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ORDERINGS AND GROUPS: A SURVEY OF RECENT RESULTS

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I will give a survey of recent results in the subject which are most likely to be of interest to universal algebraists. I will approach this by asking some natural questions and stating the answers. These often start positively but quickly come to a grinding halt. This leads to the natural suspicion that the Devil wins every time. However, in the last part of my talk I will show "*aber nicht immer*" (but not always).

Let G be a group with a total (*i.e.*, linear) order $\leq$. We say that G is an *ordered group* (or *o-group*) if $x \leq y \quad \rightarrow \quad gxh \leq gyh$.

If a group G has a total order on it making it an o-group, we say that G is *orderable*.

The obvious first question is:

(I) Is the class of orderable groups 1^{st}-order definable?

For each $n \in \mathbb{Z}_+$, let θ_n be the sentence

$$(\forall x)(x^n = 1 \rightarrow x = 1).$$

If $G \models \theta_n$ for all $n \in \mathbb{Z}_+$, then we say that G is *torsion-free*.

If $x > 1$ in an o-group G, then multiplying both sides by x gives $x^2 > x$. Hence $x^2 > 1$. Continuing in this way gives $x^m > 1$ for all $m \in \mathbb{Z}_+$. Similarly, $x^m < 1$ in G for all $m \in \mathbb{Z}_+$ if $x < 1$. Thus every orderable group is torsion-free.

In certain cases, the converse holds.

An abelian group is orderable iff it is torsion-free.

And a generalisation (definition later):

A nilpotent group is orderable iff it is torsion-free.

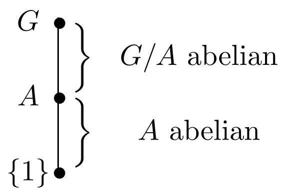
So for these classes of groups, orderable is a 1^{st}-order property, and the sentences needed involve only one letter.

More generally, the class of orderable groups is clearly closed under isomorphisms and subgroups. It is also closed under ultraproducts (an element of the ultraproduct exceeds 1 iff the set of coordinates where this occurs belongs to the ultrafilter — *i.e.*, has measure 1). So the answer to (I) is *YES.* //

Now that we have established that it is 1^{st}-order definable in the language of groups, the next natural question is:

(II) Find axioms for this class.

Having answered one question positively (and in light of what I said at the beginning), it should come as no surprise that this turns out to be rather nasty. An answer was provided by Vasily Bludov (Irkutsk) who initially worked in the area in the 1970s but only returned to it 6 years ago. When a Siberian bear wakes up from hibernation, he has a voracious appetite; and, as we shall see, Vasily has been devouring theorems and settling long-outstanding problems ever since!



metabelian

Recall that a group G is *metabelian* if it is an extension of one abelian group by another. That is, G has a subgroup A such that both G/A and A are abelian (where A is a kernel – *i.e.*, normal – subgroup of G). These are just one level up from being abelian, but they suffice to create havoc.

Theorem A. [Bludov, 2002] *For every $n \in \mathbb{Z}_+$, there is an $(n+1)$-generator metabelian group $G(n+1)$ that is not orderable but all its n-generator subgroups are orderable.*

If the class of (metabelian) orderable groups were definable by sentences using only n letters, then $G(n+1)$ would be orderable since every n-generator subgroup of it is. Hence no set of axioms in just n letters defines the class of groups that are orderable. In particular, the class is *NOT finitely axiomatisable.*

The difficulty with $\leq$ is that it is not an algebraic operation. So instead, let's consider the natural algebraic generalisation.

A *lattice-ordered group* (or *ℓ-group*) is a group which is also a lattice that satisfies

$$a(b \wedge c)d = abd \wedge acd \text{ and } a(b \vee c)d = abd \vee acd.$$

Now ℓ-groups are torsion-free since

$$(x \vee 1)^m = x^m \vee x^{m-1} \vee \cdots \vee x \vee 1.$$

So *abelian (nilpotent) groups are lattice-orderable iff orderable iff torsion-free.*

The lattice of any ℓ-group is distributive. Hence:
Any element of an ℓ-group G generated by $g_1, \ldots, g_n$ can be written in the form

$$\bigwedge_I \bigvee_J w_{i,j}(g_1, \ldots, g_n),$$

where I, J are finite and each $w_{i,j}(g_1, \ldots, g_n)$ is a group word in $g_1, \ldots, g_n$.

Whereas orderable groups were 1^{st}-order definable in the language of groups (even if we haven't found suitable axioms), matters become worse if we consider the natural class of all groups which can be made into ℓ-groups. In this case, the Devil really wins!

Theorem B. [Vinogradov, 1971] *There are metabelian groups G and H with $G \equiv H$ such that G is lattice-orderable but H is not. Hence the class of groups that are lattice-orderable is NOT 1^{st}-order definable.*

So let's stick to abelian ℓ-groups.

(III) Abelian ℓ-groups look nicer. Are they?

Digression on abelian groups:

The free abelian group on n generators is isomorphic to $\mathbb{Z}^n$.

$\mathbb{Z}^n/(2\mathbb{Z})^n \cong \mathbb{F}_2^n$, an n-dimensional vector space over the field $\mathbb{F}_2$ of 2 elements. It has size 2^n. Now every element of $\mathbb{Z}^n$ can be written as twice an n-tuple of integers added to 0 or 1 times the first generator, added to 0 or 1 times the 2^{nd} generator, ..., added to 0 or 1 times the n^{th} generator. Thus

$$\mathbb{Z}^n \models (\exists x_1, \ldots, x_n)(\forall y)(\exists z)\Big(\bigvee_{\epsilon_1, \ldots, \epsilon_n \in \{0,1\}} (y = 2z + \epsilon_1 x_1 + \cdots + \epsilon_n x_n)\Big).$$

Note that $\mathbb{Z}^m$ satisfies this sentence iff $m \leq n$. Thus

$$\mathbb{Z}^m \equiv \mathbb{Z}^n \quad \text{iff} \quad m = n.$$

Moreover, *$Th(\mathbb{Z}^m)$ is decidable for all $m \in \mathbb{Z}_+$* [Szmielew].

Every finitely generated abelian group is mechanically reducible to a unique form

$$\mathbb{Z}^r \oplus \mathbb{Z}/d_1\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/d_k\mathbb{Z},$$

where $d_1, \ldots, d_k \in \mathbb{Z}_+$ with $1 \neq d_1 | d_2 | \ldots | d_k$. Hence

the isomorphism problem for finitely generated abelian groups is decidable.

Now consider **abelian ℓ-groups**.

Since every non-trivial ℓ-group contains a copy of $\mathbb{Z}$, the smallest non-trivial variety of ℓ-groups is ℓ-$var(\mathbb{Z})$. [Weinberg, 1963] showed that ℓ-$var(\mathbb{Z})$ is the class of all abelian ℓ-groups. So what we are considering is the smallest non-trivial equational class of ℓ-groups.

To begin with, let's consider some examples.

Make the additive group $\mathbb{Z}^2 = \mathbb{Z} \oplus \mathbb{Z}$ an ℓ-group by $(m,n) \geq 0$ iff $m,n \geq 0$. It is generated by $(1,0)$ and $(0,1)$ as a group, but just by $(1,-1)$ as an ℓ-group (since $(1,-1) \vee 0 = (1,0)$).

It is the free (abelian) ℓ-group on 1 generator.

Next consider $\mathbb{Z}^n$ lattice-ordered analogously. It is generated by $f = (1,\ldots,1)$ and $g = (1,2,\ldots,n)$ since $h := (2f - g) \vee 0 = (1,0,\ldots,0)$, $f - h = (0,1,\ldots,1)$ and $g - f = (0,1,2,\ldots,n-1)$, *etc.*

Thus free abelian ℓ-groups on at least 3 generators have infinite rank as abelian groups. Indeed, this is true of the free abelian ℓ-group on 2 generators, too.

So let's refine our question a little.

($\mathbf{III'}$) What do free abelian ℓ-groups look like?

Let C be the additive group of all continuous functions from $\mathbb{R}^n$ to $\mathbb{R}$. Then C is a lattice-ordered group under the pointwise ordering; *i.e.*,

$$(f \vee g)(\mathbf{x}) = max\{f(\mathbf{x}), g(\mathbf{x})\} \text{ and } (f \wedge g)(\mathbf{x}) = min\{f(\mathbf{x}), g(\mathbf{x})\}.$$

Let $\pi_1, \ldots, \pi_n$ be the n standard projections from $\mathbb{R}^n$ to $\mathbb{R}$. So $\pi_j(\mathbf{x}) = x_j$ $(j = 1, \ldots, n)$.

Theorem C. [G. Birkhoff, 1943(*ca.*)] *The sublattice subgroup of C generated by $\pi_1, \dots, \pi_n$ is the free abelian ℓ-group, $FA\ell(n)$, on n generators.*

If $f = \sum_{i=1}^{n} m_i\pi_i$, then $Z(f)$, the zero set of f, is given by

$$Z(f) = \{(x_1, \dots, x_n) \in \mathbb{R}^n : \sum_{i=1}^{n} m_i x_i = 0\},$$

a hyperspace. Thus $Z(f \vee 0)$ is a half space of $\mathbb{R}^n$. So the join of a finite set of group words and 0 has zero set a closed convex polyhedral cone with vertex 0. And the general element of $FA\ell(n)$, being the meet of such elements, will have its zero set a finite union of such cones.

This gives rise to a duality between abelian ℓ-groups and simplicial geometry, known as the Baker-Beynon Duality [Beynon, 1977]

Abelian ℓ-groups $\Longleftrightarrow$ Simplicial Geometry.

Using this, one can show

Theorem D. [Glass, Macintyre and Point, 2005]

(i) *$FA\ell(m) \equiv FA\ell(n)$ iff $m = n$.*
(ii) *$Th(FA\ell(m))$ is decidable iff $m = 1, 2$.*

The key is to express the dimension in the 1^{st}-order language of ℓ-groups.

Note that (i) is analogous to the abelian group result, but (ii) shows that things are nastier (richer?) than in the abelian group case if $m \geq 3$.

Using the techniques of the proof and the Mundici correspondence, one can establish the analogous theorem for free MV-algebras [Glass and Point].

Whereas the isomorphism problem for finitely generated abelian groups is decidable, things go pear-shaped very quickly for abelian ℓ-groups. By a theorem of Markov and the Baker-Beynon Duality, one gets

Theorem E. [Glass and Madden, 1984]

The isomorphism problem for 10-generator 1-relator abelian ℓ-groups is undecidable.

So, although abelian ℓ-groups are reasonably civilised, there is a lot more going on (going wrong?) compared with abelian groups.

The main research on such ℓ-groups is being undertaken by the Italian school headed by Daniele Mundici. Besides Daniele's, I would especially recommend the papers of two of his ex-research students, Vincenzo Marra and Giovanni Panti. To give just one sample of their work:

Theorem F. [Manara, Marra, Mundici, 2005]

Let $m \in \mathbb{Z}_+$ and $f \in FA\ell(m)$. Then $FA\ell(m)/\langle f\rangle$ is ℓ-isomorphic to $FA\ell(n)/\langle g\rangle$ for some $n \geq m$ and pure <u>*lattice*</u> *word $g \in FA\ell(n)$.*

We next turn to the promised extension of abelian ℓ-groups, namely **nilpotent ℓ-groups**.

Let $[x, y] := x^{-1}y^{-1}xy$ and

$$[x_1, \ldots, x_{n+1}] := [[x_1, \ldots, x_n], x_{n+1}].$$

We call $[x_1, \ldots, x_{n+1}]$ a *commutator of length* $n + 1$, and say that a group is *nilpotent class* n if it satisfies the commutator identity

$$[x_1, \ldots, x_{n+1}] = 1.$$

So an abelian group is nilpotent class 1.

Let $\gamma_{n+1}(G)$ be the subgroup of G generated by all commutators of length $n + 1$. Then $G/\gamma_{n+1}(G)$ is nilpotent of class n.

In any free group F, $\bigcap_{n\in Z_+} \gamma_n(F) = \{1\}$ and $F/\gamma_m(F)$ is torsion-free for all $m \in \mathbb{Z}_+$. Hence the variety of groups generated by all (torsion-free) nilpotent groups includes all free groups (since we have just seen that F can be embedded in $\prod_{n\in\mathbb{Z}_+} F/\gamma_n(F)$). Since every group is an homomorphic image of some free group, the variety of groups generated by all (torsion-free) nilpotent groups is the class of all groups.

So no extra identities are needed to define the variety of groups generated by all (torsion-free) nilpotent groups.

(IV) What happens for nilpotent ℓ-groups?

It can be shown that every ℓ-group that is nilpotent is a subdirect product of o-groups [Kopytov, 1976] and so belongs to the variety of ℓ-groups generated by all o-groups. This is the class of all residually ordered ℓ-groups which we denote by $\mathrm{R}\mathcal{O}$ as is standard. In this terminology,

ℓ-$var(\mathcal{N}) \leq \mathrm{R}\mathcal{O}$.

The obvious next question is

(IV′) Is ℓ-$var(\mathcal{N}) = \mathrm{R}\mathcal{O}$?

In the same paper, the answer was provided (and, by now, should not be a surprise). *NO*.

Theorem G. [Kopytov, 1976] $\ell\text{-}var(\mathcal{N}) \neq {}_R\mathcal{O}$.

Indeed, every nilpotent ℓ-group satisfies

$$\mathcal{W}: \; y^{-1}|x|y \leq |x|^2,$$

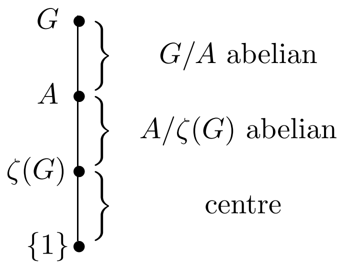
where $|x| := x \vee x^{-1} \geq 1$ (and $|x| = 1$ iff $x = 1$). So we are led to:

(**IV**$''$) (Kopytov, 1976) **Does this law define ℓ-*var*$(\mathcal{N})$?**

This would imply that $\ell\text{-}var(\mathcal{N})$ is defined by a single law. But by now we should feel confident that the answer will be *NO*, and this is indeed the case.

Theorem H. [Bludov and Glass, 2005] $\ell\text{-}var(\mathcal{N}) \neq \mathcal{W}$.

For this, metabelian does not seem to suffice to cause trouble, but something very close to it does.



centre-by-metabelian

Recall that $\zeta(G)$, the *centre* of a group G, is the set of all elements of G that commute with every element of G:

$$\zeta(G) = \{g \in G : (\forall f)(gf = fg)\}.$$

A group G is said to be *centre-by-metabelian* if $G/\zeta(G)$ is metabelian. It is a special kind of extension of an abelian group by a metabelian one.

We construct a centre-by-metabelian o-group belonging to $\mathcal{W}$ and then show that it does not belong to $\ell\text{-}var(\mathcal{N})$. To do this we actually give identities for $\ell\text{-}var(\mathcal{N})$ and show that

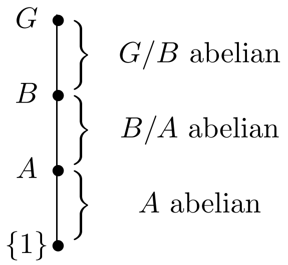
$$\ell\text{-}var(\mathcal{N}) = \ell\text{-}quasi\text{-}var(\mathcal{N}).$$

We believe that no set of identities in n letters suffices to define $\ell\text{-}var(\mathcal{N})$ but have not written down our argument carefully yet ($n \in \mathbb{Z}_+$).

For groups, if the generators commute then so do any group terms in these generators; *i.e.*, the group is abelian. The same is true for ℓ-groups: if the generators commute, then so do any ℓ-group terms in these generators; *i.e.*, the ℓ-group is abelian.

If the generators of a group satisfy the nil-2 law, then the group is nilpotent class 2. This fails for ℓ-groups as replacing generators in $[x, y, z]$ by ℓ-group terms no longer results in the identity. However,

Theorem I. [Darnel and Glass] *If G is an n-generator ℓ-group and the generators satisfy the nil-2 law, then the ℓ-group is soluble of class $n(n-1)/2$.*



soluble length 3

An abelian group is said to be *soluble of class* 1; a metabelian group is said to be *soluble of class* 2; and, more generally, a group G is said to be *soluble of class* $n+1$ if there is an abelian normal subgroup A of G such that G/A is *soluble of class n*. A group is called *soluble* or *poly-abelian* if it is soluble of some (finite) class. We denote the class of all soluble groups by $\mathrm{P}\mathcal{A}$.

Since the class of soluble groups strictly contains the class of nilpotent groups, we might hope that this larger class will suffice to get $\mathrm{R}\mathcal{O}$ (especially since we have Theorem I).

(V) Does the class of soluble o-groups generate $\mathrm{R}\mathcal{O}$?

No prizes for guessing this one; the Devil wins again.

Theorem J. [Medvedev, 2005] *ℓ-var$(\mathcal{O} \cap \mathrm{P}\mathcal{A}) \neq \mathrm{R}\mathcal{O}$.*

Indeed, the proof has almost nothing to do with the abelian property. It can easily be extended to show

Theorem K. [Glass and Medvedev, 2005] *ℓ-var$(\mathcal{O} \cap \mathrm{P}\mathcal{H}) \neq \mathrm{R}\mathcal{O}$.*

Here $\mathcal{H}$ is any variety of *hegemonic* groups defined by a single law $u(x, y) = 1$; that is, $u(x, y)$ is a product of conjugates of $x^{\pm 1}$ by powers of y (one of which dominates) and commutators in $[\langle x\rangle^{\langle y\rangle}, \langle x\rangle^{\langle y\rangle}]$, where $z^t := t^{-1}zt$. Since $\mathcal{A}$ is defined by $x^{-1}x^y$, it is hegemonic. So is $\mathcal{E}_n$ which is defined by the n-Engel law $[x, y_1, \ldots, y_n] = 1$ where all $y_j = y$ $(j = 1, \ldots, n)$.

If we do not restrict the soluble groups to being o-groups, we get a much bigger ℓ-variety. It is the unique maximal proper ℓ-variety and contains all other varieties of ℓ-groups (other than the variety of all ℓ-groups). It is called the *normal valued* ℓ-variety. These have nice permutation representations.

If we use the Dvurečenskij correspondence, then universals and permutation representations have been provided for the corresponding GMV-algebras by Glass, Rachunek and Winkler (2003).

In this context, we might wish to consider varieties of "ℓ-groups with strong order unit".

[Holland] The ℓ-variety of normal-valued ℓ-groups is no longer a maximal variety in this new context.

We next consider a natural generalisation of the fact that $(\mathbb{Z}, +)$ can be embedded in $(\mathbb{Q}, +)$. A (not necessarily abelian) group G is said to be *divisible* if for each $n \in \mathbb{Z}_+$ and $g \in G$, there is $x \in G$ with $x^n = g$. Using permutation groups (or, as we will see later, *HNN*-extensions), it is quite easy to prove that every group can be embedded in a divisible group.

B. H. Neumann posed an analogue, *ca* 1950:

(VI) Can every orderable group be embedded in a divisible orderable group ?

It is straightforward (and standard) to see that the answer is *YES* if the group is nilpotent. So we are starting out positively. The next result might lull us into great expectations.

Theorem L. [Bludov and Medvedev, 1974]

Every orderable metablian group can be embedded in a divisible such.

Unfortunately, there is no reason why the embedding should preserve the order. However, with a lot more work, this can be achieved.

Theorem M. [Bludov, 2003]

Every metabelian o-group can be embedded in a divisible metabelian o-group, the embedding preserving the order.

But now the Devil pulls the carpet out from under us with a centre-by-metabelian example motivated by (but easier than) the example used to prove Theorem H.

▶ **THEOREM N.** [Bludov, 2005]

There is a centre-by-metabelian orderable group which cannot be embedded in any divisible orderable group.

So that hope is also brought to an abrupt end.

This leads us to consider lattice-ordered groups without any extra restrictions put on them.

(VII) What if we pass to the most general ℓ-groups?

Since things have not been too nice even in controlled cases, we might expect matters to be even worse here. Surprisingly, after some intial disappointments, all will "be nice in the garden" — at least temporarily.

In group theory, one important construction involves free products with an amalgamated subgroup or, its equivalent, Higman-Neumann-Neumann (*HNN*)-extensions.

Algebras are said to satisfy the *amalgamation property* if for every G, H_1 and H_2 and injections σ_1, σ_2 of G into H_1 and H_2 respectively, we can find L and injections τ_1, τ_2 so that the diagram commutes:

$$\begin{tikzcd} G \arrow[r, tail, "\sigma_2"] \arrow[d, tail, "\sigma_1"'] \arrow[dr, phantom, "///"] & H_2 \arrow[d, tail, dashed, "\tau_2"] \\ H_1 \arrow[r, tail, dashed, "\tau_1"'] & L \end{tikzcd}$$

For groups we can always find such an L so that the only identification comes from G; *i.e.*, if $h_1\tau_1 = h_2\tau_2$, then there is $g \in G$ such that $h_j = g\sigma_j$ ($j = 1, 2$). In this case L is called *the free product of H_1 and H_2 with amalgamated subgroup G*. From it one obtains the existence of *HNN*-extensions.

Using *HNN*-extensions one can prove the promised

Theorem O. [Higman, Neumann, Neumann]

(i) *Every group can be embedded in a divisible group.*
(ii) *Every group can be embedded in one in which any two elements of the same order are conjugate.*

and, by a special form of *HNN*-extensions called Britton extensions, one can show

Theorem P. [Novikov], [Boone], [Britton] *There is a finitely presented group with insoluble word problem.*

More perspicuously

► **THEOREM Q.** [Higman, 1961]

A finitely generated group occurs as a subgroup of a finitely presented group if and only if it is defined by a recursively enumerable set of words.

Group theory swallows up recursion theory.

Emil Artin and Philip Hall gave proofs of the existence of *HNN*-extensions using permutation groups and W. Charles Holland proved an analogue of Cayley'sTheorem:

► **THEOREM R.** [Holland, 1963]

Every ℓ-group can be embedded in an ℓ-group of order-preserving permutations of some totally ordered set, the order on the permutation group being pointwise. The totally ordered set can be taken to be the rational (and hence real) line if the ℓ-group is countable.

Since the amalgamation property *fails* for ℓ-groups [K. R. Pierce, 1972], this may be the only way to proceed.

Using Holland's Theorem, one can directly prove the analogue of Theorem O(i):

Theorem S. [Holland, 1963] *Every ℓ-group can be embedded in a divisible ℓ-group.*

Holland's Theorem and more intricate constructions are needed to establish

Theorem T. [K. R. Pierce, 1972] *Every ℓ-group can be embedded in one in which any two strictly positive elements are conjugate.*

and

Theorem U. [Glass and Gurevich, 1983] *There is a 2-generator 1-relator lattice-ordered group with insoluble (group) word problem.*

This should be compared with the group-theoretic case (Theorem P) which superficially looks very similar. In contrast, every finitely generated *one-relator* group has soluble word problem by Magnus' Freiheitssatz.

So, on this one occasion, ℓ-groups look nicer.

(VIII) What about an analogue of Higman's Theorem?

By general nonsense, if a class of algebras is finitely axiomatisable, then every finitely generated subalgebra of a finitely presented algebra must be definable by a recursively enumerable set of words. The key is always to prove the converse.

The first result in this direction for ℓ-groups involves the easiest class of ℓ-groups, those abelian ℓ-groups of finite rank.

Theorem V. [Glass and Marra, 2003]

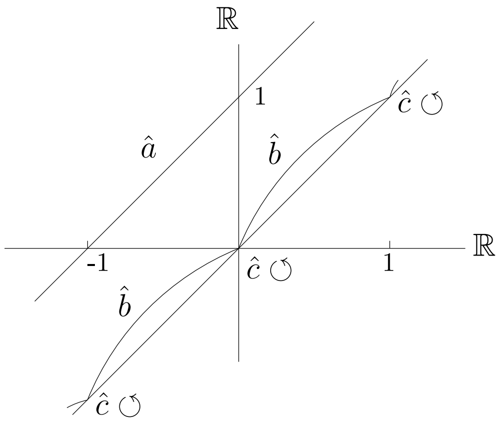
Every abelian lattice-ordered group of finite abelian rank that is defined by a recursively enumerable set of words occurs as a sublattice subgroup of a finitely presented lattice-ordered group.

Let me illustrate with a special example.

Consider the additive group $G = \mathbb{Z} \oplus \mathbb{Z}$ ordered by:

$$(m,n) > 0 \quad \text{iff} \quad (m > 0 \textit{ or } n > 0 = m),$$

the lexicographic ordering. This is generated by $x = (1,0)$ and $y = (0,1)$ with defining relations $x \geq y^n \geq 1$ $(n \in \mathbb{Z}_+)$ and $[x,y] = 1$.



We can represent G in $Aut(\mathbb{R}, \leq)$ as the sublattice subgroup generated by $\hat{a}$ and $\hat{b}$, where $\hat{a} : \alpha \mapsto \alpha + 1$ and $\hat{b} > 1$ has period 1, fixes 0 and 1 and has $(0,1)$ as an interval of support. For example, the restriction of $\hat{b}$ to $(k, k+1)$ is given by: $\alpha\hat{b} = (\sqrt{\alpha - k}) + k$ $(\alpha \in (k, k+1),\ k \in \mathbb{Z})$. An easy check shows that G is indeed isomorphic to this sublattice subgroup of $Aut(\mathbb{R}, \leq)$. One can construct $\hat{c} \in Aut(\mathbb{R}, \leq)$ so that $\hat{c}$ simultaneously conjugates $\hat{b}$ to $\hat{b}^2$ and $\hat{a}$ to itself. Let $\hat{H}$ be the sublattice subgroup of $Aut(\mathbb{R}, \leq)$ generated by $\hat{a}, \hat{b}, \hat{c}$.

Let H be the lattice-ordered group on generators a, b, c subject to the relations:

$$1 \leq b \leq a, \quad [a,b] = 1, \quad c^{-1}ac = a, \quad c^{-1}bc = b^2.$$

Then

$$1 \leq b^n \leq b^{2^n} = c^{-n}bc^n \leq c^{-n}ac^n = a \quad \text{for all } n \in \mathbb{Z}_+.$$

Thus the natural map $a \mapsto \hat{a}$, $b \mapsto \hat{b}$, $c \mapsto \hat{c}$, is a surjective homomorphism from H to $\hat{H}$. Since the representation of G in $\hat{H}$ is injective, the natural homomorphism of G into H (mapping x to a and y to b) must also be injective. Therefore we have an embedding of the finitely generated recursively defined ℓ-group G into the finitely generated finitely defined ℓ-group H. This is a specific instance of Theorem V.

Let ξ be an irrational real number and $D(\xi)$ be the group $\mathbb{Z} \oplus \mathbb{Z}$ ordered by:

$$(m,n) > 0 \quad \text{iff} \quad m - n\xi > 0.$$

Then $D(\xi)$ is finitely generated. It is defined by a recursively enumerable set of relations iff ξ is a computable real number.

For example, the defining relations for $D(\sqrt{2})$ are:

$$x < y < 2x, \quad 14x < 10y < 15x, \quad 141x < 100y < 142x, \ldots.$$

By Theorem V, if ξ is a computable real number, then the ℓ-group $D(\xi)$ is defined by a recursively enumerable set of relations. Hence it can be embedded in a finitely generated finitely related ℓ-group $L(\xi)$. Indeed, $L(\xi)$ is defined by a single relation $w_\xi(x_1, \ldots, x_\ell)$. We therefore have

$$w_\xi(x_1, \ldots, x_\ell) = 1 \rightarrow (x_1^m x_2^n > 1 \text{ iff } m > n\xi).$$

In this sense, we have captured ξ by an "ℓ-group polynomial" w_ξ, and we say that ξ is *ℓ-algebraic.* Thus, in the natural language of $(+, \leq, 0)$ (with $+$ allowed to be non-commutative and $\leq$ changed to the lattice operations $\vee$, $\wedge$ to make the relation algebraic), we have precisely captured the computable real numbers; they are precisely the real numbers that are ℓ-algebraic. So π is ℓ-algebraic (*sic*) and Theorem V implies

► **THEOREM W.** [*ibid.*] *ξ is a computable real number iff it is ℓ-algebraic.*

(VIII′) But what about the full analogue of Higman's Theorem?

I had the great fortune to be on leave in Florence for Lent Term 2004 at the kind invitation of Daniele Mundici. Since it was Lent, wrestling with the Devil seemed appropriate. I was able to extend the analogue of Higman's Theorem to cover arbitrary finitely generated ℓ-groups that were defined by recursively enumerable sets of *group* words, and then further extend that result to a near-analogue for arbitrary finitely generated ℓ-groups that were defined by recursively enumerable sets of "left strings". At Eastertide this year, I at last understood that what I had shown was enough and deduced

► **THEOREM X.** [Glass, 2005]

A finitely generated lattice-ordered group occurs as a sublattice subgroup of a finitely presented lattice-ordered group if and only if it is defined by a recursively enumerable set of relations.

Consequently,

Theorem Y. *There is a universal finitely presented lattice-ordered group. Every finitely generated ℓ-group defined by a recursively enumerable set of relations is embeddable in it.*

cf., the universal Turing machine.

So, for once, the Devil didn't win.

On that rare positive note, I think I'll end. Thank you.

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