Automorphism Groups of Models of First order Theories.

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To A. L. S. Corner, in memoriam.

Abstract

Some years ago (see [5], Problem 12.14), we asked if every first order theory having infinite models has a model whose automorphism group has undecidable theory. In this note, we prove that this is true in a strong form. It will be a consequence of our other theorem that there is a right orderable finitely presented group with insoluble word problem.

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1 Introduction

Recall that a group G is *right orderable* if there is a total order $\leq$ on G such that $f \leq g$ in G implies that $fh \leq gh$ for all $h \in G$.

We will prove

Theorem A *There is a right orderable finitely presented group with insoluble word problem.*

and deduce

Theorem B *Every first order theory having infinite models has a model whose automorphism group has undecidable existential theory. Moreover, such models exist in all infinite cardinals which are at least the cardinality of the language.*

2 Proof that Theorem A implies B.

As noted in [6], Example 1.2.3, if G is any right ordered group, then G can be embedded (as a group) in $\text{Aut}(G, \leq)$. But if $(\Omega, \leq)$ is any countable totally ordered set, then the group $\text{Aut}(\Omega, \leq)$ can be embedded in $\text{Aut}(\mathbb{Q}, \leq)$, where $(\mathbb{Q}, \leq)$ is the set of rational numbers with the usual standard ordering (see [3], Theorem 2.J). Hence we obtain the following (folklore):

Lemma 2.1 *Every countable right orderable group can be embedded in $\text{Aut}(\mathbb{Q}, \leq)$.*

We now use this to deduce Theorem B from Theorem A.

Proof: Let G be any right orderable finitely presented group with insoluble word problem; say,

$$G := \langle x_1, \dots, x_m : u_1(x_1, \dots, x_m) = 1, \dots, u_n(x_1, \dots, x_m) = 1 \rangle.$$

For any word $w(x_1, \dots, x_m)$, let ψ_w be the sentence

$$\forall x_1, \dots, x_m \left[\left(\bigwedge_{i=1}^n u_i(x_1, \dots, x_m) = 1 \right) \rightarrow w(x_1, \dots, x_m) = 1 \right].$$

Let H be any group containing G . If $H \models \psi_w$, then $G \models \psi_w$ since ψ_w is a universal sentence ([2], Theorem 5.2.4). But each $u_j(x_1, \dots, x_m) = 1$ in G ($j = 1, \dots, n$), so $w(x_1, \dots, x_m) = 1$ in G if $H \models \psi_w$. Conversely, suppose

that $w(x_1, \dots, x_m) = 1$ in G . Let $h_1, \dots, h_m \in H$ with $u_j(h_1, \dots, h_m) = 1$ in H ($j = 1, \dots, n$). Then the subgroup of H generated by $h_1, \dots, h_m$ is a homomorphic image of G . Hence $w(h_1, \dots, h_m) = 1$ in H . Thus $H \models \psi_w$. So $w(x_1, \dots, x_m) = 1$ in G if and only if $H \models \psi_w$. Since the word problem for G is insoluble, the universal theory of H is undecidable. Therefore the existential theory of H must also be undecidable. By Lemma 2.1, we may take H to be $\text{Aut}(\mathbb{Q}, \leq)$; so the latter has undecidable existential theory. But if T is any first order theory having infinite models, then it has a model $\mathcal{M}$ built using Skolem functions with indiscernibles $(\mathbb{Q}, \leq)$; moreover, every element of $\text{Aut}(\mathbb{Q}, \leq)$ extends uniquely to an element of $\text{Aut}(\mathcal{M})$ (see, [2], Section 3.3). We therefore have embeddings of G into $\text{Aut}(\mathbb{Q}, \leq)$ and $\text{Aut}(\mathbb{Q}, \leq)$ into $\text{Aut}(\mathcal{M})$. By letting H above be $\text{Aut}(\mathcal{M})$, we see that $\text{Aut}(\mathcal{M})$ has undecidable existential theory. Consequently, $\mathcal{M}$ is the desired the model of T and the first part of Theorem B follows.

For the general case, let $\mathcal{L}$ be the language of T and $\kappa \geq ||\mathcal{L}||$ be an infinite cardinal thought of as an ordinal. Let $(X, \leq)$ be the disjoint union of $\mathbb{Q}$ and κ , totally ordered by letting every rational number be less than every element of κ . Build a model $\mathcal{M}$ of T using Skolem functions and indiscernibles $(X, \leq)$. It has cardinality κ . Now $\text{Aut}(\mathbb{Q}, \leq) = \text{Aut}(X, \leq)$ which can be embedded in $\text{Aut}(\mathcal{M})$. By the same proof as in the previous paragraph, $\text{Aut}(\mathcal{M})$ has undecidable existential theory. This completes the deduction of Theorem B from Theorem A. //

3 Proof of Theorem A.

Let m be a positive integer with $m \geq 2$, and F_m be the free group on m free generators. Let $P := F_m \times F_m$. Then there is a finitely generated subgroup H of P such that the membership problem for H (in P) is insoluble ([7], Theorem 4.3). That is, there is no algorithm which determines for an arbitrary element of P whether it belongs to H or not. Let H be generated by $h_1, \dots, h_n$, say. Consider the group

$$G := \langle P, t : [t, h_i] = 1 \ (i = 1, \dots, n) \rangle.$$

We establish Theorem A by proving

Proposition 3.1 *G is a right orderable finitely presented group with insoluble word problem.*

Proof: The proof relies on a result of George Bergman ([1], Theorem 35):

Let $\{G_i : i \in I\}$ be a family of right orderable groups having a common subgroup S . Suppose that there is a group U and embeddings $\varphi_i : G_i \rightarrow U$ with $s\varphi_i = s\varphi_j$ (for all $s \in S$ and $i, j \in I$). Then the free product of $\{G_i : i \in I\}$ with S amalgamated is right orderable.

P can be given by the finite presentation

$$P = \langle a_1, \dots, a_m, b_1, \dots, b_m : [a_i, b_j] = 1 \ (i, j \in \{1, \dots, m\}) \rangle.$$

Hence G is finitely presented. Next, G has insoluble word problem, since otherwise we could solve the membership problem for H in P (an element $c \in P$ belongs to H iff $[t, c] = 1$ in G — [7], Section IV.2). Finally, we need to show that G is right orderable. Now any free group is orderable ([4], Example 1.3.24). Hence so is P (use the lexicographic order). Let N be the normal subgroup of G generated by P . Then N is a free product of orderable groups $\{t^{-k}Pt^k : k \in \mathbb{Z}\}$ with the same subgroup H of each amalgamated. For each $k \in \mathbb{Z}$, let $\varphi_k : t^{-k}Pt^k \rightarrow P$ be the natural isomorphism given by: $t^{-k}pt^k \mapsto p$. By our hypotheses, all φ_k ($k \in \mathbb{Z}$) agree on H . By Bergman's result (with $I = \mathbb{Z}$, $G_k = t^{-k}Pt^k$, $S = H$ and $U = P$), we get that N is right orderable. Let $\leq_N$ be such a right order on N . Every element g of G can be written uniquely in the form $g = xt^\ell$ with $x \in N$ and $\ell \in \mathbb{Z}$. Let $h = yt^{\ell'}$ with $y \in N$ and $\ell' \in \mathbb{Z}$. Define $g \leq h$ in G if and only if $\ell < \ell'$ (in $\mathbb{Z}$) or both $\ell = \ell'$ and $x \leq_N y$. This gives a right ordering on G and establishes the theorem. //

For future reference we observe that we have proved

Corollary 3.2 *If G is a right orderable group with subgroup H , then the group $\langle G, t : [t, h] = 1 \ (h \in H) \rangle$ is right orderable.*

Also, as mentioned in the proof of the deduction of Theorem B from Theorem A, we have that

Corollary 3.3 *$\text{Aut}(\mathbb{Q}, \leq)$ has undecidable existential theory.*

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