Sublattice Subgroups of Finitely Presented Lattice-ordered Groups: Closure under Constructions.

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July 15, 2004

Abstract

We show that the class of lattice-ordered groups that can be embedded in finitely presented lattice-ordered groups is closed under several standard constructions. Specifically,

**Theorem A.** If $G$ can be $\ell$-embedded in a finitely presented lattice-ordered group, then so can $G \wr (\mathbb{Z}, \mathbb{Z})$.

**Theorem B.** If $A$ is a recursively presented Abelian o-group and $G$ can be $\ell$-embedded in a finitely presented lattice-ordered group, then $A \oplus G$ can be $\ell$-embedded in a finitely presented lattice-ordered group.

**Theorem C.** If $G$ is a lattice-ordered group that can be $\ell$-embedded in a finitely presented lattice-ordered group and $H$ is any o-group that can be $\ell$-embedded in an o-group that is finitely presented as a lattice-ordered group and has a minimal strictly positive Archimedean class, then $H \otimes G$ can be $\ell$-embedded in a finitely presented lattice-ordered group.

AMS Classification 06F15, 06F20, 20B27, 20F60.

Keywords: lattice-ordered groups, presentations, direct systems, polyhedral geometry, recursive functions.
1 Introduction

A finitely presented lattice-ordered group with insoluble group word problem was constructed in [4]. In [5] this construction was coupled with direct limits of simplicial groups to prove that every Abelian lattice-ordered group of finite rank that is defined by a recursively enumerable set of relations can be $\ell$-embedded in a finitely presented lattice-ordered group. Clearly, the only finitely generated lattice-ordered groups that can be $\ell$-embedded in finitely presented ones must have a recursively enumerable set of defining relations.

The purpose of this note is to extend previous work to get that the class of finitely generated lattice-ordered groups that are defined by recursively enumerable sets of relations and are $\ell$-embeddable in finitely presented lattice-ordered groups is closed under several standard constructions; e.g., wreath products with $(\mathbb{Z}, \mathbb{Z})$ and lexicographic extensions. This is achieved using permutation groups and, in two proofs, the direct limit (extending the ideas in [9]).

2 Background and notation

Throughout we will use $\mathbb{N}$ for the set of non-negative integers, $\mathbb{Z}_+$ for the set of positive integers, and $\mathbb{R}$ for the set of real numbers. The only order on $\mathbb{R}$ that we will consider will be the usual one.

As is standard, in any group $G$ we write $f^g$ for $g^{-1}f^{-1}fg$, and $[f,g]$ for $f^{-1}g^{-1}fg = f^{-1}f^g$.

A lattice-ordered group is a group which is also a lattice that satisfies the identities $x(y \wedge z)t = xyt \wedge xzt$ and $x(y \vee z)t = xyt \vee xzt$. Throughout we write $x \leq y$ as a shorthand for $x \vee y = y$ or $x \wedge y = x$, $\ell$-group as a shorthand for lattice-ordered group, and $o$-group for a totally ordered group (i.e., if the $\ell$-group is totally ordered). A sublattice subgroup of an $\ell$-group is called an $\ell$-subgroup.

Lattice-ordered groups are torsion-free and $f \vee g = (f^{-1} \wedge g^{-1})^{-1}$; moreover each element of $G$ can be written in the form $fg^{-1}$ where $f, g \in G^+ = \{h \in G : h \geq 1\}$ — see, e.g., [1], Corollary 2.1.3, Lemma 2.3.2 & Lemma 2.1.8. For each $g \in G$, let $|g| = g \vee g^{-1}$. Then $|g| \in G_+$ iff $g \neq 1$, where $G_+ = G^+ \setminus \{1\}$. Therefore, $(w_1 = 1 \& \ldots \& w_n = 1)$ iff $|w_1| \vee \ldots \vee |w_n| = 1$ [ibid., Lemma 2.3.8 & Corollary 2.3.9]. Consequently, in the language of lattice-ordered groups (and in sharp contrast to group theory) any finite number of equalities can be replaced by a single equality.

We will write $f \perp g$ as a shorthand for $|f| \wedge |g| = 1$. 

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An $\ell$-homomorphism from one $\ell$-group to another is a group and a lattice homomorphism. Kernels are precisely the normal $\ell$-subgroups that are convex (if $k_1, k_2$ belong to the kernel and $k_1 \leq g \leq k_2$, then $g$ belongs to the kernel). They are called $\ell$-ideals.

Free $\ell$-groups on finite sets of generators exist by universal algebra. Finitely generated $\ell$-groups are the $\ell$-homomorphic images of free $\ell$-groups on that finite number of generators. As is standard, if the kernel is finitely generated as an $\ell$-ideal, then the $\ell$-homomorphic image is said to be finitely presented; if the kernel is generated by a recursively enumerable set of elements (as an $\ell$-ideal), then we say that the finitely generated $\ell$-homomorphic image has a recursively enumerable set of defining relations and is recursively presented (sic).

We will write $\langle Y : w_i(Y) = 1 \ (i \in I) \rangle$ for the quotient $F_Y/K$ where $F_Y$ is the free $\ell$-group on the generating set $Y$ and $K$ is the $\ell$-ideal generated by $\{w_i(Y) : i \in I\}$.

The free $\ell$-group on a single generator is $\mathbb{Z} \oplus \mathbb{Z}$ ordered by: $(m_1, m_2) \geq (0, 0)$ iff $m_1, m_2 \geq 0$; $(1, -1)$ is a generator since $(1, -1) \lor (0, 0) = (1, 0)$.

If $G_1, G_2$ are $\ell$-groups, then their cardinal product $G_1 \otimes G_2$ is their group product lattice ordered by: $(g_1, g_2) \geq (1, 1)$ iff $g_j \in G_j^+$ ($j = 1, 2$).

There is another way to partially order the direct product of partially ordered groups $G_1$ and $G_2$, namely, $G_1 \otimes G_2$:

$(g_1, g_2) \geq (1, 1)$ iff $(g_1 \in (G_1)_+ \text{ or both } g_1 = 1 \text{ and } g_2 \in G_2^+)$.

This is an $\ell$-group if $G_1$ is an o-group and $G_2$ is an $\ell$-group.

If $G_1, G_2$ are partially ordered groups with $G_1$ contained in $Aut(G_2, \cdot, \leq)$, then we can partially order the splitting extension by: $(g_2, g_1) \geq (1, 1)$ iff $(g_1 \in (G_1)_+ \text{ or both } g_1 = 1 \text{ and } g_2 \in G_2^+)$.

The amalgamation property fails miserably for $\ell$-groups: there are $\ell$-groups $G, H_1, H_2$ with $\ell$-embeddings $\sigma_j : G \to H_j$ ($j = 1, 2$) such that there is no $\ell$-group $L$ such that $H_j$ can be $\ell$-embedded in $L$ ($j = 1, 2$) so that the resulting diagram commutes (see [10] or [1], Theorem 7.C). So HNN-extension tricks cannot be used in general (see [2]). Instead we use permutation group techniques.

Let $(\Omega, \leq)$ be a totally ordered set. Then $A(\Omega) := Aut(\Omega, \leq)$ is an $\ell$-group when the group operation is composition and the lattice operations are just the pointwise supremum and infimum ($\alpha(f \lor g) = \max \{\alpha f, \alpha g\}$, etc.) There is an analogue of Cayley’s Theorem for groups, namely the Cayley-Holland Theorem ([1], Theorem 7.A):
Theorem (Holland [7]) Every lattice-ordered group can be \( \ell \)-embedded in \( A(\Omega) \) for some totally ordered set \((\Omega, \leq)\); every countable lattice-ordered group can be \( \ell \)-embedded in \( A(\mathbb{R}) \).

If \( h \in A(\Omega) \), then the support of \( h \), \( \text{supp}(h) \), is the set \( \{ \beta \in \Omega : \beta h \neq \beta \} \).

Since each real interval \((\alpha, \beta)\) is order-isomorphic to \((\mathbb{R}, \leq)\) we obtain:

Corollary 2.1 Let \( \alpha, \beta \in \mathbb{R} \) with \( \alpha < \beta \). Then every finitely generated \( \ell \)-group \( G \) can be \( \ell \)-embedded in \( A(\mathbb{R}) \) so that \( \text{supp}(g) \subseteq (\alpha, \beta) \) for all \( g \in G \).

If \( h \in A(\Omega) \) and \( \alpha \in \text{supp}(h) \), then the convexification of the \( h \)-orbit of \( \alpha \) is called the \emph{interval of support} of \( h \) containing \( \alpha \); i.e., the \emph{supporting interval} of \( h \) containing \( \alpha \) is \( \{ \beta \in \Omega : (\exists m, n \in \mathbb{Z})(\alpha h^n \leq \beta \leq \alpha h^m) \} \). So the support of an element is the disjoint union of its supporting intervals. Supporting intervals are also called \emph{bumps}.

By considering intervals of support, it is easy to establish the well-known fact

Proposition 2.2 For all \( f, g \in A(\Omega) \), \( \text{supp}(fg) = \text{supp}(f)g \). Hence if \( f^g \perp f \) and \( g \geq 1 \), then \( |f|^m \leq g \) for all \( m \in \mathbb{N} \).

3 Proof of Theorem A

**Theorem A** If \( G \) is a lattice-ordered group that can be \( \ell \)-embedded in a finitely presented lattice-ordered group, then so can \( G \wr (\mathbb{Z}, \mathbb{Z}) \).

**Proof:** (c.f., [3]) It is enough to prove the theorem when \( G \) is a finitely presented \( \ell \)-group, since \( G \wr (\mathbb{Z}, \mathbb{Z}) \) can be \( \ell \)-embedded in \( G_0 \wr (\mathbb{Z}, \mathbb{Z}) \) whenever \( G \) can be \( \ell \)-embedded in \( G_0 \). So let \( G = \langle g_1, \ldots, g_n : w(g_1, \ldots, g_n) = 1 \rangle \) be a finitely presented \( \ell \)-group. Adjoin \( g_0 = \bigvee_{i=1}^n |g_i| \) as a generator. Thus \( g_0 \) is a strong order unit in \( G \). Let

\[
H = \langle a, g_0, \ldots, g_n, h_0 : g_0 = \bigvee_{i=1}^n |g_i|, \ w(g_1, \ldots, g_n) = 1, \ h_0 \leq a, \\
g_0 \land h_0 = 1, \ h_0g_0^{-a} \land g_0^a = 1, \ h_0h_0^{-a} \land h_0^a = 1 \rangle.
\]

Then \( H \) is a finitely presented \( \ell \)-group.

Since \( h_0h_0^{-a} \geq 1 \), an easy induction shows that \( h_0 \geq h_0^m \) for all \( m \in \mathbb{N} \).

Since \( h_0g_0^{-a} \geq 1 \), we have \( h_0^m \geq g_0^{m+1} \) for all \( m \in \mathbb{N} \). Hence \( h_0 \geq g_0^m \) for all \( m \in \mathbb{Z}_+ \). Since \( g_0 \perp h_0 \), we get \( g_0 \perp g_0^m \) for all \( m \in \mathbb{Z}_+ \). Thus \( g_0^m \perp g_0^n \) for all
all distinct $m, n \in \mathbb{Z}$. By Proposition 2.2, the $\ell$-subgroup of $H$ generated by $G \cup \{a\}$ is an $\ell$-homomorphic image of $G$ wr $(\mathbb{Z}, \mathbb{Z})$.

By Corollary 2.1, we may consider $G$ as an $\ell$-subgroup of $A(\mathbb{R})$ with each element of $G$ having support contained in $(0, 1)$. Let $g_0 \in A(\mathbb{R})$ be $g_0$ on $(0, 1)$ and the identity off $(0, 1)$. Let $a_0 \in A(\mathbb{R})$ be translation by 1 ($\alpha a_0 = \alpha + 1$). Define $h_0 \in A(\mathbb{R})$ to have support contained in $\bigcup_{n \in \mathbb{Z}^+} (m, m + 1)$ and $h_0$ restricted to $(m, m + 1)$ be $g_0^m$ ($m \in \mathbb{Z}$). Then all the relations of $H$ hold with $h_0, a_0$ replaced by $h_0, a_0$, respectively. Hence the $\ell$-subgroup $H$ of $A(\mathbb{R})$ generated by $G \cup \{h_0, a_0\}$ is an $\ell$-homomorphic image of $H$. Since $G$ is $\ell$-embedded in $H$, it is $\ell$-embeddable in $H$. Consequently, the $\ell$-subgroup of $H$ generated by $g_0, g_1, \ldots, g_n, a$ is $G$ wr $(\mathbb{Z}, \mathbb{Z})$, and the $\ell$-subgroup generated by the conjugates of $g_0, g_1, \ldots, g_n$ by powers of $a$ is $\ell$-isomorphic to $\sum_{n \in \mathbb{Z}} G$. //

4 Proof of Theorem B

**Theorem B** If $A$ is a recursively presented Abelian $o$-group and $G$ is a lattice-ordered group that can be $\ell$-embedded in a finitely presented lattice-ordered group, then $A \bigotimes G$ can be $\ell$-embedded in a finitely presented lattice-ordered group.

**Proof:** If $G$ can be $\ell$-embedded in a finitely presented $\ell$-group $G_0$ by $\psi$, then $A \bigotimes G$ can be $\ell$-embedded in $A \bigotimes G_0$ via: $(a, g) \mapsto (a, g\psi)$. Hence we may assume that $G$ itself is finitely presented. As in the previous proof, we may assume that the strong order unit of $G$ is $g_0$.

Let $A$ be a finitely generated Abelian $\ell$-group of rank $n$. Then $A$ is a direct limit of simplicial groups $\mathbb{Z}^n$. Say, the direct limit of $\{A_i : i \in \mathbb{N}\}$ (where $A_i$ is the additive group $\mathbb{Z}^n$ with order given by basis $B_i$ and transition matrix an $n \times n$ integer matrix $M_f(i)$ with determinant 1 ($i \in \mathbb{N}$)) — see [5]. It is straightforward to verify that $H := A \bigotimes G$ is a direct limit of p.o. groups $A_i \bigotimes G$ (ordered by $(s, g) > 1$ iff $s > 0$ or $(s = 0 \& g > 1$)), with transition maps $\varphi_i (i \in \mathbb{N})$ given by $(s, g)\varphi_i = (sM_f(i), g)$. We will write $(M_f(i), 1)$ for $\varphi_i$ and also refer to this direct limit system as standard. We will identify $G$ with $\{0\} \times G$ in $H$.

Note that $A \cong H/G$.

As in [5], Lemma 4.3, at stage $i \in \mathbb{N}$, we can obtain a basis $B_i$ for $\mathbb{Z}^n$ such that for all $a, b \in B_i$ to obtain $H_i := A_i \bigotimes G$:

- either $a \leq b$ or $a \geq b$, or $[(h \in H_i\text{ and } 1 \leq h \leq a, b) \rightarrow h \in G]$. 

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By considering the quotient by \( G \) at each stage for the general case and the recursive set of relations for \( A^* \) ([5], Theorem B) in the recursively enumerable case (to determine if \( a, b \) have a common lower bound in \( A^* \)), we obtain the analogue of [5], Proposition 4.1:

**Proposition 4.1** If \( A^* \) is a finitely generated Abelian \( \ell \)-group of rank \( n \) and \( G^* \) is an arbitrary \( \ell \)-group, then \( A^* \otimes G^* \) is a direct limit of a standard system \( \{ S_i \otimes G^* \}_{i \in \mathbb{N}} \) with \( \{ S_i \}_{i \in \mathbb{N}} \) a standard rank \( n \) system. Moreover, this system is recursively enumerable iff \( A^* \) is recursively presented.

If \( A \) is a simplicial group and \( G \) is a finitely presented \( \ell \)-group, then the p.o. group \( H = A \otimes G \) can indeed be order-embedded in \( A(\mathbb{R}) \). For we may regard \( G \) as initially \( \ell \)-embedded in \( A((0, 1)) \), say by \( \phi_0 \). If \( A \) is \( \mathbb{Z}^2 \), for example, with \( b_1 = (1, 0) \) and \( b_2 = (0, 1) \) as basis in the usual ordering, then let \( a_1 \) have two bumps with supports \((-1, 2)\) and \((-\infty, -1)\), so that \( a_1, a_2 \) agree on \((-1, 2)\) and map 0 to 1. Then \( a_1, a_2 > g \) for all \( g \in G \). Of course \( b_1 \perp b_2 \) in \( \mathbb{Z}^2 \) but \( a_1 \not\perp a_2 \) in \( A(\mathbb{R}) \). Now \( \ell \)-embed \( G \) in \( A(\mathbb{R}) \) so that \( g \) restricted to \((0, 1)\) is \( g \phi_0 \), \( g \) restricted to \((0a_1^k, 1a_2^k)\) is \( a_1^{-k}(g \phi_0)a_2^k \) \((k \in \mathbb{Z})\), and \( g \) is the identity off \((-1, 2)\) \((g \in G)\). The homomorphism given by \( g \mapsto g \) \((g \in G)\) and \( b_i \mapsto a_i \) \((i = 1, 2)\) is the desired order-preserving embedding of \( A \otimes G \) into \( A(\mathbb{R}) \).

So the mimicking part follows as before, and we need only check that any recursive direct limit of a standard system can be encoded as in [5], Section 5. We do this so that \((**_m)\) and \((**_m)\) hold with \( g_i \) in place of \( x_j \) \((i = 0, \ldots, n)\). The key for this is that \( \mathfrak{g} \ast (e_j M_{f(m)}) \) is of the form \( x_1^r x_2^s \) with \( r_1, r_2 \in \mathbb{N} \) (not both 0) for \( j = 1, 2 \). So \( g_0 \ll \mathfrak{g} \ast (e_j M_{f(m)}) \). We will require the added relations that \([x_j, g_i] = 1 \((j = 1, 2; i = 0, \ldots, n)\) and

\[ g_i \ast (c_i^m \hat{f}) = g_i \ast c_i^m \] (\( \dagger \dagger_m \))

for all \( i = 0, \ldots, n \) and \( m \in \mathbb{N} \), all of which hold in the embedded image of \( A \otimes G \) in \( A(\mathbb{R}) \).

When \( f = \theta \), then \( \hat{f} = 1 \) obeys \((\dagger \dagger_m)\) for all \( m \in \mathbb{N} \) and \( i = 0, \ldots, n \).

If \( f = s \), then \( g_i \ast \hat{s} = g_i \ast s^2 = g_i \) together with \([\hat{s}, c_i^1] = 1\) ensure that \((\dagger \dagger_m)\) holds for all \( m \in \mathbb{N} \) and \( i = 0, \ldots, n \).

Assuming that \((\dagger \dagger_m)\) holds for \( \hat{f}, \hat{g} \) for all \( m \in \mathbb{N} \), then the definition of \( \hat{f} \hat{g} \) given in [5], page 559 is trivially shown to imply \((\dagger \dagger_m)\) for all \( m \in \mathbb{N} \) for \( \hat{f} \hat{g} \). The same is true for obtaining \( f \) by general recursion from previous “good” functions. Thus every recursive function is presentable in this broader sense: the added relations \((\dagger \dagger_m)\) \((m \in \mathbb{N})\) also hold in a finitely presented \( \ell \)-group.

Theorem B now follows. //
5 Proof of Theorem C

**Theorem C** If $G$ is a lattice-ordered group that can be $\ell$-embedded in a finitely presented lattice-ordered group and $H$ is any o-group that can be $\ell$-embedded in an o-group that is finitely presented as a lattice-ordered group and has a minimal strictly positive Archimedean class, then $H \boxtimes G$ can be $\ell$-embedded in a finitely presented lattice-ordered group.

**Proof:** If $H$ can be $\ell$-embedded in $H_0$ and $G$ in $G_0$, then $H \boxtimes G$ can be $\ell$-embedded in $H_0 \boxtimes G_0$. So again it suffices to assume that $H$ and $G$ are themselves finitely presented, $H$ is an o-group and that $H_+$ has a minimal Archimedean class.

Let $G = \langle g_0, \ldots, g_m : w_0(g_0, \ldots, g_m) = 1 \rangle$ and $H = \langle h_1, \ldots, h_n : w_1(h_1, \ldots, h_n) = 1 \rangle$, where $g_0 = \bigvee_{i=1}^m |g_i|$. Let $w_2(h_1, \ldots, h_n) \in H_+$ have minimal Archimedean class in $H_+$. Let

$$L = \langle a, g_0, \ldots, g_m, h_1, \ldots, h_n : g_0 = \bigvee_{i=1}^m |g_i|, \ w_0(g_0, \ldots, g_m) = 1, \ w_1(h_1, \ldots, h_n) = 1, \ [g_i, h_j] = 1, \ w_2(h_1, \ldots, h_n) \geq g_0, \ h^a_j = h_j, \ g^a_i = g^a_0 (i = 0, \ldots, m; j = 1, \ldots, n) \rangle.$$

Then $L$ is a finitely presented $\ell$-group and for all $k \in \mathbb{Z}_+$,

$$g^k_0 \leq g^a_k \leq w_2(h_1, \ldots, h_n)^a_k = w_2(h_1^k, \ldots, h_n^k) = w_2(h_1, \ldots, h_n);$$

so $g_0 \ll H_+$. Thus there is an $\ell$-homomorphism $\psi$ from $H \boxtimes G$ into $L$ (under the natural identification). We must show that this $\ell$-homomorphism is injective.

Let $(W, R \rightleftarrows H) = (A(\mathbb{R}), \mathbb{R}) \bar{\wr} (H, H)$ and regard $G$ as an $\ell$-subgroup of $A(\mathbb{R})$. We $\ell$-embed $G$ in $W$ diagonally: $g \mapsto \hat{g}$ where $(r, \alpha)\hat{g} = (rg, \alpha)$ and $\ell$-embed $H$ in $W$ via: $h \mapsto \hat{h}$ where $(r, \alpha)\hat{h} = (r, \alpha h)$ ($r \in \mathbb{R}; \alpha \in H$). Note that $[\hat{h}, \hat{g}] = 1$ for all $g \in G$, $h \in H$; so $H \boxtimes G$ is $\ell$-embedded in $W$ via: $hg \mapsto \hat{h}\hat{g}$.

Let $a_0 \in A(\mathbb{R})$ be such that $g_{a_0}^0 = \hat{g}_0^2$ (see [1], Lemma 8.3.3) and extend $a_0$ to $\hat{a} \in W$ diagonally: $(r, \alpha)\hat{a} = (ra_0, \alpha)$ ($r \in \mathbb{R}; \alpha \in H$). Then $[\hat{h}, \hat{a}] = 1$ for all $h \in H$ and $g_0^a = \hat{g}_0^2$. Hence $W$ contains an $\ell$-homomorphic image $L\phi$ of $L$ (the $\ell$-subgroup generated by $\hat{a}, \hat{g}_0, \ldots, \hat{g}_m, \hat{h}_1, \ldots, \hat{h}_n$) and $W$ contains $H \boxtimes G$ (to within $\ell$-isomorphism $\theta$). Moreover, the diagram is commutative; i.e., $\theta = \psi\phi$. Therefore the $\ell$-homomorphism $\psi$ of $H \boxtimes G$ into $L$ is injective. //
Every finitely generated Abelian o-group has a minimal strictly positive Archimedean class. We do not know if every finitely presented ℓ-group that is a non-Abelian o-group has a minimal strictly positive Archimedean class. If so, then the extra hypothesis in the theorem would be unnecessary. The difficulty is the paucity of examples.

6 Theorem D

In Theorem C, the elements from H acted as identity automorphisms on G. The purpose of the next result is to relax this restriction somewhat to H being a subgroup of the automorphism group of G (with a total order on H). We consider non-commutative extensions instead of the lexicographic sums. For this we must restrict to actual finitely presented ℓ-groups as the ℓ-embedding may not be feasible within Aut(G, ·, ∨).

**Theorem D** If G is a finitely presented lattice-ordered group, let H be a subgroup of the group of automorphisms of G. If H is an o-group that is finitely presented as a lattice-ordered group and has a minimal strictly positive Archimedean class, then G ⋊ H can be ℓ-embedded in a finitely presented lattice-ordered group.

**Proof:** The proof is very similar to that of Theorem C. Let

\[ G = \langle g_0, \ldots, g_m : w_0(g_0, \ldots, g_m) = 1 \rangle \]

and \[ H = \langle h_1, \ldots, h_n : w_1(h_1, \ldots, h_n) = 1 \rangle, \]

where \( g_0 = \bigvee_{i=1}^m |g_i| \).

For each \( g_i, h_j \), we have \( g_i^{h_j}, h_i^{−1} \in G \); say, \( g_i^{h_j} = w_{i,j}(g_0, \ldots, g_m) \)

and \( g_i^{−1} = u_{i,j}(g_0, \ldots, g_m) \) (\( i = 0, \ldots, m; j = 1, \ldots, n \)).

Let \( w_2(h_1, \ldots, h_n) \in H_+ \) have minimal Archimedean class in \( H_+ \). Let

\[ L = \langle a, g_0, \ldots, g_m, h_1, \ldots, h_n : w_0(g_0, \ldots, g_m) = 1, w_1(h_1, \ldots, h_n) = 1, \]

\[ g_0 = \bigvee_{i=1}^m |g_i|, \quad g_i^{h_j} = w_{i,j}(g_0, \ldots, g_m), \quad g_i^{−1} = u_{i,j}(g_0, \ldots, g_m), \]

\[ w_2(h_1, \ldots, h_n) \geq g_0, h_j^a = h_j, g_0^a = g_0^2 \quad (i = 0, \ldots, m; j = 1, \ldots, n). \]

Then L is a finitely presented ℓ-group and for all \( k \in \mathbb{Z}_+ \),

\[ g_0^k \leq g_0^{2^k} \leq w_2(h_1, \ldots, h_n)^{a^k} = w_2(h_1^{a^k}, \ldots, h_n^{a^k}) = w_2(h_1, \ldots, h_n); \]

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so $g_0 \leq H_+$. Also, $H$ acts on $G$ appropriately in $L$. Thus the natural map defines an $\ell$-homomorphism $\psi$ from $G^\infty H$ into $L$. We must show that this $\ell$-homomorphism is injective.

Let $(W, \mathbb{R} \times H) = (A(\mathbb{R}), \mathbb{R})$ Wr $(H, H)$ where we regard $G$ as an $\ell$-subgroup of $A(\mathbb{R})$. We $\ell$-embed $H$ into $W$ as before: $h \mapsto \bar{h}$ where $(r, \alpha)\bar{h} = (r, \alpha h)$ ($r \in \mathbb{R}; \; \alpha \in H$). We $\ell$-embed $G$ into $W$ diagonally modulo $H$: $g \mapsto \tilde{g}$ where $(r, \alpha)\tilde{g} = (rg^{\alpha^{-1}}, \alpha)$ ($r \in \mathbb{R}; \; \alpha \in H$), $g^\beta$ being the image of $g$ under $\beta \in H \subseteq Aut(G, \cdot, \lor)$ (see [1], proof of Theorem 7.G). These provide an $\ell$-embedding of $G^\infty H$ into $W$, for if $\alpha, f \in H$, $g \in G$ and $r \in \mathbb{R}$, then

$$(r, \alpha)f^{-1}\tilde{g}f = (r, \alpha f^{-1})\bar{g}f = (rgf^{\alpha^{-1}}, \alpha f^{-1})f = (rgf^{\alpha^{-1}}, \alpha) = (r, \alpha)\tilde{g}f;$$

so $G^\infty H$ is $\ell$-embedded in $W$ via: $hg \mapsto \bar{h}\bar{g}$.

Let $a_0 \in A(\mathbb{R})$ be such that $g_0^{a_0} = g_0$ and extend $a_0$ to $\tilde{a} \in W$ diagonally: $(r, \alpha)\tilde{a} = (ra_0, \alpha)$ ($r \in \mathbb{R}; \; \alpha \in H$). Then $[\bar{h}, \tilde{a}] = 1$ for all $h \in H$ and so

$$(r, \alpha)\tilde{g}_0 = (ra_0^{-1}g_0^{-1}a_0, \alpha) = (a_0^{-1}g_0a_0\alpha^{-1}, \alpha) = (rg_0^{2\alpha^{-1}}, \alpha) = (r, \alpha)\tilde{g}_0,$$

$r \in \mathbb{R}, \; \alpha \in H$. Thus $\tilde{a}^{-1}\tilde{g}_0\tilde{a} = \tilde{g}_0^2$, and so all the relations of $L$ hold in $W$. Hence $W$ contains an $\ell$-homomorphic image $L\phi$ of $L$ (the $\ell$-subgroup generated by $\tilde{a}, \tilde{g}_0, \ldots, \tilde{g}_n, \tilde{h}_1, \ldots, \tilde{h}_n$ and $W$ contains $G^\infty H$ (to within $\ell$-isomorphism $\theta$). Moreover, the diagram is commutative; i.e., $\theta = \psi\phi$. Therefore the $\ell$-homomorphism $\psi$ of $G^\infty H$ into $L$ is injective. $//$

Similarly, we can modify the proof of Theorem B to prove:

**Theorem E** If $G$ is a finitely presented lattice-ordered group and $A$ is a recursively presented Abelian $\alpha$-group of finite rank with $A$ a subgroup of $Aut(G, \cdot, \lor)$, then $G^\infty A$ can be $\ell$-embedded in a finitely presented lattice-ordered group.

The key here is that each simplicial group in the direct limit sequence is the same (simplicial) group and, as a group, it is the group $A$ of automorphisms of $G$; only the orders on this group change and in the limit give $A$. So $G^\infty A$ is just a direct limit of the same simplicial group acting on $G$, albeit with different partial orders.

**Acknowledgements:** This research was begun when the first author was visiting the Universities of Milano and Firenze in December 2002, and completed when he visited the latter in January and February 2004 (when the discussion and ideas begun in the hills of Tuscany eventually coalesced). It was made possible by funds from those universities in 2002, and from the Istituto Nazionale di Alta Matematica Francesco Severi, Italy, and Queens’
College, Cambridge in 2004. We are extremely grateful to them for making these visits possible and providing us with the opportunity to develop this material together.

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