * g\) for \(g^{-1}fg\), and \([f, g]\) for \(f^{-1}g^{-1}fg\). The former is often written \(f^g\), though that would be less readable here where the expressions for \(g\) are complicated.

A lattice-ordered group is a group which is also a lattice that satisfies the identities \(x(y \land z)t = xyt \land xzt\) and \(x(y \lor z)t = xyt \lor xzt\). Throughout we write \(x \leq y\) as a shorthand for \(x \lor y = y\) or \(x \land y = x\), \(\ell\)-group as a shorthand for lattice-ordered group, and \(o\)-group for a totally ordered group (i.e., if the \(\ell\)-group is totally ordered). A sublattice subgroup of an \(\ell\)-group is called an \(\ell\)-subgroup.

Lattice-ordered groups are torsion-free and \(f \lor g = (f^{-1} \land g^{-1})^{-1}\); moreover each element of \(G\) can be written in the form \(fg^{-1}\) where \(f, g \in G^+ = \{ h \in G : h \geq 1 \}\) — see, e.g., [2], Corollary 2.1.3, Lemma 2.3.2 & Lemma 2.1.8. For each \(g \in G\), let \(|g| = g \lor g^{-1}\). Then \(|g| \in G_+\) iff \(g \neq 1\), where \(G_+ = G^+ \setminus \{1\}\). Therefore, \((w_1 = 1 \& \ldots \& w_n = 1)\) iff \(|w_1| \lor \ldots \lor |w_n| = 1\) [ibid, Lemma 2.3.8 & Corollary 2.3.9]. Consequently, in the language of lattice-ordered groups (and in sharp contrast to group theory) any finite number of equalities can be replaced by a single equality.

We will write \(f \perp g\) as a shorthand for \(|f| \land |g| = 1\) and say that \(f\) and \(g\) are orthogonal.

An \(\ell\)-homomorphism from one \(\ell\)-group to another is a group and a lattice homomorphism. Kernels are precisely the normal \(\ell\)-subgroups that are convex.
(if $k_1, k_2$ belong to the kernel and $k_1 \leq g \leq k_2$, then $g$ belongs to the kernel). They are called $\ell$-ideals.

Free $\ell$-groups on finite sets of generators exist by universal algebra. Finitely generated $\ell$-groups are the $\ell$-homomorphic images of free $\ell$-groups on that finite number of generators. If the kernel is finitely generated as an $\ell$-ideal, then we call the $\ell$-homomorphic image finitely presented; if the kernel is generated by a recursively enumerable set of elements (as an $\ell$-ideal), then we say that the finitely generated $\ell$-homomorphic image has a recursively enumerable set of defining relations or is recursively presented (sic). We will write

$$\langle Y : w_i(Y) = 1 \ (i \in I) \rangle$$

for the quotient $F_Y/K$ where $F_Y$ is the free $\ell$-group on the generating set $Y$ and $K$ is the $\ell$-ideal generated by $\{w_i(Y) : i \in I\}$.

The free $\ell$-group on a single generator is $\mathbb{Z} \oplus \mathbb{Z}$ ordered by: $(m_1, m_2) \geq (0, 0)$ iff $m_1, m_2 \geq 0$; $(1, -1)$ is a generator since $(1, -1) \lor (0, 0) = (1, 0)$.

The amalgamation property fails miserably for $\ell$-groups: there are $\ell$-groups $G, H_1, H_2$ with $\ell$-embeddings $\sigma_j : G \to H_j \ (j = 1, 2)$ such that there is no $\ell$-group $L$ such that $H_j$ can be $\ell$-embedded in $L \ (j = 1, 2)$ so that the resulting diagram commutes (see [11] or [2], Theorem 7.C). So HNN-extension tricks cannot be used (see [3]). Instead we use permutation group techniques.

Let $(\Omega, \leq)$ be a totally ordered set. Then $\text{Aut}(\Omega, \leq)$ is an $\ell$-group when the group operation is composition and the lattice operations are just the pointwise supremum and infimum ($\alpha(f \lor g) = \max\{\alpha f, \alpha g\}$, etc.) There is an analogue of Cayley’s Theorem for groups, namely the Cayley-Holland Theorem ([2], Theorem 7.A):

**Theorem D** (Holland [7]) *Every lattice-ordered group can be $\ell$-embedded in $\text{Aut}(\Omega, \leq)$ for some totally ordered set $(\Omega, \leq)$; every countable lattice-ordered group can be be $\ell$-embedded in $\text{Aut}(\mathbb{R}, \leq)$.***

We will write $A(\Omega)$ as a shorthand for $\text{Aut}(\Omega, \leq)$ when the total order on $\Omega$ is clear.

If $h \in A(\Omega)$, then the support of $h$, $\text{supp}(h)$, is the set $\{\beta \in \Omega : \beta h \neq \beta\}$.

Since each real interval $(\alpha, \beta)$ is order-isomorphic to $(\mathbb{R}, \leq)$ we obtain:

**Corollary 2.1** Let $\alpha, \beta \in \mathbb{R}$ with $\alpha < \beta$. Then every countable $\ell$-group $G$ can be $\ell$-embedded in $A(\mathbb{R})$ so that $\text{supp}(g) \subseteq (\alpha, \beta)$ for all $g \in G$. 

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If \( h \in \mathcal{A}(\Omega) \) and \( \alpha \in \text{supp}(h) \), then the convexification of the \( h \)-orbit of \( \alpha \) is called the \textit{interval of support of} \( h \) \textit{containing} \( \alpha \); i.e., the supporting interval of \( h \) containing \( \alpha \) is \( \{ \beta \in \Omega : (\exists m, n \in \mathbb{Z})(\alpha h^n \leq \beta \leq \alpha h^m) \} \). So the support of an element is the disjoint union of its supporting intervals. Supporting intervals are also called \textit{bumps}.

Finally, by considering intervals of support, it is easy to establish the well-known fact that

**Proposition 2.2** For all \( f, g \in \mathcal{A}(\Omega) \), \( \text{supp}(f \cdot g) = \text{supp}(f)g \). Hence if \( f \cdot g \perp f \) and \( g \geq 1 \), then \( |f|^m \leq g \) for all \( m \in \mathbb{N} \).

### 3 Left strings

We need to define the set of expressions to which our theorem applies.

Firstly, if \( F_Y \) is the free \( \ell \)-group on a finite set \( Y \), then the word problem for \( F_Y \) is soluble (see [8] or [2], Theorem 8.E). So if \( S \) is a recursively enumerable set of \( \ell \)-group terms in the finite set of variables \( Y \), then the set of all elements of \( F_Y \) which are equal to elements of \( S \) is also recursively enumerable.

**Definition 3.1 (Left strings)** Let \( n \in \mathbb{Z}_+ \) and \( y_1, \ldots, y_n \) be a finite set of distinct formal symbols. The set of \textit{left strings} on \( Y = \{ y_1, \ldots, y_n \} \) is the smallest set of \( \ell \)-group strings in \( y_1, y_1^{-1}, \ldots, y_n, y_n^{-1} \) that contains the empty string, is closed under equality in \( F_Y \), and is such that \( wy, \ wy \vee 1 \) and \( wy \wedge 1 \) are left strings whenever \( w \) is a left string and \( y \in \{ y_1, y_1^{-1}, \ldots, y_n, y_n^{-1} \} \).

Hence any “semigroup” term in \( \{ y_1, y_1^{-1}, \ldots, y_n, y_n^{-1} \} \) is a left string; so every term in the free group on \( \{ y_1, \ldots, y_n \} \) is a left string. If \( w \) is any left string and \( y \in \{ y_1, y_1^{-1}, \ldots, y_n, y_n^{-1} \} \), then \( w \vee y = (wy^{-1} \vee 1)y \) and \( w \wedge y = (wy^{-1} \wedge 1)y \) and so are left strings. Hence so are \( w \vee 1 = (wy \vee y)y^{-1} \) and \( w \wedge 1 = (wy \wedge y)y^{-1} \). Indeed, if \( w_1, \ldots, w_k \) are group terms, then an easy exercise shows that any term of the form \( w_1 \vee \ldots \vee w_k \) or \( w_1 \wedge \ldots \wedge w_k \) is also a left string. Continuing in this way we obtain that if \( w \) is any left string, then so are \( w \vee u \) and \( w \wedge u \) for any \textit{group term} \( u \). So, for example, if \( w_1, \ldots, w_k, w_{k+1} \ldots, w_{k+r}, w_{k+r+1} \ldots, w_{k+r+s} \) are all group terms, then

\[
(\ldots((w_1 \vee \ldots \vee w_k) \wedge w_{k+1}) \wedge \ldots) \wedge w_{k+r} \vee w_{k+r+1} \vee \ldots) \vee w_{k+r+s}
\]

is a left string (hence the name, left string). In contrast, although I cannot disprove it, I can see no good reason why all \( \ell \)-group terms should be left strings. For example, why should the expression \( (y_1 \wedge y_2) \vee (y_3 \wedge y_4) \) be a left string?
string? — it certainly does not look like one! If all \(\ell\)-group terms were left strings, the theorems and proofs in this article would give the full analogue of Higman’s Theorem to lattice-ordered groups.

Since we will initially only consider group terms, we will need a Gödel numbering for these first. In this special case, let \(B = 2n + 1\) and write all numbers in base \(B\).

**Definition 3.2 (Gödel numbering of group terms)**

The Gödel number of the empty string is \(0\). That is, \(\gamma(1) = 0\).

The Gödel number of \(y_j\) is \(j\) \((\gamma(y_j) = j)\),
and that of \(y_j^{-1}\) is \(n + j\) \((\gamma(y_j^{-1}) = n + j)\) for \(j = 1, \ldots, n\).

If \(w\) has Gödel number \(\gamma(w)\), then

\(\gamma(wy) = B\gamma(w) + \gamma(y)\),
where \(y \in \{y_1, y_1^{-1}, \ldots, y_n, y_n^{-1}\}\).

More generally, let \(B = 6n + 1\) and write all numbers in base \(B\).

**Definition 3.3 (Gödel numbering of left strings)**

The Gödel number of the empty string is \(0\). That is, \(\gamma(1) = 0\).

The Gödel number of \(y_j\) is \(j\) \((\gamma(y_j) = j)\),
and that of \(y_j^{-1}\) is \(n + j\) \((\gamma(y_j^{-1}) = n + j)\) for \(j = 1, \ldots, n\).

If \(w\) has Gödel number \(\gamma(w)\), then

\(\gamma(wy) = B\gamma(w) + \gamma(y)\),
\(\gamma(wy \lor 1) = B\gamma(w) + 2n + \gamma(y)\),
\(\gamma(wy \land 1) = B\gamma(w) + 4n + \gamma(y)\),
where \(y \in \{y_1, y_1^{-1}, \ldots, y_n, y_n^{-1}\}\).

In either the group or general case, the only number in base \(B\) which has a 0 digit and is the Gödel number of a group term or left string is the number 0, the Gödel number of the empty string. This causes technical difficulties in the proof of the main theorems.

To remedy this situation, we define \(\delta(m)\) \((m \in \mathbb{Z}_+)\) as follows:

Let \(m^*\) be the number (in base \(B\)) obtained by deleting all 0s from \(m\). Then \(m^*\) is the Gödel number of a unique left string which we call \(\delta(m)\), and we call \(m\) a pseudo-Gödel number of \(\delta(m)\). Thus each non-trivial group term has an infinite (recursive) set of pseudo-Gödel numbers; \textit{mutatis mutandis}, left strings. We use pseudo-Gödel numbers for technical reasons only; they will simplify the subsequent proofs!

For example, for the left string case, if \(n = 2\) (so \(B = 13\)) and \(m = 1027040\), then \(m^* = 1274\) and \(\delta(1027040)\) is the left string \(w := (y_1y_2y_1^{-1} \lor 1)y_2^{-1}\).
Moreover, any number \( k \in \mathbb{N} \) which has \( k^* = 1274 \) has \( \delta(k) = (y_1y_2y_1^{-1} \lor 1)y_2^{-1} \) and is a pseudo-Gödel number of \( w \).

Let \( X \) be a recursively enumerable subset of \( \mathbb{N} \). Throughout we will assume that \( 0 \in X \) and if \( m \in X \) and \( k \in \mathbb{N} \) with \( m^* = k^* \), then \( k \in X \). That is, either all pseudo-Gödel numbers of a group term (respectively, left string) belong to \( X \) or none do.

It is worth noting that although every countable \( \ell \)-group \( G \) can be \( \ell \)-embedded in a two generator \( \ell \)-group \( H \) (with defining relations that are finite/recursive enumerable if those of \( G \) are) — see [2], Theorem 8.I — there is no guarantee that \( H \) has a set of left string defining relations if \( G \) does. Hence we consider \( n \) generator \( \ell \)-groups, not just 2 generator ones.

4 An infinite pairwise orthogonal set of conjugates

Given an element \( h \) of a countable \( \ell \)-group \( L \), we wish to enlarge \( L \) to \( L_\# \) by adding a finite number of new generators and relations so as to obtain an infinite pairwise orthogonal set of conjugates of \( h \) in \( L_\# \).

Specifically, \textit{inter alia}, we adjoin to \( L \) extra generators \( c_1, h_\# \) and relations

\[
h \perp h_\#, \quad h_\#(h * c_1)^{-1} \perp h * c_1, \quad h_\#(h_\# * c_1)^{-1} \perp h_\# * c_1, \quad h_\# = (h_\# * c_1) \cdot (h * c_1) \quad (1)
\]

Let \( L_\# \) be the resulting \( \ell \)-group.

Intuitively, \( h_\# \) can be thought of as all the conjugates of \( h \) by positive powers of \( c_1 \), with possibly some extra nonsense. We now make this precise. By Theorem D, we may regard \( L_\# \) as an \( \ell \)-subgroup of \( A(\mathbb{N}) \). From this identification, we obtain the intuitive description of \( h_\# \) in terms of \( h \):

**Lemma 4.1** Let \( L_\# \) be as above. If \( m \in \mathbb{Z}_+ \), then \( h * c_1^m \) and \( h_\# * c_1^m \) are sets of bumps of \( h_\# \), but \( h * c_1^{1-m} \perp h_\# \). Indeed, \( h * c_1^m \perp h * c_1^k \) if \( m, k \in \mathbb{Z} \) are distinct.

**Proof:** Let \( \alpha \in \text{supp}(h * c_1) \). Since \( h_\#(h * c_1)^{-1} \perp h * c_1 \), it follows that \( \alpha h_\# = \alpha(h * c_1) \). Thus \( h * c_1 \) is a set of bumps of \( h_\# \). Similarly, \( h_\# * c_1 \) is a set of bumps of \( h_\# \). An easy induction argument shews that \( h_\# * c_1^m \) is a set of bumps of \( h_\# \) for all \( m \in \mathbb{N} \). Therefore \( h * c_1^m \) is a set of bumps of \( h_\# \) for all \( m \in \mathbb{Z}_+ \). Since \( h \perp h_\# \), the last part of the lemma follows. //
Throughout the rest of the paper we reserve $h_#$ for just this construction. Note that $h * c_1$ is the first conjugate of $h$ by a power of $c_1$ whose polar contains the polar of $h_#$, $h * c_1^2$ the second, etc., where the polar of $x$ is the set of all elements orthogonal to $x$.

We record a triviality for later use.

**Lemma 4.2** Let $L$ and $L_#$ be as above, and $t_# \in L_#$. If $[h, t_#] = 1$ and $h_# \perp t_#$, then $[h * c_1^m, t_#] = 1 = [h_# * c_1^m, t_#]$ for all $m \in \mathbb{N}$.

**Proof:** This follows immediately from Lemma 4.1 since $\text{supp}(h * c_1^m) \subseteq \text{supp}(h_#) \cup \text{supp}(h)$ and $\text{supp}(h_# * c_1^m) \subseteq \text{supp}(h_#)$.

More generally, $L_#$ will always be obtained from $L$ by adjoining a finite number of extra generators (including $h_#$ and $c_1$) and relations (including those in the lemmata above). We will want a realisation, in $A(\mathbb{R})$, of the $\ell$-subgroup $L_*$ of $L_#$ generated by $L \cup \{h_#, c_1\}$ to see that certain expressions are not the identity; that is, a specific $\ell$-homomorphism of $L_*$ into $A(\mathbb{R})$.

Consider $L$ as an $\ell$-subgroup of $A(\mathbb{R})$ with each element of $L$ having support contained in $(0,1)$. Let $\hat{h}$ be the restriction of $h \in A(\mathbb{R})$ to $(0,1)$. Let $\hat{c}_1 \in A(\mathbb{R})_+$ be such that $0\hat{c}_1 = 1$, and $\{0\hat{c}_1^m : m \in \mathbb{Z}\}$ is bounded above and below in $\mathbb{R}$. Define $\hat{h}_# \in A(\mathbb{R})$ to have support contained in $\bigcup_{m \in \mathbb{Z}_+} (0\hat{c}_1^m, 1\hat{c}_1^m)$ and let $\hat{h}_#$ restricted to $(0\hat{c}_1^m, 1\hat{c}_1^m)$ be $\hat{h} * \hat{c}_1^m$ ($m \in \mathbb{Z}_+$). Then all the relations of $L_*$ hold with $h_#, c_1$ replaced by $\hat{h}_#, \hat{c}_1$, respectively. Hence the $\ell$-subgroup $\hat{L}_*$ of $A(\mathbb{R})$ generated by $L \cup \{\hat{h}_#, \hat{c}_1\}$ is an $\ell$-homomorphic image of $L_*$. Since $L$ is $\ell$-embedded in $\hat{L}_*$, it is $\ell$-embedded in $L_*$.

## 5 Proof of Corollary 1.1 and Theorem C: foundation

The proofs of the main results depend heavily on the machinery developed in previous papers, especially [4] and [5]. These have both a formal aspect and a visual (permutation) one. To make the present proofs more understandable, we recall the essential points from these articles (but not the details) which, if taken on trust, should make this article self-contained; the interested reader should consult [4], etc., for the proofs.
5.1 The construction in [4]

Let $\theta$ be the zero function on $\mathbb{N}$; i.e., $\theta(n) = 0$ for all $n \in \mathbb{N}$. In [4] we constructed a finitely presented $\ell$-group $G(\theta)$ which coded in the function $\theta$. Specifically, inter alia, we included in $G(\theta)$ generators $a_0, b_1, c_1, d_1$ and $a_\theta$ and a finite number of relations that included $a_\theta = 1$. [These should be thought of merely as formal symbols satisfying certain relations; the presence of $d_1$ will only appear explicitly in the last subsection.] Hence

$$a_0 * c_1^m a_\theta = a_0 * b_1^{\theta(m)} c_1^m$$
for all $m \in \mathbb{N}$.

We next formally constructed a finitely presented $\ell$-group $G(s)$ which coded in the successor function $s$ (i.e., $s(m) = m + 1$ for all $m \in \mathbb{N}$). Specifically, we included in $G(s)$ besides the generators of $G(\theta)$ and one extra one, a generator $a_s$ and a finite number of relations that implied that

$$a_0 * c_1^m a_s = a_0 * b_1^{s(m)} c_1^m$$
for all $m \in \mathbb{N}$.

Now suppose that $f_1$ and $f_2$ were recursive functions on $\mathbb{N}$ such that there are finitely presented $\ell$-groups $G(f_j)$ whose generators include the generators of $G(s)$ (which we think of as the “base” generators) as well as $a_{f_j}$, and a finite number of relations that imply

$$a_0 * c_1^m a_{f_j} = a_0 * b_1^{f_j(m)} c_1^m$$
for all $m \in \mathbb{N}$ ($j = 1, 2$). Let $f = f_1 \circ f_2$. We used the generators and relations of $G(f_j)$ ($j = 1, 2$) to formally construct a new finitely presented $\ell$-group $G(f)$ in which

$$a_0 * c_1^m a_f = a_0 * b_1^{f(m)} c_1^m$$
for all $m \in \mathbb{N}$.

Finally, we showed how to obtain a finitely presented $\ell$-group $G(f)$ in which

$$a_0 * c_1^m a_f = a_0 * b_1^{f(m)} c_1^m$$
for all $m \in \mathbb{N}$, whenever $f$ has been obtained by general recursion from functions $g, h, \ldots$ for which finitely presented $\ell$-groups $G(g), G(h), \ldots$, have been constructed with these “$\ell$-group codings” for $g, h, \ldots$.

Hence, by induction on the way that a recursive function is constructed, we obtained:

**Proposition 5.1** For each recursive function $f : \mathbb{N} \rightarrow \mathbb{N}$, there is a finitely presented $\ell$-group $G(f)$ (with generators including $a_f$ and the “base” ones) in which

$$a_0 * c_1^m a_f = a_0 * b_1^{f(m)} c_1^m$$
for all $m \in \mathbb{N}$. 

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Now let $X$ be an arbitrary recursively enumerable subset of $\mathbb{N}$. Then there is a recursive function $f : \mathbb{N} \to \mathbb{N}$ such that $f(\mathbb{N}) = X$. In [4], we formally augmented $G(f)$ by the addition of finitely many extra generators (we labelled the critical one $a_h$ where $h$ is the (not necessarily recursive) characteristic function of $\mathbb{N} \setminus X$; but, for clarity, here we use the label $t_X$ for it) and a finite number of extra defining relations to obtain a finitely presented $\ell$-group $G(X)$ in which

$$a_0 \ast c_1^m t_X = a_0 \ast c_1^m \text{ if } m \in X.$$  

Additionally, in [4] (see the pictorial representation on page 130) we constructed order-preserving permutations of $\mathbb{R}$ which satisfied, under a natural interpretation, all the defining relations of $G(X)$ hinted at above (as well as goodness knows what others). If we denote by $\hat{G}(X)$ the $\ell$-subgroup of $A(\mathbb{R})$ generated by these permutations, then $G(X)$ is an $\ell$-homomorphic image of $\hat{G}(X)$. Crucially, under the natural interpretation of the formal symbols in $\hat{G}(X) \subseteq A(\mathbb{R})$, $\hat{t}_X$ restricted to $\text{supp}(\hat{a}_0 \ast \hat{c}_1^m)$ was the identity iff $m \in X$. We then noted that, under this interpretation in $\hat{G}(X) \subseteq A(\mathbb{R})$, we have

$$\hat{a}_0 \ast \hat{c}_1^m \hat{t}_X \neq \hat{a}_0 \ast \hat{c}_1^m \text{ if } m \notin X.$$  

Since $\hat{G}(X)$ is an $\ell$-homomorphic image of $G(X)$ it follows that

$$a_0 \ast c_1^m t_X \neq a_0 \ast c_1^m \text{ if } m \notin X$$  

also holds in $G(X)$. Hence

**Proposition 5.2** Let $X$ be a recursively enumerable subset of $\mathbb{N}$. Then there is a finitely presented $\ell$-group $G(X)$ (whose generators include the “base” ones and $t_X$) in which

$$a_0 \ast c_1^m t_X = a_0 \ast c_1^m \text{ iff } m \in X.$$  

**5.2 The first “tweak”**

The following proposition was implicit in [5], Section 5.

**Proposition 5.3** If $C$ is any finitely presented $\ell$-group and $X$ is an arbitrary recursively enumerable subset of $\mathbb{N}$, then $C$ can be $\ell$-embedded in a finitely presented $\ell$-group $C(X)$ in which for each generator $g$ of $C$,

$$g \ast c_1^m t_X = g \ast c_1^m \text{ iff } m \in X.$$  

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We now outline the proof as the technique (as well as the proposition) is important for understanding the proof of the theorems of this article.

If \( C \) is any countable \( \ell \)-group, then one can use Corollary 2.1 to show that \( C \) is \( \ell \)-isomorphic to an \( \ell \)-subgroup \( \hat{C} \) of \( A(\mathbb{R}) \) so that each element \( g \) of \( C \) satisfies \( |\hat{g}| \leq \hat{a}_0 \) and

\[
\hat{g} \ast \hat{c}_1^m t_X = \hat{g} \ast \hat{c}_1^m \quad \text{iff} \quad m \in X
\]

(see below). Indeed, if \( \alpha \in supp(\hat{a}_0) \), we can further ensure that for each \( g \in C \), the restriction of \( \hat{g} \) to that interval of support of \( \hat{a}_0 \) containing \( \alpha \) satisfies \( supp(\hat{g}) \subseteq (\alpha, \alpha \hat{a}_0) \).

Now assume that \( C \) is a finitely presented \( \ell \)-group. Formally, exactly as in [4] and as outlined in the previous subsection, for each recursive function \( f : \mathbb{N} \to \mathbb{N} \), we can inductively (on the way that \( f \) is formed) construct an \( \ell \)-group \( C(f) \) with generators the union of those of \( G(f) \) and those of \( C \) and relations which include the union of those of \( G(f) \) and those of \( C \) together with

\[
g \ast a_0 \perp g \quad \text{for each generator } g \text{ of } C, \quad \text{(i)}
\]

and imply, for each generator \( g \) of \( C \) and \( m \in \mathbb{N} \), that

\[
g \ast c_1^m a_f = g \ast b_1^{f(m)} c_1^m. \quad \text{(ii)}
\]

This is achieved as follows:

for each of the defining relations with \( a_0 \) and \( a_f \), etc. that yielded \( a_0 \ast c_1^m a_f = a_0 \ast b_1^{f(m)} c_1^m \) for all \( m \in \mathbb{N} \), additionally adjoin the finite set of identities obtained by substituting each generator \( g \) of \( C \) in turn for \( a_0 \). This ensures that (ii) holds in \( C(f) \).

Note that \( C(f) \) is a finitely presented \( \ell \)-group.

In the special case that \( X \) is a recursively enumerable subset of \( \mathbb{N} \) and \( f \) is chosen so that \( X = f(\mathbb{N}) \), we can then formally construct \( C(X) \) from \( C(f) \) just as we constructed \( G(X) \) from \( G(f) \), and deduce that for each generator \( g \) of \( C \), in \( C(X) \) we get

\[
g \ast c_1^m t_X = g \ast c_1^m \quad \text{if } m \in X.
\]

Now (i) ensures that \( |g| \leq a_0 \) for all \( g \in C \) and the natural interpretation of \( t_X \) in \( A(\mathbb{R}) \) is not the identity on \( supp(\hat{a}_0 \ast \hat{c}_1^m) \) if \( m \notin X \). Indeed, as asserted above, we can ensure that for each generator \( g \) of \( C \), it is not the identity on \( supp(\hat{g} \ast \hat{c}_1^m) \) if \( m \notin X \). Hence, in \( C(X) \), for each such \( g \) we have

\[
\hat{g} \ast \hat{c}_1^m t_X \neq \hat{g} \ast \hat{c}_1^m \quad \text{if } m \notin X.
\]
As before, this gives for each generator \( g \) of \( C \) and \( m \in \mathbb{N} \), we have in \( C(X) \)
\[
g * c_1^m t_X = g * c_1^m \quad \text{iff} \quad m \in X.
\]
Since \( C \) was \( \ell \)-embedded in \( \hat{G}(X) \), an \( \ell \)-homomorphic image of \( G(X) \), the naturally induced \( \ell \)-homomorphism of \( C \) into \( G(X) \) must be an \( \ell \)-embedding. Thus we obtain Proposition 5.3.

5.3 A special case

We will want to use the above construction to prove Corollary 1.1 (and Theorem C).

Let \( F \) be the free \( \ell \)-group on \( y_1, \ldots, y_n \), and \( X \) be any recursively enumerable set of natural numbers written in base \( B \). For our purposes, as noted in Section 3, we may assume that
\[
0 \in X \quad \text{and if} \quad m \in X \quad \text{and} \quad k \in \mathbb{N} \quad \text{with} \quad m^* = k^*, \quad \text{then} \quad k \in X.
\]
Let
\[
H = \langle y_1, \ldots, y_n : w(y_1, \ldots, y_n) = 1 \ (\gamma(w) \in X) \rangle
\]
be a recursively presented \( \ell \)-group with all \( w(y_1, \ldots, y_n) \) being group terms in the first instance and left strings in the second. So \( H \) is the quotient of \( F \) by the \( \ell \)-ideal \( K \) generated by \( \{ w(y_1, \ldots, y_n) : \gamma(w) \in X \} \). Indeed, by enlarging \( X \) if necessary and using [8], we may assume that if \( w = w(y_1, \ldots, y_n) \) is a group term (respectively, a left string), then
\[
w \in K \quad \text{iff} \quad \gamma(w) \in X.
\]
In the absence of the HNN property, we cannot, a priori, keep sufficient control if we simply form \( H(X) \). So we take a more circuitous approach.

Let \( F^* \) be the free \( \ell \)-group on \( h, y_1, \ldots, y_n \), thus \( F^* \) is the \( \ell \)-group free product of \( F \) and the free \( \ell \)-group on one generator \( (h) \) — existence by universal algebra. By Proposition 5.3, since \( F^* \) is a finitely presented \( \ell \)-group, we can construct \( F^*(X) \) as in the previous subsection (using \( F^* \) in place of \( C \) and the recursively enumerable set \( X \) giving rise to the recursively presented \( \ell \)-group \( H \)) so that, for each \( z \in \{ h, y_1, \ldots, y_n \} \),
\[
z * c_1^m t_X = z * c_1^m \quad \text{iff} \quad m \in X.
\] (2)
Note that \( F^*(X) \) is a finitely presented \( \ell \)-group, and we regard \( F^* \) as an \( \ell \)-subgroup of \( F^*(X) \) in the natural way. We fix this notation for the remainder of the article.

Let \( H^* \) be the \( \ell \)-group free product of \( H \) and the free \( \ell \)-group on the generator \( h \) — existence by universal algebra. Then \( H^* \) is an \( \ell \)-homomorphic image of \( F^* \).
5.4 The second “tweak”

We now formally form an $\ell$-homomorphic image of $F^*(X)$ by adjoining the extra recursively enumerable set of group term (respectively, left string) relations $w(y_1, \ldots, y_n) = 1$ ($\gamma(w) \in X$) to the defining relations of $F^*(X)$. We will denote this recursively presented $\ell$-group by $H^*[X]$ and let $H[X]$ be the $\ell$-subgroup of $H^*[X]$ generated by the images of $y_1, \ldots, y_n$.

Since all the defining relations of $H$ have been imposed on $H^*[X]$, it follows that the $\ell$-surjection from $F^*$ onto $H^*$ extends naturally to an $\ell$-surjection from $F^*(X)$ onto $H^*[X]$.

Caution: In the absence of the HNN property, although $F^*$ is an $\ell$-subgroup of $F^*(X)$, I see no a priori reason why either the $\ell$-homomorphism from $H$ to $H^*[X]$ or its extension from $H^*$ to $H^*[X]$ should be $\ell$-embeddings. This seems a critical difficulty to me; although both are true and follow as a consequence of our proof of Corollary 1.1 and Theorem C, they cannot be assumed at this stage of the argument. It may have seemed more natural to write $H(X)$ instead of $H^*[X]$. However, this might have caused the reader to assume during the proof that the $\ell$-homomorphism from $H$ was injective, so I have instead used the notation $H^*[X]$ and $H^*[X]$ for clarity.

5.5 The third “tweak”

In order to obtain the $\ell$-embedding and the main results, we will need to adjoin further elements $t := t\#, h\#$ as well as a finite number of extra generators and relations to obtain $H^*[X]\#$. These extra relations may cause some “collapsing”; hence we will obtain an $\ell$-homomorphism of $H^*[X]$ into $H^*[X]\#$ that is not necessarily injective. The relations will include the four equations of (1), $[h, t] = 1$ and $h\# \perp t$. By Lemma 4.2, we will have in $H^*[X]\#$

$$[h \circ c^m_1 t, t] = 1 \quad \text{for all } m \in \mathbb{N}. \quad (3)$$

From these and extra data we will deduce in the next three sections (for all group terms, respectively left strings, $w \in F$ and $m \in \mathbb{N}$ with $\delta(m) = w$)

$$wh \circ c^m t = h \circ c^m t X. \quad (4)$$

In Subsection 5.2 we outlined an interpretation $\hat{F}^*(X)$ (an $\ell$-homomorphic image) of $F^*(X)$ in $A(\mathbb{R})$. As in [11], we can $\ell$-embed $A(\mathbb{R})$ in $A(\Omega)$ where $(\Omega, \leq)$ is a dense totally ordered set without endpoints that is highly saturated in the model-theoretic sense so that there is $\hat{h} \in A(\Omega)$ with $supp(\hat{h}) \subseteq (\alpha_0, \alpha_0\alpha_0)$ (with “many” supporting intervals which move points up, likewise down, “many” being disjoint from $\bigcup_{j=1}^n supp(y_j)$ — see Sections 7 and 8 for
added details) so that \( \hat{w} \hat{h} \) and \( \hat{h} \) are conjugate in \( A(\Omega) \) for all \( w \in H \). We extend the \( \ell \)-embedding of \( H \) into \( A(\Omega) \) to an \( \ell \)-homomorphism of \( H^* \) into \( A(\Omega) \) by mapping \( h \) to \( \hat{h} \). This gives an interpretation of \( H^*[X] \) in \( A(\Omega) \).

[Since \( H^*[X] \) is countable, it is possible to provide such an interpretation in \( A(\mathbb{R}) \), though I have been unable to keep the above tight control which I will need later in the proof.]

As in the previous subsection (or see [5]), we have an \( \ell \)-homomorphic image of \( H^*[X] \) in \( A(\Omega) \) in which 

\[
\hat{h} \cdot \hat{c}_1^m \hat{t}_X \neq \hat{h} \cdot \hat{c}_1^m \quad \text{if} \ m \notin X
\]

(indeed, the left and right hand sides are incomparable in the order on \( A(\Omega) \) if \( m \notin X \)).

As indicated in Section 4, we let \( \hat{h}_\# \) be the “union” of the \( \hat{h} \cdot \hat{c}_1^m \hat{t}_X \) \( (m \in \mathbb{Z}_+; \ k \in \mathbb{Z}) \); i.e., \( \hat{h}_\# \) is \( \hat{h} \cdot \hat{c}_1^m \hat{t}_X \) on \( \text{supp}(\hat{a}_0 \cdot \hat{c}_1^m \hat{d}_k^m) \) \( (m \in \mathbb{Z}_+; \ k \in \mathbb{Z}) \) and the identity off \( \bigcup \{ \text{supp}(\hat{a}_0 \cdot \hat{c}_1^m \hat{d}_k^m) : m \in \mathbb{Z}_+ ; k \in \mathbb{Z} \} \). We will define \( \hat{t} \in A(\Omega) \) to be the identity on \( \text{supp}(\hat{h}) \cup \text{supp}(\hat{h}_\#) \) and extend this appropriately later so as to satisfy (10) and (11) in the group term case and (15) and (16) in the left string case (see Sections 7 and 8).

Although we may have further “collapsing” when we interpret \( H^*[X] \# \) in \( A(\Omega) \), we will still be able to ensure that

\[
\hat{h} \cdot \hat{c}_1^m \hat{t}_X \neq \hat{h} \cdot \hat{c}_1^m \quad \text{if} \ m \notin X.
\]

Hence, if \( m \in \mathbb{N} \) and \( w = \delta(m) \), we obtain

\[
w \hat{h} \cdot \hat{c}_1^m \hat{t} = \hat{h} \cdot \hat{c}_1^m \text{ in } H^*[X]_\# \quad \text{iff} \ m \in X.
\]

Therefore, by (3), for any group term (respectively left string) \( w \), we have

\[
w(y_1, \ldots, y_n) = 1 \text{ in } H^*[X]_\# \quad \text{iff} \ \gamma(w) \in X.
\]

Consequently, the relations

\[
w(y_1, \ldots, y_n) = 1 \quad \text{if} \ \gamma(w) \in X
\]

are unnecessary and can be removed from the list of defining relations of \( H^*[X]_\# \). Let \( L[X] \) be the resulting finitely presented \( \ell \)-group. Thus \( L[X] \) is \( \ell \)-isomorphic to \( H^*[X]_\# \) under the natural map and we will have an \( \ell \)-homomorphism of \( H \) into the finitely presented \( \ell \)-group \( L[X] \). Moreover, \( w = 1 \) in \( L[X] \) iff \( \gamma(w) \in X \), so the \( \ell \)-subgroup of \( L[X] \) generated by the images
of \{y_1, \ldots, y_n\} satisfies exactly the same left string relations as \(H\) by the
definition of \(H\). Thus the restriction of this \(\ell\)-homomorphism to \(H\) is “injective
on left strings”. This suffices for Theorem C (with \(G^{\flat} = L[X]\)) and the
corollaries. //

I suspect that this implies that the restriction of the \(\ell\)-homomorphism to
\(H\) is actually injective (whence \(G \cong G^{\flat}\) in Theorem C) but I have been unable
to deduce it.

6 Proof of Theorem: further construction

In order to obtain (4) from a finite set of defining relations, we need further
generators for \(H^*\) [\(X\)]_{\#}; these are patterned on an idea of Valiev (see [9], p.226).
In analogy with Valiev’s proof of Higman’s Theorem for groups, consider
4\(n\) new elements \(t_\lambda, v_\lambda\) for \(\lambda \in \{y_1, y_1^{-1}, \ldots, y_n, y_n^{-1}\}\), each commuting with
d1 and satisfying:

\[
\begin{align*}
  h * t_\lambda &= \lambda h, & y_j * t_\lambda &= y_j, & c_1 * t_\lambda &= c_1^B, \\
  h * v_\lambda &= h * c_1^{\gamma(\lambda)}, & y_j * v_\lambda &= y_j * c_1^{\gamma(\lambda)}, & c_1 * v_\lambda &= c_1^B,
\end{align*}
\]

(6)

\[
( j = 1, \ldots, n ).
\]

We also add one extra element \(u\) commuting with \(d_1\) and satisfying

\[
\begin{align*}
  z * u &= z & \text{and} & c_1 * u &= c_1^B,
\end{align*}
\]

where \(z \in \{h, y_1, \ldots, y_n\}\).

Aside: Let \(\lambda \in \{y_1, y_1^{-1}, \ldots, y_n, y_n^{-1}\}\). Since \(H^*\) is the \(\ell\)-group free product
of \(H\) and the free \(\ell\)-group on one generator (\(h\)), it is also the \(\ell\)-group free
product of \(H\) and the free \(\ell\)-group on one generator (\(\lambda h\)). Each \(t_\lambda\) will actually
be an \(\ell\)-automorphism of \(H^*\), though we will not need this fact.

We now show how to interpret these in \(A(\Omega)\) (see [2] Section 8.3):

Demonstration for \(t_\lambda\): We consider the \(\ell\)-homomorphic image \(\hat{H}^*[X]_{\#}\) of
\(H^*[X]_{\#}\) in \(A(\Omega)\) in the previous section, and assume that \(t_\lambda\) has only been
defined on the convexification of the support of \(\hat{H}^*\); i.e., \(\tilde{t}_\lambda\) is defined on
an interval \((\alpha_0, \alpha_0 \hat{a}_0)\) so that \(h * \tilde{t}_\lambda = \lambda h\) and \([y_j, \tilde{t}_\lambda] = 1\) \((j = 1, \ldots, n)\).
Since \(\alpha_0 \hat{a}_0 < \alpha_0 \hat{c}_1\), there is an order-preserving bijection \(t_\lambda^*\) between \([\alpha_0, \alpha_0 \hat{c}_1]\)
and \([\alpha_0, \alpha_0 \hat{c}_1^B]\) extending \(\tilde{t}_\lambda\). We extend this to the interval of support of \(\hat{c}_1\)
containing \(\alpha_0\) as follows. Let \(t_{\lambda,m} : [\alpha_0 \hat{c}_1^m, \alpha_0 \hat{c}_1^{m+1}] \rightarrow [\alpha_0 \hat{c}_1^{Bm}, \alpha_0 \hat{c}_1^{B(m+1)}]\) be
given by \(t_{\lambda,m} = \hat{c}_1^{-m} t_\lambda \hat{c}_1^{Bm}\) \((m \in \mathbb{Z})\). Then \(t_\lambda^* = \bigcup \{t_{\lambda,m} : m \in \mathbb{Z}\}\) is the
desired extension. For all other supporting intervals of \( \hat{c}_1 \), we can similarly construct such an order-preserving bijection of the supporting interval of \( \hat{c}_1 \) (containing, say, \( \beta \)) to the same supporting interval of \( \hat{c}_1^B \) by starting with any order-preserving bijection from \( [\beta, \beta \hat{c}_1] \) to \( [\beta, \beta \hat{c}_1^B] \) such that the resulting permutation commutes with \( \hat{d}_1 \). Finally, we extend this map to \( \hat{t}_\lambda \in A(\Omega) \); it is the identity off the intervals of support of \( \hat{c}_1 \ast \hat{d}_m^m \) (\( m \in \mathbb{Z} \)). By construction, \( \hat{t}_\lambda \) conjugates \( \hat{h} \) to \( \lambda \hat{h} \), \( \hat{c}_1 \) to \( \hat{c}_1^B \) and commutes with \( y_1, \ldots, y_n \) and \( \hat{d}_1 \). //

The reader is encouraged to adapt this demonstration to \( v_\lambda \) and \( u \).

7 The group term case

Besides the relations

\[
[h, t] = 1 \quad \text{and} \quad h_\# \bot t, \quad (9)
\]

we adjoin \( 2n + 1 \) further relations to complete the construction of \( H^*[X]_\# \):

\[
h_\# \bot t_X t^{-1} t_\lambda c_1^{(\lambda)} t \cdot t_X^{-1} v_\lambda^{-1}, \quad (10)
\]

where \( \lambda \in \{ y_1, y_1^{-1}, \ldots, y_n, y_n^{-1} \} \); and

\[
h_\# \bot [t^{-1} u^{-1} t, t_X^{-1}]. \quad (11)
\]

We define \( \hat{t} \) and \( \hat{t}_X \) so that these relations hold in our interpretation in \( A(\Omega) \). If necessary, we make minor perturbations of these on the intervals of support \( \text{supp}(\hat{h} \ast \hat{c}_1^m \hat{d}_1^k) \) to ensure that \( \hat{h} \ast \hat{c}_1^m \hat{t}_X \neq \hat{h} \ast \hat{c}_1^m \) remains true if \( m \notin X \). This is always possible as the “positive” and “negative” bumps can be made into 1-sets of size \( \aleph_1 \) so that each of them and the fixed point sets together form 1-sets of degree 3 — for the technicalities, see [1], p.199 ff. This was the first purpose of “many” in Subsection 5.5. [Indeed, \( \hat{w} \hat{h} \) and \( \hat{h} \) are conjugate in \( A(\Omega) \).] We can further ensure that if \( X_+ \cup X_- \cup X_0 = \mathbb{N} \) with \( X_0 \) disjoint from \( X_+ \cup X_- \supseteq X \) and \( X_+ \cap X_- \subseteq X \), then

\[
\hat{h} \ast \hat{c}_1^m \hat{t}_X \geq \hat{h} \ast \hat{c}_1^m \quad \text{if} \ m \in X_+, \quad (12)
\]

\[
\hat{h} \ast \hat{c}_1^m \hat{t}_X \leq \hat{h} \ast \hat{c}_1^m \quad \text{if} \ m \in X_-, \quad (13)
\]

\[
\hat{h} \ast \hat{c}_1^m \hat{t}_X \quad \text{and} \quad \hat{h} \ast \hat{c}_1^m \quad \text{are incomparable if} \ m \in X_0. \quad (14)
\]
This will only be needed in the left string case and was the second purpose of “many” in Subsection 5.5.

We first show by induction on \( k \in \mathbb{N} \), that if \( w(y_1, \ldots, y_n) \) is a group term of length at most \( k \), then

\[ wh * c_1^{\gamma(w)} \lambda = h * c_1^{\gamma(w)} \lambda \ast t_X. \]

**Proof:** The result is obvious if \( w(y_1, \ldots, y_n) \) has length 0 (it is the identity) using (2) and (9), since \( \gamma(1) = 0 \in X \).

Assume that the result holds for group terms of length at most \( k \in \mathbb{N} \) and consider group terms \( w(y_1, \ldots, y_n) \) of length \( k + 1 \). So \( w(y_1, \ldots, y_n) = w_1(y_1, \ldots, y_n) \lambda \) for some group term \( w_1(y_1, \ldots, y_n) \) of length \( k \) and \( \lambda \in \{y_1, y_1^{-1}, \ldots, y_n, y_n^{-1}\} \).

By our induction hypothesis,

\[ w_1 h * c_1^{\gamma(w_1)} i = h * c_1^{\gamma(w_1)} t_X. \]

So

\[ w_1 h * c_1^{\gamma(w_1)} = h * c_1^{\gamma(w_1)} t_X t^{-1}. \]

By (9), Lemma 4.2 and (10) we get

\[ h * (c_1^{\gamma(w_1)} t_X t^{-1} t_\lambda c_1^{\gamma(\lambda)} t \cdot t_X^{-1} \lambda^{-1}) = h * c_1^{\gamma(w_1)}. \]

Hence

\[ h * c_1^{\gamma(w_1)} t_X t^{-1} t_\lambda c_1^{\gamma(\lambda)} t = h * c_1^{\gamma(w_1)} t_\lambda t_X. \]

Thus (by our inductive hypothesis)

\[ w_1 h * c_1^{\gamma(w_1)} t_\lambda c_1^{\gamma(\lambda)} t = h * c_1^{\gamma(w_1)} t_\lambda t_X. \]

By (6) and (7), we obtain

\[ w_1 \lambda h * c_1^{B\gamma(w_1) + \gamma(\lambda)} t = h * c_1^{\gamma(\lambda) + B\gamma(w_1)} t_X. \]

That is,

\[ wh * c_1^{\gamma(w)} \lambda = h * c_1^{\gamma(w)} t_X, \]

as desired. //

We can now conclude the proof of (4) that for all \( m \in \mathbb{N} \),

\[ \delta(m) h * c_1^{\gamma(w)} \lambda = h * c_1^{\gamma(w)} t_X; \]

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by induction on the number of digits of \( m \) in base \( B \).

**Proof:** This is true if \( m \) has just one digit in base \( B \) by what we have just established. Now if \( m \in \mathbb{Z}_+ \) and

\[
\delta(m)h \ast c_1^m t = h \ast c_1^m t_X,
\]

then we must consider the numbers \( Bm + \gamma(\lambda) \) and \( Bm \). As

\[
h \ast c_1^m t_X t^{-1} t_\lambda c_1^m = h \ast c_1^m v_\lambda t_X
\]

by Lemma 4.2 and (10), we deduce that

\[
wh \ast c_1^Bm + \gamma(\lambda) = h \ast c_1^Bm + \gamma(\lambda) t_X,
\]

where \( w = \delta(m) \) — so \( w\lambda = \delta(Bm + \gamma(\lambda)) \).

Finally,

\[
h \ast c_1^m t_X t^{-1} ut = h \ast (c_1^m (t^{-1} ut) t_X)
\]

by (11). Therefore, \( \delta(Bm) = w \) and, by (8),

\[
wh \ast c_1^Bm = wh \ast (c_1^m t \cdot t^{-1} ut) = h \ast c_1^m t_X t^{-1} ut = h \ast (c_1^m (t^{-1} ut) t_X) = h \ast c_1^Bm t_X.
\]

This completes the deduction of (4) from the finite number of remaining defining relations of \( H^*[X]_\# \).

---

Let \( L[X] \) be the finitely presented \( \ell \)-group with the same finite generating set as \( H^*[X]_\# \) and defining relations those given for \( H^*[X]_\# \) (with the set \( \{w(y_1, \ldots, y_n) = 1 : \gamma(w) \in X \} \) removed). So \( L[X] \) and \( H^*[X]_\# \) are \( \ell \)-isomorphic under the natural map.

As already noted in the deduction at the end of Section 5, the satisfaction of (4) in \( L[X] \) suffices to ensure that \( w(y_1, \ldots, y_n) = 1 \) in \( L[X] \) iff \( \gamma(w) \in X \), whence the \( \ell \)-homomorphism of \( H \) into \( L[X] \) is an \( \ell \)-embedding.

This completes the proof of Corollary 1.1.

---

**8 The left string case**

We now consider the general left string case. So write all numbers in base \( B = 6n + 1 \) and assume that \( X \) is a recursively enumerable subset of \( \mathbb{N} \) such that \( 0 \in X \) and if \( m \in X \) and \( k \in \mathbb{N} \) with \( m^* = k^* \), then \( k \in X \). Since \( w = 1 \) implies \( w \lor 1 = 1 = w \land 1 \), we will also assume that if \( \delta(m) = wy \) and \( m \in X \), then \( m + 2n \in X \) and \( m + 4n \in X \) (\( y \in \{y_1, y_1^{-1}, \ldots, y_n, y_n^{-1}\} \)).
We let $L[X]$ be as above, but initially we add $2n$ extra generators $h_{\lambda,\#}$ and $10n$ extra relations

$$h_{\lambda,\#} \perp h \ast c_1^{\gamma(\lambda)}, \quad (12)$$

$$h_{\lambda,\#}(h \ast c_1^{\gamma(\lambda)+B})^{-1} \perp h \ast c_1^{\gamma(\lambda)+B}, \quad (13)$$

$$h_{\lambda,\#}(h_{\lambda,\#} \ast c_1^B)^{-1} \perp h_{\lambda,\#} \ast c_1^B, \quad (14)$$

$$h_{\lambda,\#} \perp t_X t^{-1} c_1^{2n} t \cdot t_X^{-1} c_1^{-2n}, \quad (15)$$

$$h_{\lambda,\#} \perp t_X t^{-1} c_1^{4n} t \cdot t_X^{-1} c_1^{-4n}, \quad (16)$$

where $\lambda \in \{y_1, y_1^{-1}, \ldots, y_n, y_n^{-1}\}$.

Identities (12), (13), and (14) allow us to use Lemma 4.2, but with $c_1^B$ in place of $c_1$, $h \ast c_1^{\gamma}$ in place of $h$, and $h_{\lambda,\#}$ in place of $h_{\#}$.

We complete the definition of $L[X]$ by adding $4n$ relations

$$h_{\#} \ast u c_1^k t_X \geq h_{\#} \ast u c_1^k, \quad (17)$$

$$h_{\#} \ast u c_1^{2n+k} t_X \leq h_{\#} \ast u c_1^{2n+k}, \quad (18)$$

$k = 2n + 1, \ldots, 4n$.

The purpose of (17) is to ensure that

$$h \ast c_1^m t_X \geq h \ast c_1^m \quad \text{if the unit digit of } m \text{ is } 2n + 1, \ldots, 4n. \quad (19)$$

Proof: By Theorem D, we can $\ell$-embed $L[X]$ in $A(\mathbb{R})$. Let $m \in \mathbb{N}$ have $k \in \{2n + 1, \ldots, 4n\}$ as unit digit (base $B$) and $\alpha \in \text{ supp}(h \ast c_1^m)$. Thus, by Lemma 4.1 and (17),

$$\alpha(h \ast c_1^m) = \alpha h_{\#} \ast u c_1^k \leq \alpha h_{\#} \ast u c_1^k t_X = \alpha(h \ast c_1^m) t_X.$$ 

Consequently, $h \ast c_1^m t_X \geq h \ast c_1^m$ if $m$ is of the described form (in base $B$). \hfill \Box \hfill \Box

Similarly, using (18), we obtain

$$h \ast c_1^m t_X \leq h \ast c_1^m \quad \text{if the unit digit of } m \text{ is } 4n + 1, \ldots, 6n. \quad (20)$$
The desired interpretation holds in $A(\Omega)$ with $h_\lambda,\#$ the “union” of

$$(\hat{h} * \hat{c}_1^{(\lambda)} * \hat{c}_1^{Bm} \hat{d}_1^k \ (m \in \mathbb{Z}_+; \ k \in \mathbb{Z}) \text{ and } \hat{i}_X, \ \hat{i} \text{ defined similarly to before with } X_+ \text{ the union of } X \text{ and the set of numbers in base } B \text{ with unit digit in } \{2n+1, \ldots, 4n\}, \text{ with } X_- \text{ the union of } X \text{ and the set of numbers in base } B \text{ with unit digit in } \{4n+1, \ldots, 6n\}, \text{ and } X_0 \text{ the complement of } X_+ \cup X_- \text{ in } \mathbb{N}.$$  

We must now verify that (4) holds in $L[X]$ for all left strings $w$; that is, for all left strings $w$ and $m \in \mathbb{N}$ with $\delta(m) = w$,

$$wh * c_m^t = h * c_1^m t_X.$$  

As for group terms, we begin by showing that for all left strings $w$, we have

$$wh * c_1^{\gamma(w)} t = h * c_1^{\gamma(w)} t_X.$$  

Again we assume the result for left string $w$ and shew it follows for $wy$, $wy \lor 1$ and $wy \land 1$, where $y \in \{y_1, y_1^{-1}, \ldots, y_n, y_n^{-1}\}$.

**Proof:** The proof for $wy$ is as before.

Now $\gamma(wy \lor 1) = B\gamma(w) + 2n + \gamma(y)$ and

$$wyh * c_1^{\gamma(wy)} t = h * c_1^{\gamma(wy)} t_X$$

by the first case (the conclusion for $wy$ from that for $w$).

We therefore obtain by (15) that

$$wyh * c_1^{\gamma(wy \lor 1)} t = h * c_1^{B\gamma(w) + \gamma(y)} t_X t^{-1} c_1^{2n} t = h * c_1^{B\gamma(w) + \gamma(y)} c_1^{2n} t_X = h * c_1^{\gamma(wy \lor 1)} t_X.$$  

Thus, by (19),

$$(wy \lor 1)h * c_1^{\gamma(wy \lor 1)} t = h * c_1^{\gamma(wy \lor 1)} t_X \lor h * c_1^{\gamma(wy \lor 1)} = h * c_1^{\gamma(wy \lor 1)} t_X.$$  

The proof for $wy \land 1$ is similar using (16) and (20).  

The deduction of (4) follows from this just as in the group term case.

The proof of Theorem C (and consequently of all the results claimed) follows exactly as the deduction of Corollary 1.1. from the corresponding result in the group term case.
9 Concluding Remarks

(I) The only obstacle to proving the analogue of Higman’s Theorem now seems to be technical: use base $B = 6n + 5$, the extra numbers encoding the symbols ($,$, $\vee$ and $\wedge$) for more complicated terms than left strings. I have simply not yet been able to find a finite number of extra relations that allow one to pass from arbitrary $w_1$ and $w_2$ to $(w_1) \vee (w_2)$ and $(w_1) \wedge (w_2)$ without some “collapsing” of $H$.

(II) The seeming ugliness of Theorem C should be removable, so that $G_y$ is unnecessary. I have been unable to achieve this yet.

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Dedication:

This paper is dedicated to W. Charles Holland on his 70th birthday. I am most grateful to him for his guidance and help over the years, as a Research Director, then as a Colleague, and throughout as a mathematician and a friend.

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