Word Problems, Embeddings, and Free Products of Right Ordered Groups with Amalgamated Subgroup

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Dedicated to Graham Higman, in memoriam, with gratitude for his research.

Abstract

We use permutation groups to give necessary and sufficient conditions for the free product of right ordered groups with amalgamated subgroup to be right orderable. We obtain several consequences answering previously posed problems and also prove the right orderable analogues of the Higman Embedding Theorem and the Boone–Higman Theorem.

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1 Introduction

The class of groups is closed under weak amalgamation. That is, if H, G_1, G_2 are groups with embeddings $\sigma_j : H \rightarrow G_j$ ($j = 1, 2$), then there is a group L and embeddings $\tau_j : G_j \rightarrow L$ ($j = 1, 2$) such that $h\sigma_1\tau_1 = h\sigma_2\tau_2$ for all $h \in H$. Indeed, we can take L to be the free product of G_1 and G_2 with precisely H amalgamated. In contrast, the classes of orderable groups and right orderable groups fail to be closed under weak amalgamation [10], though the free product of orderable groups is orderable [22] and the free product of right orderable groups is right orderable [17], Corollary *****. However, we showed that amalgamation is possible in the class of right ordered groups when either H is a cyclic group [5], or the amalgamating isomorphism preserves the order on H and H satisfies the condition $h > 1 \Rightarrow g^{-1}hg > 1$ for all $h \in H, g \in G_1 \cup G_2$ [6].

We use infinite permutation groups to give necessary and sufficient conditions for the free product of right orderable groups with amalgamated subgroup to be right orderable (see Theorem A). This result provides a new tool to obtain right orderable free products with amalgamated subgroups and so solve several problems. For example, Theorem B, Example 6.1, Corollaries 5.2, 5.3 and 5.9 answer Problems 2.2–2.6 (Section 6.2) of [18] and Corollary 5.5 answers Problem 2.13 of [18] for abelian subgroups of right ordered groups. We also prove that the finitely presented Boone–Britton group in [8] with insoluble word problem is right orderable (Theorem C and Corollary 7.1), and obtain the full right orderable analogues of the Higman Embedding Theorem and the Boone–Higman Theorem. These are, respectively,

Theorem D: A finitely generated group can be embedded in a right orderable finitely presented group iff it is right orderable and can be defined by a recursively enumerable set of relations.

Theorem E: A finitely generated right orderable group has soluble word problem iff it can be embedded in a simple right orderable group which can be embedded in a finitely presented right orderable group.

The key technical result needed to prove Theorem A concerns extensions of representations (see Lemma 4.4).

2 Background

Preliminary facts can be found in [9, 10, 15, 16, 17, 20].

As is standard, if G_i is a group with subgroup H_i ($i = 1, 2$), we will write $G_1 * G_2 (H_1 \overset{\varphi}{\cong} H_2)$ for the free product of G_1 and G_2 with subgroups H_1 and H_2 amalgamated by the isomorphism φ . If $G_1 = G_2 = G$ and $H_1 = H_2 = H$ with φ the identity isomorphism, we will simply write $G * G (H = H)$ for this group. In general, we write $*_{i \in I} G_i (H_i \overset{\varphi_i}{\cong} H)$ for the free product of the family of groups $\{G_i \mid i \in I\}$ with amalgamated subgroups $\{H_i \mid i \in I\}$ under the isomorphisms $H_i \overset{\varphi_i}{\cong} H$ ($i \in I$). As is also standard, we write $\langle G, t \mid h_1^t = h_1^\varphi (h_1 \in H_1) \rangle$ for the HNN-extension of a group G with isomorphic subgroups $H_1 \overset{\varphi}{\cong} H_2$.

Recall that a group G is said to be *right orderable* if there is a total order $\leq$ on G such that $fh \leq gh$ whenever $f \leq g$ ($f, g, h \in G$). If $\leq$ is a right order that is also preserved by multiplication on the left, we will say that $\leq$ is a *two-sided order* on G .

If G is a group, let $f^g := g^{-1}fg$; if G is right ordered, let $|h| := \max\{h, h^{-1}\}$ and write $f \ll g$ iff $f^n < g$ for all $n \in \mathbb{Z}$.

Throughout this paper, if G is a right ordered group with right order $\leq$, we will write $(G, \leq)$ for the *set* G totally ordered by $\leq$.

A subgroup H of a right ordered group G is called *convex* if $h_1, h_2 \in H$ and $g \in G$ with $h_1 \leq g \leq h_2$ imply that $g \in H$. A subgroup H of a right orderable group G is called *right relatively convex* if it is convex in some right order on G .

We write $P = \{g \in G \mid g \geq 1\}$ for the *positive cone* of a right ordered group G . P is a subsemigroup of G such that $P \cup P^{-1} = G$ and $P \cap P^{-1} = \{1\}$. There is a one-to-one correspondence between right orderings on a group and their positive cones: given a subsemigroup P of G such that $P \cup P^{-1} = G$ and $P \cap P^{-1} = \{1\}$, define $g_1 \leq g_2$ if and only if $g_1 g_2^{-1} \in P$. The right order is two-sided iff $P^g = P$ for all $g \in G$.

Right orderability is a local property (see [10] Lemma 2.2.2, or [17] Lemma 3.1.1). We will use this fact in the following form:

If a partial right order $\leq$ of a group G does not extend to a right order on G , then there is a finitely generated subgroup H of G such that the

induced partial right order $\leq_H$ on H does not extend to a right order on H .

We will frequently use the following well-known fact

Lemma 2.1 *If G and H are right ordered groups and $\varphi : G \rightarrow H$ is a homomorphism with kernel N , then there is a right order $\prec$ on G in which N inherits its original order and is convex, and $g \succ 1$ if $g^\varphi > 1$ (in H) for $g \in G \setminus N$.*

Proof: Let $<$ be the original right order on G . Define $g \succ 1$ iff either $g^\varphi > 1$ in H or $g \in N$ and $g > 1$. It is easy to verify that $\prec$ is a right order on G with the desired properties. //

If $(\Omega, \leq)$ is a totally ordered set, then $A(\Omega) := \text{Aut}(\Omega, \leq)$ is right orderable: well order $(\Omega, \leq)$; declare $f \prec g$ in G iff $\beta f < \beta g$, where β is the least element of $\text{supp}(gf^{-1}) = \{\alpha \in \Omega \mid \alpha gf^{-1} \neq \alpha\}$ under the well-ordering of Ω . This is a right ordering on $A(\Omega)$. Using the right regular representation, if $\leq$ is a right order on a group G , then G can be embedded (as a group) in $A(G)$. In this case, G acts fixed point free on the ordered set $(G, \leq)$. Throughout the paper we will let $A^*(\Omega)$ denote the subset of $A(\Omega)$ consisting of all automorphisms that are fixed point free; i.e., $A^*(\Omega) = \{f \in A(\Omega) \mid \alpha f \neq \alpha \text{ for all } \alpha \in \Omega\}$. If $G \leq A(\Omega)$ and $\Delta \subseteq \Omega$, then we write $G_{(\Delta)}$ for the setwise stabiliser of Δ under G .

If Γ, Δ are initial intervals (i.e., left cuts) of Ω , then $\Gamma \subset \Delta$ iff there is $\delta_0 \in \Delta \setminus \Gamma$ with $\Gamma < \delta_0$ (i.e., $\gamma < \delta_0$ for all $\gamma \in \Gamma$).

3 Orders on right orderable groups

In this section we examine four special types of right orderings on a right orderable group.

(I) *Conjugate orders.* If $\leq$ is a right order on a group G , then for any $f \in G$, define $\leq^f$ by:

$$1 \leq^f g \iff f \leq fg.$$

Then $\leq^f$ is a right order on G . We call $\leq$ and $\leq^f$ *conjugate orders*, because if P is the positive cone determined by $\leq$, then P^f is the positive cone determined by $\leq^f$.

(II) *Ultraproducts of orders.* If $\{\leq_i \mid i \in I\}$ is a family of right orders on G and $\mathcal{U}$ is an ultrafilter on I , then

$$1 \leq_{\mathcal{U}} g \iff \{i \in I \mid 1 \leq_i g\} \in \mathcal{U}$$

is a right order on G . We call $\leq_{\mathcal{U}}$ the *ultraproduct* of the family of orders $\{\leq_i \mid i \in I\}$. If P_i is the positive cone determined by $\leq_i$ ($i \in I$), then $\prod_{i \in I} P_i / \mathcal{U}$ is the positive cone determined by $\leq_{\mathcal{U}}$.

(III) *D-orders.* If $\leq_1, \leq_2$ are right orders on G and Δ is an initial interval in G under the order $\leq_1$ (i.e., a left cut in $(G, \leq_1)$), then

$$1 \leq_{\Delta} g \iff \text{either } \Delta \subset \Delta g \text{ or both } \Delta = \Delta g \text{ and } 1 \leq_2 g$$

is a right order on G . We call $\leq_{\Delta}$ the *D-order* determined by the orders $\leq_1, \leq_2$ and the initial interval Δ under $\leq_1$.

(IV) *α -orders.* Let G be a group of order-preserving permutations of an ordered set Ω . If $\alpha \in \Omega$ and $\alpha g \neq \alpha$ for all $g \in G$, then

$$1 <^{\alpha} g \iff \alpha < \alpha g \tag{1}$$

is a right order on G . We call $<^{\alpha}$ the *α -order* on G .

There are close relations between orders (I) – (IV). If G acts by right translation on $(G, \leq)$, the orders $\leq^f$ determined in (I) and (IV) are the same; and if $\Delta = \{x \in G \mid x \leq f\}$, then $\leq_{\Delta} = \leq^f$. Moreover, if $\leq$ is a right order on G , $\{\leq^g \mid g \in G\}$ is the family of all conjugate orders of $\leq$, and $\mathcal{U}$ is an ultrafilter on G containing the singleton $\{f\}$, then $\leq_{\mathcal{U}} = \leq^f$.

The next two facts are obvious.

(F1) If $\leq_{\mathcal{U}}$ is an ultraproduct of orders $\{\leq_i \mid i \in I\}$, then the conjugate order $(\leq_{\mathcal{U}})^f$ is the ultraproduct of orders $\leq_i^f$ with respect to the same ultrafilter $\mathcal{U}$.

(F2) If $\leq_{\Delta}$ is a D-order determined by orders $\leq_1, \leq_2$ and an initial interval Δ under $\leq_1$, then the conjugate order $(\leq_{\Delta})^f$ is the D-order determined by the orders $\leq_1, \leq_2^f$ and the interval Δf .

Let Ω be a totally ordered set and $G \leq A(\Omega)$. It is well known that for any initial interval $\Delta \subseteq \Omega$, the setwise stabiliser $G_{(\Delta)}$ is a right relatively convex subgroup of G . On the other hand, for any right relatively convex subgroup K of a right orderable group G , there is a totally ordered set Ω and an initial interval $\Delta \subseteq \Omega$ such that $G \leq A(\Omega)$ and $K = G_{(\Delta)}$ (see [17], Corollary 5.1.4). We will use these facts but in a more precise form:

Proposition 3.1 *Let G be a right orderable group, $\leq_1, \leq_2$ be right orders of G and Δ be an initial interval under the order $\leq_1$. Then $G_{(\Delta)}$ is convex with respect to the D -order $\leq_\Delta$ determined by the orders $\leq_1, \leq_2$ and the interval Δ .*

Proof: Assume $1 \leq_\Delta g \leq_\Delta h$ and $h \in G_{(\Delta)}$. Then $1 \leq_\Delta hg^{-1}$ and so $\Delta \subseteq \Delta hg^{-1} = \Delta g^{-1}$. Hence $\Delta g \subseteq \Delta$. Since $1 \leq_\Delta g$, we have $\Delta \subseteq \Delta g$. Thus $g \in G_{(\Delta)}$. //

Proposition 3.2 *Let G be a right ordered group with right order $\leq$, and H be a subgroup of G . Let $\leq_H$ be the restriction of $\leq$ to H . If K is a convex subgroup of H under $\leq_H$, then there is an initial interval $\Delta = \Delta(K) \subseteq G$ (under the order $\leq$) such that $G_{(\Delta)} \cap H = K$.*

Proof: Let $\Delta = \{g \in G \mid (\exists h \in K)(g \leq h)\}$ and $h \in K$. If $g \in \Delta$, then $g \leq h_1$ for some $h_1 \in K$. So $gh \leq h_1 h \in K$ and $gh^{-1} \leq h_1 h^{-1} \in K$. Thus $\Delta h \subseteq \Delta$ and $\Delta h^{-1} \subseteq \Delta$, whence $\Delta h = \Delta$. So $K \leq G_{(\Delta)} \cap H$. Conversely, let $h \in G_{(\Delta)} \cap H$. Since $G_{(\Delta)} \cap H$ is a subgroup, we may assume that $1 \leq h$. But $1 \in \Delta$. Therefore, $h = 1 \cdot h \in \Delta$. So $1 \leq h \leq h_1$ for some $h_1 \in K$. This gives $1 \leq_H h \leq_H h_1$ whence $h \in K$ (as K is convex in H under $\leq_H$). //

The relation between D -orders and ultraproduct orders constitutes an essential part of our analysis, so we examine it in detail.

Lemma 3.3 *Let G be a right orderable group and let $\leq_1, \leq_2$ be right orders on G . Let Δ be an initial interval in G under the order $\leq_1$ and let $K = G_{(\Delta)}$. Let $I \subseteq \Delta$ and $\mathcal{U} := \mathcal{U}(I)$ be an ultrafilter on I satisfying the property: for all $x \in \Delta$, there exists $J \in \mathcal{U}$ such that $x \leq_1 J$. Then K is convex in the ultraproduct of the orders $\leq_1^f$ ($f \in I$). The order $\leq_{K, \mathcal{U}}$ given by*

$$1 \leq_{K, \mathcal{U}} g \iff \begin{cases} \{f \in I \mid 1 \leq_1^f g\} \in \mathcal{U} & \text{for } g \in G \setminus K \\ 1 \leq_2 g & \text{for } g \in K \end{cases} \quad (2)$$

coincides with $\leq_\Delta$ determined by the orders $\leq_1, \leq_2$ and initial interval Δ under $\leq_1$.

Proof: Assume $x \in G \setminus K$. If $\Delta x \subset \Delta$, there is $a \in \Delta \setminus \Delta x$ and so $\Delta x <_1 a$. By the hypothesis of the lemma, there exists $J \in \mathcal{U}$ such that $a \leq_1 J$. This gives $\Delta x <_1 J$. Since $J \subseteq I \subseteq \Delta$, we get $bx <_1 b$ for all $b \in J$. So $\Delta x \subset \Delta$ implies $x <_{K, \mathcal{U}} 1$. If, on the other hand, $\Delta \subset \Delta x$, then there is $a \in \Delta$ with $ax \in \Delta x \setminus \Delta$; so $\Delta <_1 ax$. By the hypothesis of the lemma, there exists $J \in \mathcal{U}$ such that $a \leq_1 J$. This gives $\Delta <_1 Jx$ and so $b <_1 bx$ for all $b \in J$. Thus $\Delta \subset \Delta x$ implies $1 <_{K, \mathcal{U}} x$. Consequently, the positive cones of the orders $\leq_\Delta$ and $\leq_{K, \mathcal{U}}$ coincide on $G \setminus K$. By (2), the positive cones of the orders $\leq_\Delta$ and $\leq_{K, \mathcal{U}}$ coincide on K as well. So $\leq_\Delta = \leq_{K, \mathcal{U}}$.

By Proposition 3.1, K is convex in this order. //

Definition 3.4 A family $\mathcal{R}$ of right orders on a group G is called:

normal (or *G-invariant*) if it contains all orders conjugate to orders in $\mathcal{R}$; i.e., $\leq \in \mathcal{R}$ implies $\leq^g \in \mathcal{R}$ for all $g \in G$.

$\mathfrak{u}$ -closed if it closed under ultraproducts of orders from $\mathcal{R}$;

$\mathcal{D}$ -invariant if it contains all $\mathcal{D}$ -orders determined by orders from $\mathcal{R}$ and their initial intervals;

$\mathcal{A}$ -invariant (appropriate for amalgamation) if it is normal, $\mathfrak{u}$ -closed, and $\mathcal{D}$ -invariant.

Ω -invariant if it contains all α -orders when $\alpha \in (\Omega, \leq)$ and $G \leq A(\Omega)$.

Remark 3.5 If $\mathcal{R}$ contains all right orders on a right orderable group G , then $\mathcal{R}$ is $\mathcal{A}$ -invariant and Ω -invariant for any ordered set Ω on which G acts as order-preserving permutations. On the other hand, the singleton $\mathcal{R} = \{\leq\}$ may be $\mathcal{A}$ -invariant; this is the case if $\leq$ is a two-sided order on G .

Lemma 3.6 Let G be a right orderable group and $\mathcal{R}$ be a G -invariant family of right orders on G . If $\mathcal{R}$ induces $\mathcal{A}$ -invariant families $\mathcal{R}_i$ of right orders on subgroups $H_i \leq G$ ($i \in I$), then $\mathcal{R}$ is contained in an $\mathcal{A}$ -invariant family $\overline{\mathcal{R}}$ of right orders on G such that the restrictions to H_i of all orders from $\overline{\mathcal{R}}$ belong to $\mathcal{R}_i$ for each $i \in I$.

Proof: We will be interested in families $\mathcal{S}$ of right orders on G satisfying two crucial properties:

- (A1) $\mathcal{S}$ is G -invariant; and
 (A2) for each $i \in I$, the restriction to H_i of any order from $\mathcal{S}$ belongs to $\mathcal{R}_i$.

By the hypothesis on $\mathcal{R}$, the set of such $\mathcal{S}$ is non-empty. By Zorn's Lemma, there is a family of right orders on G that is maximal subject to satisfying conditions (A1) and (A2). We denote such a family by $\overline{\mathcal{R}}$. By construction, $\overline{\mathcal{R}}$ contains $\mathcal{R}$.

Let $\overline{\mathcal{R}}^U$ be the set of all ultraproduct orders from $\overline{\mathcal{R}}$. By (F1), $\overline{\mathcal{R}}^U$ is G -invariant and so satisfies (A1). Let $\leq_U$ be the ultraproduct of orders $\{\leq_j \mid j \in J\} \subseteq \overline{\mathcal{R}}$ and $P = \prod_{j \in J} P_j / \mathcal{U}$ be the positive cone for the order $\leq_U$ (where P_j is the positive cone for the order $\leq_j$). Fix $i \in I$. For each $j \in J$, $P_j \cap H_i$ defines a total order belonging to $\mathcal{R}_i$. Since $\mathcal{R}_i$ is $\mathcal{A}$ -invariant in H_i , we have $P \cap H_i = \prod_{j \in J} (P_j \cap H_i) / \mathcal{U}$ defines a total order belonging to $\mathcal{R}_i$. This holds for all $i \in I$. Thus $\overline{\mathcal{R}}^U$ satisfies (A2) and so $\overline{\mathcal{R}}^U = \overline{\mathcal{R}}$.

Now consider a family $\overline{\mathcal{R}}^D$ comprising all D -orders arising from orders belonging to $\overline{\mathcal{R}}$ and their initial intervals. By (F2), $\overline{\mathcal{R}}^D$ is G -invariant and so satisfies (A1). Let $\leq_1, \leq_2 \in \overline{\mathcal{R}}$ and Δ be an initial interval in G under the order $\leq_1$. Consider the order $\leq_\Delta \in \overline{\mathcal{R}}^D$ built from $\leq_1, \leq_2$ and Δ . Let $\Delta_g = \{a \in \Delta \mid g \leq_1 a\}$ ($g \in G$). Then $\mathcal{F} = \{\Delta_g \mid g \in \Delta\}$ generates a filter on Δ . Let $\mathcal{U}$ be an ultrafilter on Δ which contains $\mathcal{F}$. By Lemma 3.3, $G_{(\Delta)}$ is convex under $\leq_U$ and $\leq_\Delta$ coincides with the order (2). Since $\overline{\mathcal{R}}$ is $\mathfrak{u}$ -closed, we have $\leq_U \in \overline{\mathcal{R}}$ and so $\leq_U$ induces a right order $\leq_{U,i}$ on H_i which belongs to $\mathcal{R}_i$. Fix $i \in I$ and let $K_i = G_{(\Delta)} \cap H_i$. Then K_i is convex in H_i under $\leq_{U,i}$. This provides an initial interval Δ_i in H_i given by $\Delta_i := \{g \in H_i \mid (\exists h \in K_i)(g \leq_{U,i} h)\}$. Now $\leq_2$ induces a right order $\leq_{2,i}$ on H_i which belongs to $\mathcal{R}_i$. Let $\leq_{\Delta_i}$ be the D -order on H_i built from $\leq_{U,i}, \leq_{2,i}$ and Δ_i . Since $\mathcal{R}_i$ is $\mathcal{A}$ -invariant in H_i , we have $\leq_{\Delta_i} \in \mathcal{R}_i$. It only remains to verify that the restriction of $\leq_\Delta$ to H_i coincides with $\leq_{\Delta_i}$. Assume that $1 \leq_\Delta h \in H_i$. By Lemma 3.3, we get that $1 \leq_U h$ and so $1 \leq_{U,i} h$. One more application of Lemma 3.3 yields $1 \leq_{\Delta_i} h$, as required. //

Lemma 3.7 *If $\mathcal{R}$ is an $\mathcal{A}$ -invariant family of right orders on a right orderable group G , then the restrictions of the orders from $\mathcal{R}$ to any subgroup H of G is an $\mathcal{A}$ -invariant family $\mathcal{R}_H$ of right orders on H .*

Proof: Let $\leq_H$ be the restriction to H of an order $\leq$ belonging to $\mathcal{R}$. Then $\leq_H^h$ is the restriction of $\leq^h$ from $\mathcal{R}$. For each $i \in I$, let $\leq_{H,i}$ be the restriction of an order $\leq_i$ belonging to $\mathcal{R}$. If $\mathcal{U}$ is an ultrafilter on I , then the ultraproduct of the set of orders $\{\leq_{H,i} \mid i \in I\}$ with respect to $\mathcal{U}$ is the restriction of the ultraproduct of the set of orders $\{\leq_i \mid i \in I\}$ with respect to the same ultrafilter. Let $\leq_{H,1}, \leq_{H,2}$ be the restriction of orders $\leq_1, \leq_2$ from $\mathcal{R}$ to H , and Δ_1 be an initial interval in H under $\leq_{H,1}$. Let $\leq_{\Delta_1}$ be the D -order on H built from $\leq_{H,1}, \leq_{H,2}$ and Δ_1 . Let Δ be the initial interval in G generated by Δ_1 ; *i.e.*, $\Delta := \{g \in G \mid (\exists h \in \Delta_1)(g \leq h)\}$. Let $\leq_\Delta$ denote the order on G built from $\leq_1, \leq_2$ and Δ . Let $h \in H$ and assume $h \in G_{(\Delta)}$. Since $\Delta_1 \subseteq \Delta$, we get $xh \in \Delta$ for all $x \in \Delta_1$. Thus for all $x \in \Delta_1$, we have $xh \leq h_1$ for some $h_1 = h_1(x) \in H$. Since $xh \in H$, it follows that $xh \in \Delta_1$. Hence $h \in G_{(\Delta)} \cap H$ implies $h \in H_{(\Delta_1)}$. Now assume $\Delta \subset \Delta h$. In this case $\Delta < gh$ for some $g \in \Delta$. So $g \leq h_1 \in H$ for some $h_1 \in \Delta_1$, whence $\Delta < h_1 h$. But $\Delta_1 \subseteq \Delta$. Thus $(\Delta \subset \Delta h \text{ and } h \in H)$ implies $\Delta_1 \subset \Delta_1 h$. By the definition of D -orders, we deduce that $\leq_{\Delta_1}$ is the restriction of $\leq_\Delta$ to H . Since $\mathcal{R}_H$ is H -invariant, $\mathfrak{u}$ -closed, and $\mathcal{D}$ -invariant, it is $\mathcal{A}$ -invariant. //

4 Necessary and sufficient conditions

Assume that $L = *_{i \in I} G_i (H_i \xrightarrow{\varphi_i} H)$ is right orderable and denote by $\mathcal{R}_0$ the set of all possible right orders on L . Then $\mathcal{R}_0$ induces a family of right orders $\mathcal{R}_i$ on G_i for each $i \in I$. Since $h^{\varphi_i^{-1}} = h^{\varphi_j^{-1}}$ for all $h \in H$, it follows that if $i \in I$ and $\leq_i \in \mathcal{R}_i$, then there is $\leq_j \in \mathcal{R}_j$ such that $1 \leq_i h^{\varphi_i^{-1}}$ implies $1 \leq_j h^{\varphi_j^{-1}}$. This motivates our next definition.

Let H_i be a subgroup of a right orderable group G_i and $\mathcal{R}_i$ be a family of right orders on G_i ($i \in I$). Suppose that $\varphi_i : H_i \cong H$. Let $i, j \in I$. We say that $\varphi_{i,j} := \varphi_i \varphi_j^{-1}$ is *compatible for the pair* $(\mathcal{R}_i, \mathcal{R}_j)$ iff

$$(\forall \leq_i \in \mathcal{R}_i)(\exists \leq_j \in \mathcal{R}_j)(\forall h_i \in H_i)(1 \leq_i h_i \implies 1 \leq_j h_i^{\varphi_{i,j}}) \quad (3)$$

We also say that $\varphi_{i,j}$ is *compatible for the orders* $\leq_i, \leq_j$, or that $\leq_i$ and $\leq_j$ are *associated orders via* $\varphi_{i,j}$. If $\varphi_{i,j}$ is compatible for $(\mathcal{R}_i, \mathcal{R}_j)$ for all $i, j \in I$, we will say that $\{\mathcal{R}_i \mid i \in I\}$ is *compatible for* $\{\varphi_i \mid i \in I\}$.

Proposition 4.1 *Let G_i be a right ordered group with order $\leq_i$ ($i \in I$). Let H_i be a subgroup of G_i ($i \in I$) and suppose that each H_i is isomorphic to H (say by φ_i). Let $L = \ast_{i \in I} G_i$ ($H_i \stackrel{\varphi_i}{\cong} H$). Suppose that there are right orders $\leq_i$ on G_i that can be extended to a right order on L ($i \in I$). Then there are $\mathcal{A}$ -invariant families of right orders $\mathcal{R}_i$ on G_i with $\leq_i \in \mathcal{R}_i$ ($i \in I$) so that $\{\mathcal{R}_i \mid i \in I\}$ is compatible for $\{\varphi_i \mid i \in I\}$.*

Proof: Let $\mathcal{R}$ consist of all right orders on L and $\mathcal{R}_i$ denote the induced family of right orders on G_i ($i \in I$). As was shown above, $(\mathcal{R}_i, \mathcal{R}_j)$ is compatible for $\varphi_{i,j}$. By Remark 3.5, $\mathcal{R}$ is $\mathcal{A}$ -invariant on L and, by Lemma 3.7, $\mathcal{R}_i$ is $\mathcal{A}$ -invariant on G_i ($i \in I$). //

To illustrate Proposition 4.1, consider the following

Example 4.2 Let $G_1 = \langle a_1, b_1 \mid [b_1, a_1, a_1] = 1 = [b_1, a_1, b_1] \rangle$ and $G_2 = \langle a_2, b_2 \mid a_2 a_2^{b_2} = 1 \rangle$. So G_1 is the free nilpotent group of class two with generators a_1, b_1 . Let $c_1 = [b_1, a_1]$, $c_2 = b_2^2$, $H_1 = \langle a_1, c_1 \rangle$ and $H_2 = \langle a_2, c_2 \rangle$. Both subgroups H_1 and H_2 are free abelian groups of rank two and $\varphi : a_1 \mapsto a_2, c_1 \mapsto c_2$ defines an isomorphism $H_1 \stackrel{\varphi}{\cong} H_2$. There are right orders on G_1 and G_2 defined by, respectively,

$$1 < a_1 \ll c_1 \ll b_1, \quad \text{and} \quad 1 < a_2 \ll b_2. \quad (4)$$

Clearly, φ preserves the induced orders on H_1 and H_2 . Consider the conjugate order $\leq^{b_1}$ on G_1 . Then $b_1 < b_1 c_1 < b_1 a_1$ (because $b_1 c_1 a_1^{-1} b_1^{-1} = a_1^{-1} < 1$), and so $1 <^{b_1} c_1 <^{b_1} a_1$. But there is no right order on G_2 which extends the order $1 <_2 c_2 <_2 a_2$ from H_2 to G_2 (otherwise, $1 < b_2 < a_2$ and so $1 < a_2 b_2^{-1} a_2 b_2 = 1$). It follows that there are no compatible families of normal orders containing the initial orders on G_1 and G_2 . By the same calculation, $1 < |a_2| \ll |b_2|$ in any right order on G_2 . Thus if $G_1 \ast G_2$ ($H_1 \stackrel{\varphi}{\cong} H_2$) were right orderable, there must be a normal family $\mathcal{R}_i$ of right orders on G_i ($i = 1, 2$) with $(\mathcal{R}_1, \mathcal{R}_2)$ compatible for φ . After suitable conjugation in G_1 , we obtain $|b_1| |c_1| < |b_1| |a_1|$ and so a contradiction. Hence $G_1 \ast G_2$ ($H_1 \stackrel{\varphi}{\cong} H_2$) is not right orderable.

The first examples of a non-right orderable free product of right ordered groups with amalgamated subgroup were provided in [3]. They were based on the same reasoning as used in the above example; namely,

some conjugate right order on one of the groups does not extend (after amalgamation via φ) to any right order on the other group. In [3], Bludov asked if this is the only obstacle to $G_1 * G_2 (H_1 \cong H_2)$ being right orderable. The main problem has been to find necessary and sufficient conditions for the right orderability of $G_1 * G_2 (H_1 \cong H_2)$. We provide such conditions and show that all obstacles to right ordering the free product of right ordered groups with amalgamation indeed boil down to the compatibility of the amalgamating isomorphism φ with conjugate orders on the factor groups.

To prove our results we need two lemmata. In the first we reduce the condition of $\mathcal{A}$ -invariance to normality; and in the second we give a method to extend the right regular representations of G_1 and G_2 to fixed point free representations on a common totally ordered set.

Let $\{(\mathcal{R}_{1,s}, \mathcal{R}_{2,s}) \mid s \in S\}$ be a set of pairs of families of right orders, the first on G_1 and the second on G_2 . Inclusion provides a partial order on this set of pairs:

$$(\mathcal{R}_{1,s}, \mathcal{R}_{2,s}) \leq (\mathcal{R}_{1,t}, \mathcal{R}_{2,t}) \iff (\mathcal{R}_{1,s} \subseteq \mathcal{R}_{1,t} \text{ and } \mathcal{R}_{2,s} \subseteq \mathcal{R}_{2,t}).$$

Suppose that the set $\{(\mathcal{R}_{1,s}, \mathcal{R}_{2,s}) \mid s \in S\}$ contains all pairs of normal families of right orders which are compatible for some given isomorphism φ between subgroups $H_1 \leq G_1$ and $H_2 \leq G_2$. Consider the pair

$$(\overline{\mathcal{R}}_1, \overline{\mathcal{R}}_2) = \left(\bigcup_{s \in S} \mathcal{R}_{1,s}, \bigcup_{s \in S} \mathcal{R}_{2,s} \right). \quad (5)$$

Clearly, $\overline{\mathcal{R}}_1$ and $\overline{\mathcal{R}}_2$ are normal families of right orders for their groups and are compatible for φ . By construction, the pair $(\overline{\mathcal{R}}_1, \overline{\mathcal{R}}_2)$ is a maximal element in the set of pairs of normal families of right orders that are compatible for φ .

Lemma 4.3 *Let G_i be a right orderable group with subgroup H_i ($i \in I$). If each H_i is isomorphic to a group H , let $\varphi_i : H_i \cong H$. For each $i \in I$, let $\mathcal{R}_i$ be a normal family of right orders on G_i such that $\{\mathcal{R}_i \mid i \in I\}$ is compatible for $\{\varphi_i \mid i \in I\}$. Then $\mathcal{R}_i$ is contained in an $\mathcal{A}$ -invariant family $\overline{\mathcal{R}}_i$ of right orders on G_i for all $i \in I$ so that $\{\overline{\mathcal{R}}_i \mid i \in I\}$ is compatible for $\{\varphi_i \mid i \in I\}$.*

Proof: Since the definition of compatibility is pairwise, it is sufficient to consider the case that $I = \{1, 2\}$ and $\varphi = \varphi_1 \varphi_2^{-1}$. We prove that $(\overline{\mathcal{R}}_1, \overline{\mathcal{R}}_2)$ defined in (5) is $\mathcal{A}$ -invariant. Clearly, $\overline{\mathcal{R}}_i$ is G_i -invariant ($i = 1, 2$).

Let $\mathcal{R}_i^U$ consist of all ultraproducts of orders from $\overline{\mathcal{R}}_i$ ($i = 1, 2$). By (F1), $\mathcal{R}_i^U$ is normal in G_i ($i = 1, 2$).

Assume $1 \leq_{1, \mathcal{U}} h \in H_1$ for some $\leq_{1, \mathcal{U}} \in \mathcal{R}_1^U$, where $\leq_{1, \mathcal{U}}$ is an ultraproduct of orders $\{\leq_{1, x} \mid x \in X\} \subseteq \overline{\mathcal{R}}_1$. Then there is $Y \in \mathcal{U}$ such that $1 \leq_{1, y} h$ for all $y \in Y$. Since $(\overline{\mathcal{R}}_1, \overline{\mathcal{R}}_2)$ is compatible for φ , for each $x \in X$ there is an order $\leq_{2, x} \in \overline{\mathcal{R}}_2$ such that $1 \leq_{1, x} h$ implies $1 \leq_{2, x} h^\varphi$. Thus $1 \leq_{2, x} h^\varphi$ for all $x \in Y$. For the same ultrafilter $\mathcal{U}$ on X , $1 \leq_{2, \mathcal{U}} h^\varphi$, where $\leq_{2, \mathcal{U}} \in \mathcal{R}_2^U$ is the ultraproduct of orders $\{\leq_{2, x} \mid x \in X\} \subseteq \overline{\mathcal{R}}_2$. So $(\mathcal{R}_1^U, \mathcal{R}_2^U)$ is compatible for φ . Since $(\overline{\mathcal{R}}_1, \overline{\mathcal{R}}_2)$ is a maximal pair of normal orders that is compatible for φ , we have $\mathcal{R}_1^U \subseteq \overline{\mathcal{R}}_1$ and $\overline{\mathcal{R}}_1$ is $\mathfrak{U}$ -closed. Similarly, $\overline{\mathcal{R}}_2$ is $\mathfrak{U}$ -closed.

Let $\mathcal{R}_i^D$ consist of all D -orders arising from the orders belonging to $\overline{\mathcal{R}}_i$ ($i = 1, 2$). By (F2), $\mathcal{R}_i^D$ is normal in G_i ($i = 1, 2$). Let $\leq_{1,1}, \leq_{1,2} \in \overline{\mathcal{R}}_1$ and Δ be an initial interval in G_1 under order $\leq_{1,1}$. Consider the order $\leq_\Delta \in \mathcal{R}_1^D$ built from $\leq_{1,1}, \leq_{1,2}$ and Δ . Let $\Delta_g = \{a \in \Delta \mid g \leq_{1,1} a\}$ ($g \in G_1$). Then $\mathcal{F} = \{\Delta_g \mid g \in \Delta\}$ generates a filter on Δ . Let $\mathcal{U}$ be an ultrafilter on Δ which contains $\mathcal{F}$. By Lemma 3.3, $(G_1)_{(\Delta)}$ is convex under $\leq_\mathcal{U}$ and $\leq_\Delta$ coincides with the order (2). Since $\overline{\mathcal{R}}_1$ is $\mathfrak{U}$ -closed, we have $\leq_\mathcal{U} \in \overline{\mathcal{R}}_1$. Let $\leq_{2,1}$ be an order in $\overline{\mathcal{R}}_2$ associated with $\leq_\mathcal{U}$ via φ . Let $T_1 = (G_1)_{(\Delta)} \cap H_1$; then T_1 is convex in H_1 . By (3), $T_2 = T_1^\varphi$ is convex in $H_2 = H_1^\varphi$ under $\leq_{2,1}$. Let $\Delta^{(\varphi)} = \{g \in G_2 \mid (\exists h \in T_2)(g \leq_{2,1} h)\}$ be the initial interval in G_2 generated by T_2 . If $a \in \Delta^{(\varphi)}$, then $a \leq_{2,1} h_2$ for some $h_2 \in T_2$. Hence $ah \leq_{2,1} h_2 h$ and $ah^{-1} \leq_{2,1} h_2 h^{-1}$ for all $h \in T_2$. Thus $\Delta^{(\varphi)} T_2 = \Delta^{(\varphi)}$ and $T_2 \leq S_2 := (G_2)_{(\Delta^{(\varphi)})}$. If $f \in S_2 \cap H_2$ and $1 \leq_{2,1} f$, then $f = 1 \cdot f \in \Delta^{(\varphi)}$ and so $1 \leq_{2,1} f \leq_{2,1} h$ for some $h \in T_2$. Since T_2 is convex in H_2 , we deduce that $f \in T_2$; so $S_2 \cap H_2 = T_2$. Now take an order $\leq_{2,2} \in \overline{\mathcal{R}}_2$ associated with $\leq_{1,2} \in \overline{\mathcal{R}}_1$ via φ and form the order $\leq_{\Delta^{(\varphi)}} \in \mathcal{R}_2^D$ using $\leq_{2,1}, \leq_{2,2}$ and $\Delta^{(\varphi)}$. By construction, $1 \leq_\Delta h_1$ implies $1 \leq_{\Delta^{(\varphi)}} h_1^\varphi$ for all $h_1 \in H_1$. Similarly, for every $\leq_\Delta \in \mathcal{R}_2^D$, there exists $\leq_{\Delta^{(\varphi^{-1})}} \in \mathcal{R}_1^D$ such that $1 \leq_\Delta h_2$ implies $1 \leq_{\Delta^{(\varphi^{-1})}} h_2^{\varphi^{-1}}$ for all $h_2 \in H_2$. Thus $(\mathcal{R}_1^D, \mathcal{R}_2^D)$ is compatible for φ . This gives $\mathcal{R}_i^D \subseteq \overline{\mathcal{R}}_i$ ($i = 1, 2$); so $\overline{\mathcal{R}}_1$ and $\overline{\mathcal{R}}_2$ are $\mathcal{D}$ -invariant. Consequently, $\overline{\mathcal{R}}_1$ and $\overline{\mathcal{R}}_2$ are $\mathcal{A}$ -invariant. //

We now prove an important representation lemma.

Lemma 4.4 *Let $(\Omega_2 \leq)$ be a totally ordered set and $\Omega_1 \subseteq \Omega_2$. Let $G \leq A^*(\Omega_1)$ be a right orderable group with an $\mathcal{A}$ -invariant family $\mathcal{R}$ of right orders. Suppose that the α -order on G belongs to $\mathcal{R}$ for all $\alpha \in \Omega_1$. Let H be a subgroup of G that can be represented in $A^*(\Omega_2)$ so that the α -orders on H extend to orders on G belonging to $\mathcal{R}$ (for all $\alpha \in \Omega_2$) and the restriction of H to Ω_1 coincides with the induced action of H as a subgroup of G . Then there exists a totally ordered set $\Omega_3 \supseteq \Omega_2$ such that the action of G on Ω_1 extends to Ω_3 with $G \leq A^*(\Omega_3)$, and the α -orders on G belong to $\mathcal{R}$ for all $\alpha \in \Omega_3$.*

Proof: Since $\Omega_1 H = \Omega_1$, the H -orbits βH ($\beta \in \Omega_2 \setminus \Omega_1$) are disjoint from Ω_1 . Let $\{\beta_j : j \in J\}$ be chosen so that $\{\beta_j H \mid j \in J\}$ partitions $\Omega_2 \setminus \Omega_1$. Well order J . We construct two chains of totally ordered sets $\{\Omega_{1,j} : j \in J\}$ and $\{\Omega_{2,j} : j \in J\}$ with $\Omega_{s,j} \supseteq \Omega_s$ ($s = 1, 2, j \in J$) such that

$$\Omega_{1,i} \subseteq \Omega_{1,j} \iff i \leq j \quad (6)$$

$$\Omega_{2,i} \subseteq \Omega_{2,j} \iff i \leq j \quad (7)$$

and

$$\beta_j \in \Omega_{1,j} \subseteq \Omega_{2,j} \quad \text{for all } j \in J. \quad (8)$$

In addition we will ensure that:

(I) the order on $\Omega_{2,j}$ extends the order on $\Omega_{1,j}$ for all $j \in J$; and for all $i \leq j$, the orders on $\Omega_{1,i}$, $\Omega_{2,i}$ are the restrictions of the orders on $\Omega_{1,j}$ and $\Omega_{2,j}$ respectively;

(II) $G \leq A^*(\Omega_{1,j})$ for all $j \in J$; if $i \leq j$, $g \in G$ and $\alpha \in \Omega_{1,i} \subseteq \Omega_{1,j}$, then αg is the same point in $\Omega_{1,i}$ and $\Omega_{1,j}$;

(III) $H \leq A^*(\Omega_{2,j})$ for all $j \in J$; if $i \leq j$, $h \in H$ and $\alpha \in \Omega_{2,i} \subseteq \Omega_{2,j}$ (or $\alpha \in \Omega_{1,i} \subseteq \Omega_{2,j}$), then αh is the same point in $\Omega_{2,i}$ and $\Omega_{2,j}$ (and $\Omega_{1,i}$).

(IV) α -orders on G belong to $\mathcal{R}$ for all $\alpha \in \Omega_{1,j}$ and α -orders on H extend to orders on G belonging to $\mathcal{R}$ for all $\alpha \in \Omega_{2,j}$.

We set up some notation for the construction.

Let j_0 be the least element of J . Let

$$\Omega'_{1,j_0} := \Omega_1 \quad \text{and} \quad \Omega'_{2,j_0} := \Omega_2.$$

Suppose that $\Omega_{1,i}$ and $\Omega_{2,i}$ have been defined for all $i < j$ and satisfy (6), (7), (8), and (I) – (IV). For $j = i + 1$, let

$$\Omega'_{1,j} := \Omega_{1,i} \quad \text{and} \quad \Omega'_{2,j} := \Omega_{2,i};$$

and for a limit point $j \in J$, let

$$\Omega'_{1,j} := \bigcup_{i < j} \Omega_{1,i} \quad \text{and} \quad \Omega'_{2,j} := \bigcup_{i < j} \Omega_{2,i}.$$

We extend the total orders and define the actions of G and H on these sets in the natural way. Note that (IV) holds with $\Omega'_{s,j}$ in place of $\Omega_{s,j}$ ($s = 1, 2$).

Induction Step:

Let $\beta_j G$ be an isomorphic copy of the set G ,

$$\Omega_{1,j} := \Omega'_{1,j} \cup \beta_j, \quad \text{and} \quad \Omega_{2,j} := \Omega'_{2,j} \cup \beta_j(G \setminus H).$$

We now establish orders on $\Omega_{s,j}$ and actions of G and H on $\Omega_{s,j}$ ($s = 1, 2$).

Let $\Delta_j := \{\alpha \in \Omega'_{1,j} : \alpha < \beta_j\}$ and $\Delta_j^+ := \{\alpha \in \Omega'_{1,j} : \alpha > \beta_j\}$. If $\Delta_j \neq \emptyset$, set $K_j := \{g \in G : \Delta_j g = \Delta_j\} = G_{(\Delta_j)}$. If $\Delta_j = \emptyset$, set $K_j := G$. Since $H \leq A^*(\Omega'_{2,j})$, the β_j -order on H given by

$$1 \leq_j h \iff \beta_j \leq \beta_j h \text{ in } \Omega_2 \tag{9}$$

is a right order $\leq_j$ on H . This order extends to a right order $\leq_j$ on G belonging to $\mathcal{R}$ by the hypotheses of the lemma (and (IV) if $j > j_0$). We obtain a D -order $\leq_{\Delta_j}$ on G by

$$1 <_{\Delta_j} g \iff \begin{cases} \Delta_j \subset \Delta_j g & \text{for } g \in G \setminus K_j \\ 1 \leq_j g & \text{for } g \in K_j \end{cases}$$

Since $\mathcal{R}$ is $\mathcal{D}$ -invariant, we have $\leq_{\Delta_j} \in \mathcal{R}$.

We use the $\leq_{\Delta_j}$ -order on G to order $\beta_j G$; so

$$\beta_j g_1 \leq \beta_j g_2 \iff g_1 \leq_{\Delta_j} g_2. \quad (g_1, g_2 \in G). \tag{10}$$

We see that the total order (10) is compatible with (9). For if $h \in H$ and $\Delta_j \subset \Delta_j h$, then $\beta_j < \beta_j h$; and if $\Delta_j = \Delta_j h$, then $h \in K_j$. So $\beta_j \leq \beta_j h$

iff $1 \leq_j h$ by (9). Moreover, $G \leq A^*(\beta_j G)$ and the α -orders on G belong to $\mathcal{R}$ for all $\alpha \in \beta_j G$ by construction.

In order to order $\Omega_{1,j}$, we must “insert” the points $\beta_j g$ ($g \in G$) between points of $\Omega'_{1,j}$. Consider the right cosets $K_j g_t \subseteq G$ (where $\{g_t : t \in T_j\}$ is a right transversal of K_j in G) and the cuts $(\Delta_j g_t, \Delta_j^+ g_t)$ in $\Omega'_{1,j}$. For $g \in K_j g_t \setminus H$, we put:

$$\Delta_j g_t < \beta_j g < \Delta_j^+ g_t. \quad (11)$$

This, together with (10), gives a total order on $\Omega'_{1,j} \cup \beta_j G$.

So we have “inserted” the points of $\beta_j G$ between appropriate cuts in $\Omega'_{1,j}$. We now need to insert them at appropriate places between the points of $\Omega_{2,j} \setminus \Omega'_{2,j}$ and so order $\Omega_{2,j}$.

Let $g \in G \setminus H$. So $g \in K_j g_t \setminus H$ for some fixed $t \in T_j$. Let

$$\begin{aligned} H_{j,t} &= H \cap K_j g_t = \{h \in H : \Delta_j g_t < \beta_j h < \Delta_j^+ g_t\}, \\ H_{j,t}^-(g) &= \{h \in H_{j,t} : h <_{\Delta_j} g\} \quad \text{and} \\ H_{j,t}^+(g) &= \{h \in H_{j,t} : g <_{\Delta_j} h\}. \end{aligned}$$

If $\Gamma \subseteq \Omega'_{1,j}$, let $\text{Conv}(\Gamma)$ be the convexification of Γ in $\Omega'_{2,j}$; i.e., $\lambda \in \text{Conv}(\Gamma)$ iff there are $\gamma_1, \gamma_2 \in \Gamma$ such that $\gamma_1 \leq \lambda \leq \gamma_2$. We extend the order on $\Omega'_{2,j}$ by setting

$$\alpha_1 \leq \beta_j h_1 < \beta_j g < \alpha_2 \leq \beta_j h_2 \leq \alpha_3 \quad (12)$$

for all $\alpha_1, \alpha_2, \alpha_3 \in \Omega'_{2,j}$ for which $\alpha_1 \in \text{Conv}(\Delta_j)g_t$, $\alpha_3 \in \text{Conv}(\Delta_j^+)g_t$, $h_1 \in H_{j,t}^-(g)$, $h_2 \in H_{j,t}^+(g)$, and $\beta_j H_{j,t}^-(g) < \alpha_2 < \Delta_j^+ g_t$. (If there are no points $\beta_j h_1$ with $h_1 \in H_{j,t}^-(g)$ between $\Delta_j g_t$ and $\Delta_j^+ g_t$, then $\beta_j g < \alpha$ for all $\alpha \in \Omega'_{2,j} \setminus \Omega'_{1,j}$ such that $\Delta_j g_t < \alpha < \Delta_j^+ g_t$, etc.)

This completes the construction.

We now show that all the desired properties are satisfied.

By construction, G acts fixed point free on $\Omega_{1,j} = \Omega'_{1,j} \cup \beta_j G$ and H acts fixed point free on $\Omega_{2,j} = \Omega'_{2,j} \cup \beta_j(G \setminus H)$.

We next verify that G and H preserve the orders on $\Omega_{1,j}$ and $\Omega_{2,j}$, respectively.

Let $\alpha_1 < \alpha_2$ in $\Omega_{1,j}$ and $g \in G$. If $\alpha_1, \alpha_2 \in \Omega'_{1,j}$, then $\alpha_1 g < \alpha_2 g$ by the induction hypotheses. If $\alpha_1, \alpha_2 \in \beta_j G$, then $\alpha_1 g < \alpha_2 g$ by (10).

Let $\alpha_1 \in \Omega'_{1,j}$ and $\alpha_2 \in \beta_j G$. Then $\alpha_1 g \in \Omega'_{1,j}$ and $\alpha_2 g \in \beta_j G$. By construction, $\alpha_1 \in \Delta_j g_t < \alpha_2$ for a suitable transversal element g_t ($t \in T_j$). But $g_t g = k g_s$ for a unique $k \in K_j$ and $s \in T_j$. Then $\alpha_1 g \in \Delta_j g_s$ and $\Delta_j g_s < \alpha_2 g$. Hence $\alpha_1 g < \alpha_2 g$. Similarly, $\alpha_1 < \alpha_2$ implies $\alpha_1 g < \alpha_2 g$ when $\alpha_2 \in \Omega'_{1,j}$ and $\alpha_1 \in \beta_j G$.

Let $\alpha_1 < \alpha_2$ in $\Omega_{2,j}$ and $h \in H$. Again we have $\alpha_1 h < \alpha_2 h$ if $\alpha_1, \alpha_2 \in \Omega'_{2,j}$ by the induction hypotheses; and $\alpha_1 h < \alpha_2 h$ if $\alpha_1, \alpha_2 \in \beta_j G$ by (10). Let $\alpha_1 \in \Omega'_{2,j}$ and $\alpha_2 \in \beta_j G \setminus \Omega'_{2,j}$. Then $\alpha_1 h \in \Omega'_{2,j}$ and $\alpha_2 h \in \beta_j G \setminus \Omega'_{2,j}$. By construction, $\alpha_2 = \beta_j g$ and $\Delta_j g_t < \beta_j g < \Delta_j^+ g_t$ for some $t \in T_j$. If $\alpha_1 \in \text{Conv}(\Delta_j) g_t$, then $\alpha_1 h \in \Delta_j g_s$ where $K_j g_s = K_j g_t h$; let $k \in K_j$ be such that $k g_s = g_t h$. So $\alpha_1 h \in \text{Conv}(\Delta_j) g_s < \beta_j k g_s = \beta_j g_t h = \alpha_2 h$. If $\Delta_j g_t < \alpha_1 < \beta_j g$, then by (12) $\alpha_1 < \beta_j h_1$ for some $h_1 \in H_{j,t}^-(g)$; so $\alpha_1 h < \beta_j h_1 h < \beta_j g h = \alpha_2 h$.

If $\alpha_1 \in \beta_j G$, $\alpha_2 h \in \Omega'_{2,j} \setminus \Omega'_{1,j}$, then $\alpha_1 = \beta_j g$ and the only interesting remaining case is $\beta_j H_{j,t}^-(g) < \alpha_2 < \Delta_j^+ g_t$ (or there are no points $\beta_j h_1$ with $h_1 \in H_{j,t}^-(g)$ between $\Delta_j g_t$ and $\Delta_j^+ g_t$; in this case there are no such points between $\Delta_j g_t h$ and $\Delta_j^+ g_t h$). Let $K_j g_s = K_j g_t h$. Now $\Delta_j g_s < \alpha_1 h, \alpha_2 h < \Delta_j^+ g_s$, $H_{j,i}^-(g) h = H_{j,s}^-(g h)$ and $H_{j,i}^+(g) h = H_{j,s}^+(g h)$. This implies $\alpha_1 = \beta_j g h < \alpha_2 h$. Thus G and H preserve the orders on $\Omega_{1,j}$ and $\Omega_{2,j}$ respectively.

By construction, $\Omega_{1,j}$ and $\Omega_{2,j}$ satisfy (6), (7), (8), and (I) – (IV) for all $i \leq j$.

Finally,

$$\Omega_3 := \bigcup_{j \in J} \Omega_{1,j}$$

satisfies the conclusions of the lemma and the proof is complete. //

Theorem A *Let H_i be a subgroup of a right orderable group G_i ($i \in I$). Assume that all H_i ($i \in I$) are isomorphic to a group H . Let $\varphi_i : H_i \cong H$ be isomorphisms. Then $L := *G_i$ ($H_i \xrightarrow{\varphi_i} H$) ($i \in I$) is right orderable iff for each $i \in I$, there is a normal family $\mathcal{R}_i$ of right orders on G_i such that $\{\mathcal{R}_i \mid i \in I\}$ is compatible for $\{\varphi_i \mid i \in I\}$. Moreover, the set of right orders $\leq_i$ on G_i ($i \in I$) extends to a right order on L iff $\{\leq_i \mid i \in I\}$ is compatible with $\{\varphi_i \mid i \in I\}$ and contained in normal families $\mathcal{R}_i$ of right orders on G_i ($i \in I$) such that $\{\mathcal{R}_i \mid i \in I\}$ is compatible for $\{\varphi_i \mid i \in I\}$.*

Proof: Necessity was proved in Proposition 4.1.

By Lemma 4.3, we may assume that all $\mathcal{R}_i$ are $\mathcal{A}$ -invariant ($i \in I$).

First consider case $I = \{1, 2\}$ and let $\varphi = \varphi_1 \varphi_2^{-1}$. We wish to construct (by induction) a chain of totally ordered sets

$$\Gamma_1 \subset \Gamma_2 \subset \dots \subset \Gamma_k \subset \dots \subset \bigcup_{k \in \mathbb{Z}_+} \Gamma_k = \Gamma^\# \quad (13)$$

such that:

($\Gamma 1$) the order on Γ_{k+1} extends the order on Γ_k for all $k \in \mathbb{Z}_+$;

($\Gamma 2$) $G_1 \leq A^*(\Gamma_{2k+1})$, $G_2 \leq A^*(\Gamma_{2k+2})$ for all $k \geq 0$; if $0 \leq k_1 \leq k_2$, $g_s \in G_s$ and $\alpha \in \Gamma_{2k_1+s} \subseteq \Gamma_{2k_2+s}$, then αg_s is the same point in Γ_{2k_1+s} and Γ_{2k_2+s} ($s = 1, 2$);

($\Gamma 3$) $\alpha h_1 = \alpha h_1^\varphi$, $\alpha h_2 = \alpha h_2^{\varphi^{-1}}$ for all $h_1 \in H_1$, $h_2 \in H_2$ and $\alpha \in \Gamma_k$ ($k \in \mathbb{Z}_+$);

($\Gamma 4$) if $\alpha \in \Gamma_{2k+1+s} \setminus \Gamma_{2k+s}$ and $g_s \in G_s \setminus H_s$, then $\alpha g_s \in \Gamma_{2k+2+s} \setminus \Gamma_{2k+1+s}$ for $s = 1, 2$ and for all $k \in \mathbb{Z}_+$.

Let $\leq_s \in \mathcal{R}_s$ ($s = 1, 2$) and φ be compatible with $\leq_1, \leq_2$.

Let $(\Gamma_1, \leq) := (G_1, \leq_1)$ and use the right regular action of G_1 on Γ_1 . Write Γ_1 as a union of disjoint H_1 -orbits: $\Gamma_1 = \bigcup_{j \in J} B_j$ and choose $\beta_j \in B_j$ ($j \in J$) with $\beta_1 = 1$ (where 1 is the least element of J in some well ordering of J). In this case, for all $g_1 \in G_1$,

$$1 \leq_1 g_1 \text{ in } G_1 \iff \beta_1 \leq \beta_1 g_1 \text{ in } \Gamma_1. \quad (14)$$

Using the isomorphism $\varphi : H_1 \cong H_2$, we define an action of H_2 on B_j ($j \in J$) in the natural way:

$$\beta h_2 := \beta h_2^{\varphi^{-1}} \quad (h_2 \in H_2, \beta \in B_j). \quad (15)$$

Let $h_1 = h_2^{\varphi^{-1}}$ (i.e., $h_2 = h_1^\varphi$). Thus (15) leads to

$$\beta h_1^\varphi = \beta h_1 \quad (h_1 \in H_1, \beta \in B_j).$$

Since $H_1 \leq A^*(\Gamma_1)$, the isomorphism φ and (15) define an action of H_2 on Γ_1 such that $H_2 \leq A^*(\Gamma_1)$. By (14), for all $h_1 \in H_1$,

$$1 \leq_1 h_1 \text{ in } G_1 \iff \beta_1 \leq \beta_1 h_1 \text{ in } \Gamma_1.$$

Since φ is compatible with $\leq_1, \leq_2$, for all $h_2 \in H_2$:

$$1 \leq_2 h_2 \text{ in } G_2 \iff \beta_1 \leq \beta_1 h_2 \text{ in } \Gamma_1.$$

Extend Γ_1 to $\Gamma_1 \cup \beta_1(G_2 \setminus H_2)$ and order the set of points $\{\beta_1 g \mid g \in G_2 \setminus H_2\}$ by

$$\beta_1 f_2 \leq \beta_1 g_2 \iff f_2 \leq_2 g_2. \quad (16)$$

Let $\alpha \in \Gamma_1$ and set

$$\alpha \leq \beta_1 g_2 \iff (\exists h_2 \in H_2)(\alpha \leq \beta_1 h_2 \ \& \ h_2 \leq_2 g_2); \quad (17)$$

$$\beta_1 g_2 \leq \alpha \iff (\forall h_2 \in H_2)(h_2 \leq_2 g_2 \rightarrow \beta_1 h_2 \leq \alpha). \quad (18)$$

We see that conditions (16) – (18) ensure that $G_2 \leq A^*(\beta_1 G_2)$ and

$$1 \leq_2 g_2 \text{ in } G_2 \iff \beta_1 \leq \beta_1 g_2 \text{ in } \beta_1 G_2 \quad (19)$$

for all $g_2 \in G_2$.

Apply Lemma 4.4 with $G = G_2$, $\Omega_1 = \beta_1 G_2$ and $\Omega_2 = \Gamma_1 \cup \beta_1(G_2 \setminus H_2)$ to obtain Ω_3 such that $G_2 \leq A^*(\Omega_3)$, and let $\Gamma_2 = \Omega_3$. Once Γ_{2k} has been defined, apply Lemma 4.4 to $G = G_1$, $\Omega_1 = \Gamma_{2k-1}$ and $\Omega_2 = \Gamma_{2k}$, and let $\Gamma_{2k+1} := \Omega_3$. Once Γ_{2k+1} has been defined, apply Lemma 4.4 to $G = G_2$, $\Omega_1 = \Gamma_{2k}$ and $\Omega_2 = \Gamma_{2k+1}$, and let $\Gamma_{2k+2} := \Omega_3$. The chain (13) is defined inductively; $\Gamma^\#$ satisfies all the requirements $(\Gamma 1) - (\Gamma 4)$. By $(\Gamma 2)$, $G_1, G_2 \leq A^*(\Gamma^\#)$. Indeed, by construction, the subgroup of $A(\Gamma^\#)$ generated by G_1 and G_2 is a subset of $A^*(\Gamma^\#)$.

Consider the subgroup $\langle G_1, G_2 \rangle$ of $A(\Gamma^\#)$ generated by G_1 and G_2 ; so $\langle G_1, G_2 \rangle$ is a subset of $A^*(\Gamma^\#)$. By $(\Gamma 3)$, at least H_1 and H_2 are amalgamated in $\langle G_1, G_2 \rangle$ via φ . It only remains to verify that $\langle G_1, G_2 \rangle = L := G_1 * G_2 (H_1 \overset{\varphi}{\cong} H_2)$.

By [20], Theorem 2.6, non-trivial elements from L have the canonical representation

$$g = h g_0 g_1 \cdots g_{2k} g_{2k+1}$$

where $h \in H_1$; $g_{2j+1} \in G_1 \setminus H_1$ ($j = 0, \dots, k-1$), and if $g_{2k+1} \neq 1$ then $g_{2k+1} \in G_1 \setminus H_1$; $g_{2j} \in G_2 \setminus H_2$ ($j = 1, \dots, k$), and if $g_0 \neq 1$ then $g_0 \in G_2 \setminus H_2$. Consider βg with $\beta \in \Gamma_1 \subset \Gamma^\#$. If $g = h g_0 \neq 1$ then $g \in G_2$ and $\beta g \neq \beta$ because G_2 acts fixed point free on $\Gamma^\#$. For $g = h g_0 g_1$ and $g_1 \in G_1 \setminus H_1$, by construction $\beta h g_0 g_1 \in \Gamma_3 \setminus \Gamma_2$ and so $\beta g \neq \beta$. By

induction on k , $\beta h g_0 g_1 \cdots g_k \in \Gamma_{k+2} \setminus \Gamma_{k+1}$ and again $\beta g \neq \beta$. Thus $\langle G_1, G_2 \rangle$ is indeed isomorphic to L .

By the construction, all α -orders on L ($\alpha \in \Gamma^\#$) induce orders on G_1 and G_2 belonging to $\mathcal{R}_1$ and $\mathcal{R}_2$, respectively. So $(\mathcal{R}_1, \mathcal{R}_2)$ extends to an L -invariant family $\mathcal{R}$ of orders on L . By Lemma 3.6, we can choose $\mathcal{R}$ to be $\mathcal{A}$ -invariant. This completes the proof of the theorem when $I = \{1, 2\}$.

Next suppose that $n \geq 2$ and $I = \{1, 2, \dots, n\}$. Let $L_1 := G_1$ and $L_{j+1} := L_j * G_{j+1}$ ($H_{j+1} \xrightarrow{\varphi_{j+1}} H$) ($j = 1, \dots, n-1$). By induction and the previous step, for each $j \in \{1, \dots, n\}$, L_j is right orderable and has an $\mathcal{A}$ -invariant family of right orders that is compatible for all φ_i and containing the G_i -invariant families $\mathcal{R}_i$ on G_i ($i = 1, \dots, j$). Since $L_n \cong *_{i \in I} G_i$ ($H_i \xrightarrow{\varphi_i} H$), the theorem is proved for any free product of a finite set of components with amalgamated subgroups.

By the local property highlighted in Section 2, if the theorem were to fail for an infinite index set I , then the theorem would fail for some finite index set $I' \subseteq I$. By the above argument, this cannot happen. Consequently, the theorem is proved (for an arbitrary index set I). //

5 Corollaries

We now provide important consequences of Theorem A when the group $*_{i \in I} G_i$ ($H_i \xrightarrow{\varphi_i} H$) is right orderable and the compatibility condition is easily verified. Corollaries 5.1 and 5.4 illustrate relatively general situations and produce obvious but useful consequences (Corollaries 5.2, 5.3, 5.5, 5.6). Note that Corollary 5.3 generalises Theorem B from [5], Corollary 5.2 solves Problem 2.6 of [18], Corollary 5.5 solves Problem 2.13 of [18] for abelian right ordered groups, and Corollaries 5.8, 5.9 and 5.10 give alternative proofs of corresponding results from [6] and [2].

Corollary 5.1 *Let H_i be a subgroup of a right ordered group G_i such that any right order on H_i is extendable to a right order on G_i ($i \in I$). Suppose that each H_i is isomorphic to a group H . Then $L := *_{i \in I} G_i$ ($H_i \xrightarrow{\varphi_i} H$) is right orderable for any amalgamating isomorphisms φ_i ($i \in I$). Moreover,*

the initial right orders on G_i ($i \in I$) extend to a right order on L if $\varphi_i \varphi_j^{-1}$ preserves the induced orders on H_i and H_j for all $i, j \in I$.

Proof: Let $\mathcal{R}_i$ be the set of all right orders on G_i ($i \in I$). By Remark 3.5, the families $\mathcal{R}_i$ are $\mathcal{A}$ -invariant ($i \in I$). Let $\leq_i \in \mathcal{R}_i$. Then $\varphi_{i,j} := \varphi_i \varphi_j^{-1}$ induces an order $\leq_j$ on H_j given by: $1 \leq_j h_j$ iff $1 \leq_i h_j^{\varphi_{i,j}^{-1}}$ ($h_j \in H_j$). By the hypothesis of the corollary, $\leq_j$ extends to a right order on G_j . Thus the compatibility condition is satisfied and the corollary follows from Theorem A. //

The next two corollaries follow immediately from Corollary 5.1.

Corollary 5.2 *Let G_i be a right ordered group with convex subgroup H_i ($i = 1, 2$). Suppose that H_1 and H_2 are isomorphic; say, $\varphi : H_1 \cong H_2$. Then $L := G_1 * G_2$ ($H_1 \overset{\varphi}{\cong} H_2$) is right orderable. Moreover, the initial right orders on G_1 and G_2 extend to a right order on L if φ preserves the induced orders on H_1 and H_2 .*

Corollary 5.3 *Let G_i be a right ordered group containing a rank one abelian subgroup H_i ($i = 1, 2$). Suppose that H_1 and H_2 are isomorphic; say, $\varphi : H_1 \cong H_2$. Then $L := G_1 * G_2$ ($H_1 \overset{\varphi}{\cong} H_2$) is right orderable. Moreover, the initial right orders on G_1 and G_2 extend to a right order on L if φ preserves the induced orders on H_1 and H_2 .*

Proof: The only two possible right orders on a rank 1 abelian group are the reverses of each other. //

Corollary 5.4 *Let G_i be a right ordered group with $H_i \leq G_i$, ($i = 1, 2$) and $H_1 \cong H_2$. Suppose that any right order on H_2 is extendable to a right order on G_2 . Assume further that for any right order $\leq_2$ on G_2 and $g_2 \in G_2$, there is $h_2 \in H_2$ such that $P_2^{g_2} \cap H_2 = P_2^{h_2} \cap H_2$, where P_2 is the positive cone of $\leq_2$. Then $L := G_1 * G_2$ ($H_1 \overset{\varphi}{\cong} H_2$) is right orderable for any amalgamating isomorphism φ . Moreover, the initial right orders on G_1 and G_2 extend to a right order on L if φ preserves the original right orders on H_1 and H_2 .*

Proof: Let $\mathcal{R}_1$ be the set of all right orders on G_1 . Then $\mathcal{R}_1$ is G_1 -invariant. Thus $\mathcal{R}_{H_1}$, the family of restrictions of right orders from $\mathcal{R}_1$ to H_1 , is H_1 -invariant and φ induces an H_2 -invariant family of right orders $\mathcal{R}_{H_2}$ on H_2 : for if $\leq \in \mathcal{R}_{H_1}$, let $\leq^\varphi \in \mathcal{R}_{H_2}$ be defined by

$$1 \leq^\varphi h_2 \iff 1 \leq h_2^{\varphi^{-1}} \quad (h_2 \in H_2). \quad (20)$$

By the hypothesis of the corollary, each order on H_2 extends to a right order on G_2 . Let $\mathcal{R}_2$ be all extensions of orders from $\mathcal{R}_{H_2}$ to G_2 . If $\leq_2 \in \mathcal{R}_2$ and $g_2 \in G_2$, there is $h_2 \in H_2$ such that $P_2^{g_2} \cap H_2 = P_2^{h_2} \cap H_2$. But $(\leq_2^{h_2})_{H_2} \in \mathcal{R}_{H_2}$ as $\mathcal{R}_{H_2}$ is H_2 -invariant. So $\mathcal{R}_2$ is G_2 -invariant. By (20), φ is compatible for $(\mathcal{R}_1, \mathcal{R}_2)$. Hence the corollary follows from Theorem A. //

Corollary 5.4 produces two useful consequences.

Corollary 5.5 *Let G be right ordered group with abelian subgroup A . Then G is embeddable in a right ordered group $G^\#$ with divisible abelian subgroup $A^\#$ such that A embeds in $A^\#$. Moreover, every right order on G extends to a right order on $G^\#$.*

Corollary 5.6 *Let G be a right ordered group, $H \leq G$, and φ be an isomorphism between H and K . Then $G * (K \times T)$ ($H \cong^\varphi K$) is right orderable for any right orderable group T .*

Corollary 5.7 *Let G be a right ordered group, $H \leq G$, and φ be an order-preserving automorphism of H . Assume further that*

$$g \cdot h > g \iff g \cdot h^\varphi > g \quad \text{for all } g \in G, h \in H. \quad (21)$$

Then $G * G$ ($H \cong^\varphi H$) is right orderable.

Proof: Let $\mathcal{R}$ be the set of all orders conjugate to the initial right order of G . Then $\mathcal{R}$ is normal and $(\mathcal{R}, \mathcal{R})$ is compatible for φ by (21). //

Corollary 5.8 (Bludov and Glass [6]) *Let G_i be a right ordered group with subgroup H_i ($i = 1, 2$) and suppose that $\varphi : H_1 \cong H_2$ is an order-preserving isomorphism. Assume further that $h_i > 1$ iff $g_i^{-1} h_i g_i > 1$ ($g_i \in G_i$; $h_i \in H_i$) ($i = 1, 2$). Then $L := G_1 * G_2$ ($H_1 \cong^\varphi H_2$) is right orderable and the initial orders on G_1 and G_2 extend to a right order on L .*

Corollary 5.9 (Bludov and Glass [6]) *If G_i is a two-sided ordered group with subgroup H_i ($i = 1, 2$) and $\varphi : H_1 \cong H_2$ is an order-preserving isomorphism, then $L := G_1 * G_2 (H_1 \stackrel{\varphi}{\cong} H_2)$ is right orderable and the initial orders on G_1 and G_2 extend to a right order on L .*

Corollary 5.10 (Bergman [2]) *If G is a right orderable group with subgroup H , then $L := G * G (H = H)$ is right orderable and each right order on G extends to a right order on L .*

The next corollary is folklore and can also easily be proved without the added machinery we have developed (see, *e.g.*, [17], Corollary *****).

Corollary 5.11 *If $\{G_i \mid i \in I\}$ is a family of right ordered groups, then $L := *_{i \in I} G_i$ is right orderable with a right order extending that on each G_i .*

6 HNN-extensions

Let A and B be isomorphic subgroups of a group G , and $\varphi : A \cong B$. Let $K = \langle G, t \mid a^t = a^\varphi (a \in A) \rangle$ be the corresponding HNN-extension of G with stable letter t . We will consider normal families $\mathcal{R}$ of right orders on G such that φ is compatible for $(\mathcal{R}, \mathcal{R})$ or, briefly, φ and $\mathcal{R}$ are compatible.

Theorem B *Let G be a right orderable group, and A and B be isomorphic subgroups of G ; say, $\varphi : A \cong B$. Then $K = \langle G, t \mid a^t = a^\varphi (a \in A) \rangle$ is right orderable iff there is a normal family $\mathcal{R}$ of right orders on G compatible with φ . If $\mathcal{R}$ is such a family of right orders on G , then each right order belonging to $\mathcal{R}$ extends to a right order on K .*

Proof: Assume that K is right orderable and let $\mathcal{R}_K$ be the family of all right orders on K . Then $\mathcal{R}_K$ induces a family $\mathcal{R}_G$ of right orders on G . Hence a right order $\leq_G$ on G extends to a right order $\leq_K$ iff $\leq_G \in \mathcal{R}_G$. Since $\mathcal{R}_K$ is a normal family of right orders on K , $\mathcal{R}_G$ is a normal family of right orders on G . Moreover, if $\leq_G \in \mathcal{R}_G$ and $\leq_K$ is one of its extensions to $\mathcal{R}_K$, then $\leq_K^t \in \mathcal{R}_K$. Now $\leq_K^t$ induces a right order $\leq_G^t \in \mathcal{R}_G$. We show that $\leq_G$ and $\leq_G^t$ are associated orders via φ . Let $a \in A$. If $1 \leq_G a$ in G , then $1 \leq_K a$ in K ; so $1 \leq_K^t a^t$ in K . This

gives $1 \leq_G^t a^\varphi$ in G . Taking $b \in B$ and t^{-1} instead of t , we obtain that $1 \leq_G^t b$ implies $1 \leq_G b^{\varphi^{-1}}$. Thus for any $\leq_1 \in \mathcal{R}_G$, there is $\leq_2 \in \mathcal{R}_G$ such that $1 \leq_1 a$ implies $1 \leq_2 a^\varphi$ for all $a \in A$ and for any $\leq_2 \in \mathcal{R}_G$ there is $\leq_1 \in \mathcal{R}_G$ such that $1 \leq_2 b$ implies $1 \leq_1 b^{\varphi^{-1}}$ for all $b \in B$. So the orders are associated via φ and necessity is proved.

The proof of sufficiency relies on the fact that an HNN-extension can be realised via a direct limit of free products with amalgamated subgroups.

For each $n \in \mathbb{Z}$, let G_n , A_n and B_n be isomorphic copies of G , A and B respectively, and let g_n be the element of G_n corresponding to $g \in G$. Moreover, φ induces an isomorphism from A_n to B_{n+1} in the natural way; *i.e.*, $a_n^\varphi := (a^\varphi)_{n+1}$ ($a \in A$). Let

$$G_{(0)} := G_0 \text{ and } G_{(m+1)} := G_{(m)} * G_{m+1} \text{ } (A_m \stackrel{\varphi}{\cong} B_{m+1}) \text{ } (m \in \mathbb{N} := \mathbb{Z}_+ \cup \{0\}).$$

Let $G_{(\mathbb{N})}$ be the direct limit of $\{G_{(n)} \mid n \in \mathbb{N}\}$. Let

$$G_{(-1)} := G_{(\mathbb{N})} * G_{-1} \text{ } (A_{-1} \stackrel{\varphi}{\cong} B_0) \text{ and}$$

$$G_{(m)} := G_{(m+1)} * G_m \text{ } (A_m \stackrel{\varphi}{\cong} B_{m+1}) \text{ } (m \in \mathbb{Z}, m \leq -2).$$

Let $\tilde{G}$ be the direct limit of $\{G_{(n)} \mid -n \in \mathbb{Z}_+\}$ and t be the automorphism of $\tilde{G}$ defined by $g_n^t = g_{n+1}$ ($n \in \mathbb{Z}$). Then $K = \langle G, t \mid a^t = a^\varphi (a \in A) \rangle$ is isomorphic to $\tilde{G} \rtimes \langle t \rangle$. By Theorem A, $G_{(1)}$ is right orderable iff there is a normal family $\mathcal{R}$ of right orders on G such that φ and $\mathcal{R}$ are compatible; in this case, $\mathcal{R}$ extends to a normal family $\mathcal{R}_1$ of right orders on $G_{(1)}$. Similarly, $\mathcal{R}_1$ extends to a normal family $\mathcal{R}_2$ of right orders on $G_{(2)}$. Moreover, $(\mathcal{R}_1, \mathcal{R}_2)$ is compatible for φ on G_2 . It follows by induction that all the normal families of right orders are pairwise compatible with φ and extend to a normal family of right orders $\mathcal{R}_{\mathbb{N}}$ on $G_{(\mathbb{N})}$. We repeat the process to get that $\tilde{G}$ has such a normal family of right orders. Thus $K \cong \tilde{G} \rtimes \langle t \rangle$ is right orderable (with the right order extending the original one on G). //

To see the need for the hypotheses in Theorem B, consider our previous example again.

Example 6.1 Let G_1, G_2 be as in Example 4.2:

$$G_1 = \langle a_1, b_1 \mid [b_1, a_1, a_1] = [b_1, a_1, b_1] \rangle, \quad G_2 = \langle a_2, b_2 \mid a_2 a_2^{b_2} = 1 \rangle.$$

Let $c_1 = [b_1, a_1]$ and $c_2 = b_2^2$. Let $G = G_1 \times G_2$, $H_1 = \langle a_1, c_1 \rangle < G_1 < G$, and $H_2 = \langle a_2, c_2 \rangle < G_2 < G$. Let $\varphi : H_1 \cong H_2$ be the isomorphism defined by $\varphi : a_1 \mapsto a_2, c_1 \mapsto c_2$. The right orders on G_1 and G_2 defined by (4) in Example 4.2 extend to the lexicographic right order on G . In this case φ preserves the induced orders on H_1 and H_2 but is not compatible with the conjugate right order $\leq^{b_1}$ (see Example 4.2). Thus $\langle G, t \mid h^t = h^\varphi (h \in H_1) \rangle$ is not right orderable, although the right order on G is a Conrad right order (see [10], Section 6.6).

Corollary 6.2 *Let H_1 and H_2 be isomorphic subgroups of a right ordered group G ; say $\varphi : H_1 \cong H_2$. Let N be a convex subgroup of G such that $(N \cap H_1)^\varphi = N \cap H_2$. Let $\bar{G} := G/N$ and $\bar{H}_i := H_i N/N$ ($i = 1, 2$). Define $\bar{\varphi} : \bar{H}_1 \cong \bar{H}_2$ by $\bar{h}_1^{\bar{\varphi}} := (h_1^\varphi)N$, where $\bar{h}_1 := h_1 N \in \bar{H}_1$. If $K_1 := \langle \bar{G}, \bar{t} \mid \bar{h}_1^{\bar{t}} = \bar{h}_1^{\bar{\varphi}} (\bar{h}_1 \in \bar{H}_1) \rangle$ and $K_2 := \langle G, t \mid h_1^t = h_1^\varphi (h_1 \in H_1 \cap N) \rangle$ are right orderable, then $K := \langle G, t \mid h_1^t = h_1^\varphi (h_1 \in H_1) \rangle$ is right orderable with a right order extending that on G .*

Proof: By Theorem B, there is a normal family of right orders $\mathcal{R}_1 = \{\leq_i \mid i \in I\}$ on $\bar{G}$ such that $\bar{\varphi}$ and $\mathcal{R}_1$ are compatible, and there is a normal family of right orders $\mathcal{R}_2 = \{\leq_j \mid j \in J\}$ on G such that the restriction of φ to $H_1 \cap N$ and $\mathcal{R}_2$ are compatible. For each $(i, j) \in I \times J$, we obtain an order on G given by

$$1 \leq_{i,j} g \Leftrightarrow \begin{cases} \bar{1} \leq_i \bar{g} & \text{if } g \in G \setminus N \\ 1 \leq_j g & \text{if } g \in N \end{cases} \quad (22)$$

It is easy to see that $\mathcal{R} := \{\leq_{i,j} \mid (i, j) \in I \times J\}$ is a family of right orders on G . Indeed, let $1 \leq_{i,j} g, f$. If $g, f \notin N$, then by (22), $\bar{1} < \bar{g}\bar{f}$. It follows that $gf \notin N$ and (22) gives $1 \leq_{i,j} gf$. If $g \notin N$ and $f \in N$, then $gf \notin N$ and $\bar{g}\bar{f} = \bar{g}$. By (22), $\bar{1} < \bar{g} = \bar{g}\bar{f}$. Hence $1 \leq_{i,j} gf$. Similarly, $1 \leq_{i,j} gf$ if $g \in N$ and $f \notin N$. If $g, f \in N$, then $gf \in N$; so $1 \leq_j gf$. This immediately gives $1 \leq_j gf$ and so $1 \leq_{i,j} gf$ by (22).

We now verify that $\mathcal{R}$ is a normal family of right orders on G . Since $\mathcal{R}_1$ and $\mathcal{R}_2$ are normal families of right orders on their groups, for any $(i, j) \in I \times J$, and $g \in G$, we have $\leq_i^{\bar{g}} \in \mathcal{R}_1$ and $\leq_j^g \in \mathcal{R}_2$. Write $\leq_{i(g)}$ for

$\leq_i^{\bar{g}}$ and $\leq_{j(g)}$ for $\leq_j^g$; then $\leq_{i,j}^g$ is $\leq_{i(g),j(g)}$. Let $1 \leq_{i,j} f$. If $f \notin N$, then $\bar{1} <_i \bar{f}$. But $f^g \notin N$ since N is normal, whence $\bar{1} <_i^{\bar{g}} \bar{f}^{\bar{g}}$. It follows that $1 \leq_{i,j}^g f^g$. On the other and, if $f \in N$, then $f^g \in N$. Thus $1 \leq_j^g f^g$; so $1 \leq_{i,j}^g f^g$.

It remains to show that φ and $\mathcal{R}$ are compatible. Let $\leq_{i,j} \in \mathcal{R}$. Since $\bar{\varphi}$ and $\mathcal{R}_1$ are compatible, there is $\leq_{i'} \in \mathcal{R}_1$ such that $\bar{1} \leq_i \bar{h}$ iff $\bar{1} \leq_{i'} \bar{h}^{\bar{\varphi}}$ for all $h \in H \setminus N$. There is also $\leq_{j'} \in \mathcal{R}_2$ such that $1 \leq_j h$ iff $1 \leq_{j'} h^{\varphi}$ for all $h \in H \cap N$. Now $1 \leq_{i,j} h$ iff $1 \leq_{i',j'} h^{\varphi}$ by (22). The proof of the corollary now follows from Theorem B. //

The conditions in Corollary 6.2 are not easy to verify so we provide special cases where the verification is easier. These will suffice for our main applications; all follow immediately from Corollary 6.2.

Corollary 6.3 *Let G be a right ordered group with normal convex subgroup N . Let $\varphi : H_1 \cong H_2$ be an order-preserving isomorphism between subgroups H_1 and H_2 . Assume that $H_1 \cap N = H_2 \cap N = \{1\}$ and the induced isomorphism $\bar{\varphi} : H_1 N / N \cong H_2 N / N$ is the identity. Then $K = \langle G, t \mid h_1^t = h_1^{\varphi} \ (h_1 \in H_1) \rangle$ is right orderable with an order extending that of G .*

As a special case (putting $N = \{1\}$, $H_1 = H_2$, and φ the identity), we obtain the HNN-analogues of Corollary 5.10.

Corollary 6.4 *Let G be a right ordered group with subgroup H . Then $K = \langle G, t \mid [h, t] = 1 \ (h \in H) \rangle$ is right orderable with an order extending that of G .*

Repeating the process, we get an equivalent formulation of Corollary 5.6.

Corollary 6.5 *Let G be a right ordered group with subgroup H . Then $K = \langle G, t_1, \dots, t_m \mid [h, t_j] = 1 \ (h \in H : j = 1, \dots, m) \rangle$ is right orderable with an order extending that of G .*

We can extend Corollary 6.3 further by dropping the condition that $\bar{\varphi}$ is the identity.

Corollary 6.6 *Let H_1 and H_2 be isomorphic subgroups of a right ordered group G , say $\varphi : H_1 \cong H_2$. Let N be a convex normal subgroup of G such that $N \cap H_1 = \{1\} = N \cap H_2$. Let $\bar{\varphi} : H_1N/N \cong H_2N/N$ be the isomorphism induced by φ . If there is a normal family of right orders on G/N that is compatible for $\bar{\varphi}$, then $\langle G, t \mid h^t = h^\varphi \ (h \in H_1) \rangle$ is right orderable.*

Analogues of many corollaries from the previous section can be obtained using Theorem B instead of Theorem A. For example, instead of Corollary 5.3, we can derive

Corollary 6.7 (Bludov and Glass [5]) *If G is a right ordered group and $f, g \in G \setminus \{1\}$, then $\langle G, t \mid f^t = g \rangle$ is right orderable with a right order extending that on G .*

We close this section with the analogue of a result about countable groups. Using Wreath products, one can actually replace *three* by *two* (cf. [9], Theorem 10.A) but we want to demonstrate a further application of our necessary and sufficient conditions for right orderability.

Corollary 6.8 (cf. [20], Corollary IV.3.1) *Every countable right ordered group C can be embedded in a three-generator right ordered group which is defined by a finite or recursively enumerable set of relations if C is.*

Proof: Let F be the free group on free generators a, b . Then F can be made into a (two-sided) ordered group using the lower central series of F with $1 < b \ll a$. By Corollary 5.11, $G := C * F$ can be right ordered so that C and F inherit their original right orders. By Lemma 2.1, we may further assume that $N := C^G$ is convex in G . Enumerate $C = \{c_n \mid n \in \mathbb{N}\}$ with $c_0 = 1$. Then $H_1 := \langle b^{-n}ab^n \mid n \in \mathbb{N} \rangle$ and $H_2 := \langle c_nb^{-n}ab^n \mid n \in \mathbb{N} \rangle$ are free groups on the indicated generators, all of which are strictly positive. So $H_1 \cap N = H_2 \cap N = \{1\}$. The isomorphism given by $b^{-n}ab^n \mapsto c_nb^{-n}ab^n$ ($n \in \mathbb{N}$) is order-preserving and the induced isomorphism $H_1N/N \cong H_2N/N$ is the identity. By Corollary 6.3, $T := \langle H, t \mid (b^{-n}ab^n)^t = c_nb^{-n}ab^n \ (n \in \mathbb{N}) \rangle$ is right orderable with an order extending that of H . By the definition, T is generated by a, b, t . //

7 Right orderable analogues of theorems of Boone, Britton, Higman and Novikov

We now apply Theorem B to obtain the main consequences of our theorems. Specifically we use Corollaries 6.2 - 6.8.

If T is any Turing machine, let $E := E(T)$ be the recursively enumerable set of semigroup words (numbers) in the alphabet of symbols $\{a_1, \dots, a_m\}$ associated with T . Let $\gamma(T)$ be the finitely presented semigroup associated with T (see [21], Chapter 12). It has generators $s_1, \dots, s_M$ (symbols) and $q_0, \dots, q_N$ (states), and defining relations $\Sigma_i = \Gamma_i$ ($i = 1, \dots, I$), where each Σ_i and Γ_i is special; *i.e.*, has the form wq_jw' where w, w' are semigroup words in $\{a_1, \dots, a_m\} \subseteq \{s_1, \dots, s_M\}$ and $j \in \{0, \dots, N\}$. One can build a finitely presented group $B(T)$ associated with $\gamma(T)$. This was done originally by P. S. Novikov and independently by W. W. Boone. We will use the finitely presented group $B(T)$ of J. L. Britton in [8] which is a homomorphic image of Boone's original group. We describe it in the course of the proof of the following theorem. It is generated by $x, y, s_1, \dots, s_M, q_0, \dots, q_N, \ell_1, \dots, \ell_I, r_1, \dots, r_I, t, k$ subject to a finite set of relations, where $q_0, \dots, q_N, s_1, \dots, s_M$ and I are as above.

Theorem C *Let T be a Turing machine and $B(T)$ be the Britton group associated with T . Then $B(T)$ is right orderable.*

Proof: Let

$$V_1 := \langle x, s_1 \mid x^{s_1} = x^2 \rangle \quad \text{and} \quad V_2 := \langle y, s_1 \mid y^{2s_1} = y \rangle.$$

Then V_1 and V_2 are two-sided orderable ([16], Chapter 1 Example 5). Now

$$G_1 := \langle x, y, s_1 \mid x^{s_1} = x^2, y^{2s_1} = y \rangle \cong V_1 * V_2 \ (\langle s_1 \rangle = \langle s_1 \rangle)$$

and so is right orderable by Corollary 5.3. For $j = 2, \dots, M$, let $s'_j := s_1^{-1}s_j$. Let

$$G_M := \langle x, y, s_1, \dots, s_M \mid x^{s_j} = x^2, y^{2s_j} = y \ (j = 1, \dots, M) \rangle \cong \\ \langle G_1, s'_2, \dots, s'_M \mid x^{2s'_j} = x^2, y^{s'_j} = y \ (j = 2, \dots, M) \rangle.$$

By Corollary 6.5, G_M can be right ordered. By taking the obvious homomorphism of G_M to the free group $\langle s_1, \dots, s_M \rangle$ of rank M and noting that the images of V_1 and V_2 are $\langle s_1 \rangle$, we can use Lemma 2.1 to ensure that $F := \langle x, y \rangle$ inherits its original order, F^{G_M} is convex in G_M and $1 < s_1 \ll s_2 \ll \dots \ll s_M$.

Since $\langle s_1, \dots, s_M \rangle$ is a free group of rank M and is a homomorphic image of each of $H_1 = \langle y^{-1}s_1, \dots, y^{-1}s_M \rangle$ and $H_2 := \langle ys_1, \dots, ys_M \rangle$, H_1 and H_2 are free groups on the indicated generators. Note that fs_j and s_jf are strictly positive in this order for all $f \in F$ and $j \in \{1, \dots, M\}$; so $1 < ys_j$ and $1 < y^{-1}s_j$ for all $j = 1, \dots, M$. Let $N = F^{G_M}$ (which is convex in G_M) and $\varphi : H_1 \cong H_2$ be given by $y^{-1}s_j \mapsto ys_j$ ($j = 1, \dots, M$). So $H_1 \cap N = H_2 \cap N = \{1\}$ and $\bar{\varphi} : H_1N/N \cong H_2N/N$ induced by φ is the identity. Let

$$G_M^* := \langle G_M, \ell_1 \mid (y^{-1}s_j)^{\ell_1} = ys_j \ (j = 1, \dots, M) \rangle,$$

a split extension of $G_M^{G_M^*}$ by $\langle \ell_1 \rangle$. By Corollary 6.3, G_M^* can be right ordered so that G_M inherits its original right order. By Lemma 2.1, we can further ensure that $G_M^{G_M^*}$ is convex with $G_M^{G_M^*} < \ell_1^{-1}$. Let

$$G^\# := \langle G_M, \ell_1, \dots, \ell_I \mid (y^{-1}s_j)^{\ell_i} = ys_j \ (j = 1, \dots, M; \ i = 1, \dots, I) \rangle \cong \\ \langle G_M^*, \ell'_2, \dots, \ell'_I \mid (ys_j)^{\ell'_i} = ys_j \ (j = 1, \dots, M; \ i = 2, \dots, I) \rangle,$$

where $\ell'_i := \ell_1^{-1}\ell_i$ ($i = 2, \dots, I$). We repeat our previous argument. By Corollary 6.5 and Lemma 2.1, $G^\#$ can be right ordered so that G_M inherits its original right order, $G_M^{G_M^*}$ is convex in $G^\#$ and $1 < \ell_1^{-1} \ll \dots \ll \ell_I^{-1}$. Let

$$D := \langle G^\#, r_1, \dots, r_I : (s_jx)^{r_i} = s_jx^{-1} \ (i = 1, \dots, I; \ j = 1, \dots, M) \rangle.$$

By the same argument, D can be right ordered so that $G^\#$ inherits its original right order, $(G^\#)^D$ is convex and $1 < r_1 \ll \dots \ll r_I$.

Let

$$C := \langle D, q_0, \dots, q_N \mid \Sigma_i = \ell_i \Gamma_i r_i \ (i = 1, \dots, I) \rangle,$$

where $\{\Sigma_i = \Gamma_i \mid i = 1, \dots, I\}$ is the finite set of defining relations of $\gamma(T)$.

Let z be a new symbol and $p_j := q_j z^{-1}$ ($j = 0, \dots, N$). Each relation $\Sigma_i = \ell_i \Gamma_i r_i$ can be written in the form

$$w_{i,1} q_{j_i} w_{i,2} = \ell_i w'_{i,1} q_{j'_i} w'_{i,2} r_i$$

for some semigroup words $w_{i,1}, w_{i,2}, w'_{i,1}, w'_{i,2}$ in the alphabet $\{a_1, \dots, a_m\} \subseteq \{s_1, \dots, s_M\}$ and for some $j_i, j'_i \in \{0, \dots, N\}$. These defining relations become

$$(p_{j'_i}^{-1} (w'_{i,1})^{-1} \ell_i^{-1} w_{i,1} p_{j_i})^z = w'_{i,2} r_i w_{i,2}^{-1} \quad (i = 1, \dots, I).$$

Now $\bar{K} := C * \langle z \rangle$ contains the subgroup $\bar{D} := \langle D, p_0, \dots, p_N \rangle$ which is isomorphic to $D * \langle p_0, \dots, p_N \rangle$. By Corollary 5.11 and Lemma 2.1, $\bar{D}$ can be right ordered so that D inherits its original order, $\langle p_0, \dots, p_N \rangle^{\bar{D}}$ is convex, and $1 < p_0 \ll \dots \ll p_N$.

$$\text{Now } \bar{K} \cong \langle \bar{D}, z \mid (p_{j'_i}^{-1} (w'_{i,1})^{-1} \ell_i^{-1} w_{i,1} p_{j_i})^z = w'_{i,2} r_i w_{i,2}^{-1} \ (i = 1, \dots, I) \rangle.$$

Thus $\bar{K}$ can be viewed as an HNN-extension of $\bar{D}$.

Let N be the convex normal subgroup of $\bar{D}$ generated by G_M . By the choice of the right ordering of $\bar{D}$,

$$N p_{j'_i}^{-1} (w'_{i,1})^{-1} \ell_i^{-1} w_{i,1} p_{j_i} = N \ell_i^{-1} > N \quad (i = 1, \dots, I),$$

$$\text{and } N w'_{i,2} r_i w_{i,2}^{-1} = N r_i > N \quad (i = 1, \dots, I).$$

Moreover, $L := \langle \ell_1, \dots, \ell_I \rangle$ and $R := \langle r_1, \dots, r_I \rangle$ are free on the indicated generators, and $N \cap \langle L, R \rangle = \{1\}$. Let $\bar{\varphi} : LN/N \cong RN/N$ be given by $\ell_i^{-1} \mapsto r_i$ ($i = 1, \dots, I$) and $\mathcal{R}$ be the normal family of right orders on $\bar{D}/N$ conjugate to the one induced by our previous right order. Then $\mathcal{R}$ is compatible for $\bar{\varphi}$. Hence the hypotheses of Corollary 6.6 are satisfied and, using Lemma 2.1, we obtain a right order on $\bar{K}$ so that $\bar{D}$ inherits its original right order, $\bar{D}^{\bar{K}}$ is convex and $z > 1$. Since $q_j = p_j z$ ($j = 0, \dots, N$), C inherits a right order from $\bar{K}$ in which D has its original right ordering, D^C is convex and $1 < q_0 \ll \dots \ll q_N$.

$$\text{Let } A := \langle C, t \mid \ell_i^{-t} = \ell_i^{-1}, y^t = y \ (i = 1, \dots, I) \rangle \quad \text{and}$$

$$B(T) := B := \langle A, k \mid r_i^k = r_i, x^k = x, (q_0^{-1} t q_0)^k = q_0^{-1} t q_0 \ (i = 1, \dots, I) \rangle.$$

By Corollary 6.4 and Lemma 2.1, A can be right ordered so that C inherits its original right order, C^A is convex and $t > 1$. By Corollary 6.4 and Lemma 2.1 again, B can be right ordered so that A inherits its original right order, A^B is convex and $k > 1$. This completes the proof of Theorem C that $B(T)$ is right orderable. //

Remark. $x^{s_1^{-1}s_2}$ and x are distinct in $B(T)$ but have the same square. In any (two-sided) orderable group, $a = b$ whenever $a^2 = b^2$. Hence $B(T)$ cannot be (two-sided) orderable for any Turing machine T .

By taking T to be any Turing machine yielding a recursively enumerable but non-recursive set $E(T)$, we obtain a right orderable finitely presented group with insoluble word problem. Thus

Corollary 7.1 (cf. [4]) *There is a right orderable finitely presented group with insoluble word problem.*

Theorem D (cf. [14]) *A finitely generated group can be embedded in a right orderable finitely presented group iff it is right orderable and can be defined by a recursively enumerable set of relations.*

Proof: It is immediate that any finitely generated subgroup of a finitely presented right orderable group must be right orderable and definable by a recursively enumerable set of relations.

To prove the converse, we follow Aanderaa's proof of the Higman Embedding Theorem (see [1]). Let U be any finitely generated right orderable group that is defined by a recursively enumerable set of relations. By increasing the set of generators, we may assume that each of these relations is a semigroup word in the generators. Let the resulting set of generators for U be $\{u_1, \dots, u_m\}$. Let $a_1, \dots, a_m$ be formal symbols. For w any word in $a_1, \dots, a_m$, let w_u be the word obtained from w by replacing each a_i by u_i ($i = 1, \dots, m$). Let T be the Turing machine that enumerates the set of all words w in $a_1, \dots, a_m$ such that $w_u = 1$ in U . So $E = E(T)$ and

$$U := \langle u_1, \dots, u_m \mid w_u = 1 \ (w \in E) \rangle.$$

Let $B(T)$ be the Britton group of the previous proof with the right order so constructed. So $a_1, \dots, a_m \in \{s_0, \dots, s_M\}$. In $\gamma(T)$, there was a

generator $h = s_M$ that played an important role but was not needed for our proof of Theorem C. It figures prominently in Aanderaa's proof. Let $k_0 := hkh^{-1}$ and $t_0 := (hq_1)^{-1}t(hq_1)$. Note that in the right order that we put on $B := B(T)$ in the proof of Theorem C, both k_0 and t_0 were strictly positive. Indeed, the normal subgroups C^B and A^B in the proof of Theorem C were convex with $C^B \subseteq A^B$. Moreover, $t_0 \in A^B \setminus C^B$ and $k_0 \in B \setminus A^B$ with $1 < t_0 \ll k_0$.

Let

$$B(T)_2 := U * B(T),$$

$$A_0 := \langle a_1, \dots, a_m, k_0 \rangle \leq B(T) \quad \text{and} \quad \tilde{U} = *_{w \in A_0} U^w.$$

Then $B(T)_2$ is a split extension of a free product $\tilde{U} * V$ by $B(T)$, where V is the free product of the remaining set of conjugates of U in $B(T)_2$. By Corollary 5.11 and Lemma 2.1, we can right order $\tilde{U} * V$ (and hence $B(T)_2$) so that U and $B(T)$ inherit their given right orders and $U^{B(T)_2}$ is convex in $B(T)_2$.

Let $B(T)_3 :=$

$$\langle B(T)_2, c_1, \dots, c_m \mid u_j^{c_i} = u_j, a_j^{c_i} = a_j, k_0^{c_i} = k_0 u_i^{-1} \ (i, j \in \{1, \dots, m\}) \rangle,$$

$$B(T)_4 := \langle B(T)_3, d \mid k_0^d = k_0, (a_j c_j)^d = a_j \ (j = 1, \dots, m) \rangle, \quad \text{and}$$

$$B(T)_5 := \langle B(T)_4, p \mid t_0^p = t_0 d, k_0^p = k_0, a_j^p = a_j \ (j = 1, \dots, m) \rangle.$$

Aanderaa's very clever proof relies on showing that (like the Higman Rope Trick – see [20], Lemma IV.7.6.) the relations defining U are not necessary in $B(T)_5$ and so $B(T)_5$ can actually be finitely presented. Since U embeds in $B(T)_5$, Higman's Embedding Theorem follows.

$$H_1 := \langle u_1, \dots, u_m, a_1, \dots, a_m, k_0 \rangle \quad \text{and}$$

$$H_2 := \langle u_1, \dots, u_m, a_1, \dots, a_m, k_0 u_1^{-1} \rangle$$

are the free products of U and the free group on the remaining indicated generators (see [1], Lemma 7). Moreover, each of these free generators

is strictly positive in the right order we have imposed. Let N be the convex normal subgroup of $B(T)_2$ generated by U . The order-preserving isomorphism φ between H_1 and H_2 given by: $u_i \mapsto u_i$, $a_i \mapsto a_i$, $k_0 \mapsto k_0 u_1^{-1}$ ($i = 1, \dots, m$) induces the identity isomorphism between $H_1 N / N$ and $H_2 N / N$. By Theorem B, $\langle B(T)_2 / N, \bar{t} \mid u_j N^{\bar{t}} = u_j N, a_j N^{\bar{t}} = a_j N, k_0 N^{\bar{t}} = (k_0 u_1^{-1}) N \ (j = 1, \dots, m) \rangle$ is right orderable with a right order inducing the original induced right order on $B(T)_2 / N$ since $\bar{\varphi}$ is the identity. Now

$$H_1 \cap N = H_2 \cap N = U^{U * A} \cong \tilde{U}.$$

But $\tilde{U}$ is a free factor in the free product $U^{B(T)_2} = \tilde{U} * V$, so any right order on $\tilde{U}$ is extendable to a right order of $U^{B(T)_2}$ and hence to its splitting extension by $B(T)$. Since this is $B(T)_2$, all the hypotheses of Corollary 6.2 hold. Hence

$$C_1 := \langle B(T)_2, c_1 \mid u_j^{c_1} = u_j, a_j^{c_1} = a_j, k_0^{c_1} = k_0 u_1^{-1} \ (j \in \{1, \dots, m\}) \rangle$$

can be right ordered. By Lemma 2.1, this can be done so that $B(T)_2$ inherits its original order, U^{C_1} is convex, and $1 < c_1 \ll a_1 \ll \dots \ll a_m$.

We adjoin c_2 to C_1 to get C_2 , c_3 to C_2 to get C_3 , and continue. At each stage we can use the argument in the previous paragraph and obtain that $C_2, \dots, C_m$ ($= B(T)_3$) can be right ordered so that $B(T)_2$ inherits its original right order, $U^{B(T)_3}$ is convex and $1 < c_1 \ll \dots \ll c_m \ll a_1$.

The subgroups $H_1 := \langle k_0, a_1, \dots, a_m \rangle$ and $H_2 = \langle k_0, a_1 c_1, \dots, a_m c_m \rangle$ of $B(T)_4$ are free on the indicated generators. Let N be the convex normal subgroup of $B(T)_3$ generated by $c_1, \dots, c_m$. Then $H_1 \cap N = H_2 \cap N = \{1\}$ and the isomorphism between $H_1 N / N$ and $H_2 N / N$ induced by the order-preserving isomorphism $k_0 \mapsto k_0$, $a_i \mapsto a_i c_i$ ($i = 1, \dots, m$), is the identity. By Corollary 6.3, $B(T)_4$ can be right ordered so that $B(T)_3$ inherits its original right order. By Lemma 2.1, this can be achieved with $c_m \ll d \ll a_1$.

We need just one more application of Corollary 6.3 to complete the proof of the theorem. As was shown in [1], Lemma 17, the groups $\langle a_1, \dots, a_m, k_0, t_0 \rangle$ and $\langle a_1, \dots, a_m, k_0, t_0 d \rangle$ are isomorphic, the former having defining relations $t_0^{w k_0 w^{-1}} = t_0$ and the latter $(t_0 d)^{w k_0 w^{-1}} = t_0 d$ ($w \in E$). This time, let N be the convex normal subgroup of $B(T)_4$ generated by d , $H_1 := \langle a_1, \dots, a_m, k_0, t_0 \rangle$ and $H_2 := \langle a_1, \dots, a_m, k_0, t_0 d \rangle$

with the order-preserving isomorphism given by $a_i \mapsto a_i$, $k_0 \mapsto k_0$, $t_0 \mapsto t_0 d$ ($i = 1, \dots, m$). So Corollary 6.3 again pertains. Thus we obtain that $B(T)_5$ can be right ordered and Theorem D follows. //

The proof of Theorem D actually gives

Corollary 7.2 *A finitely generated right ordered group can be embedded (as a right ordered group) in some finitely presented right ordered group iff it can be defined by a recursively enumerable set of relations.*

We obtain the standard consequence of the Higman Embedding Theorem by taking the free product of all finitely presented right ordered groups (to within isomorphism), embedding this countable recursively generated and related right ordered countable group in a three-generator recursively defined right ordered group (Corollary 6.8), and then applying Theorem D.

Corollary 7.3 *There is a finitely presented right ordered group in which every right ordered finitely presented group can be embedded (as a right ordered group).*

Theorem E (cf. [7]) *A finitely generated right orderable group has soluble word problem iff it can be embedded in a simple group which can be embedded in a finitely presented right orderable group.*

Proof. The algebraic condition clearly implies the logical one (see the proof of Theorem IV.7.4 in [20]). To prove the converse, let G be a right ordered finitely generated group with soluble word problem. Then we can recursively enumerate all pairs (u, v) in G with $u, v \neq 1$. Let $\{(u_n, v_n) \mid n \in \mathbb{Z}_+\}$ be such a listing. We can right order $C_1 := \langle G, t_1 \mid u_1^{t_1} = v_1 \rangle$ so that the right order on G extends to one on C_1 by Corollary 6.7. Moreover, C_1 has soluble word problem if G does as was shown in [7] (or see the proof of Theorem IV.7.4 in [20]). We define C_{n+1} inductively by

$$C_{n+1} := \langle C_n, t_{n+1} \mid u_{n+1}^{t_{n+1}} = v_{n+1} \rangle.$$

For all $m \in \mathbb{Z}_+$ we get that C_m is right orderable with a right order extending to a right order on C_{m+1} by Corollary 6.7. Let $G^\#$ be the countable union of such HNN-extensions (as m runs through $\mathbb{Z}_+$). Let $S_0 := G$ and $S_{k+1} := S_k^\#$ ($k \in \mathbb{N}$). Define $S := \bigcup \{S_n : n \in \mathbb{N}\}$. Then

G is embedded in the right ordered simple group S (which has soluble word problem as G has soluble word problem). We can then embed S in a three-generator right ordered group T by Corollary 6.8 (which can be defined by a recursively enumerable set of relations because G can). We can embed T in a finitely presented right ordered group by Theorem D. This proves the theorem. //

Again we can milk the proof and obtain

Corollary 7.4 *A finitely generated right ordered group has soluble word problem iff it can be embedded in a right ordered simple group which can be embedded in a right ordered finitely presented group (all the embeddings also preserving order).*

Being right orderable is a Markov property for finitely presented groups (see [20], Section IV.4): isomorphic finitely presented groups are either both right orderable or neither is, there is a right orderable finitely presented group (e.g., $\langle x \mid x = 1 \rangle$) and there is a finitely presented group (e.g., $\langle y \mid y^2 = 1 \rangle$) that cannot be embedded in any right orderable group. Therefore, there is no algorithm to determine if an arbitrary finitely presented group is right orderable or not ([20], Theorem IV.4.1).

The question arises whether we could solve the isomorphism problem for right orderable finitely presented groups, assuming that we were provided with an oracle which would tell us (truthfully) that the finitely presented groups we are considering are right orderable. Unfortunately, the construction given by Rabin (see the proof of Theorem IV.4.1. in [20]) does not fit the hypotheses of our corollaries, so this remains open.

8 Concluding Remarks and Questions

We close this article with two questions which we have been unable to answer and which may need very different techniques.

Question 8.1 *Is there a two-sided orderable finitely presented group with insoluble word problem?*

Question 8.2 *Is there a lattice-orderable finitely presented group with insoluble word problem?*

Given the interrelation between right orderable and lattice-orderable groups, the second seems more tractable. We showed that the finitely presented group given in the proof of Corollary 7.1 was not orderable because we had distinct elements g, h with $g^2 = h^2$ (an impossibility in any two-sided orderable group). We do not know if the groups $B(T)$ are lattice-orderable; they are certainly not if there are elements g, h which are not conjugate with $g^2 = h^2$. Indeed, Theorems D and E have lattice-ordered group analogues (see [11] and [12]). A finitely presented lattice-ordered group with insoluble group word problem was constructed in [13]. Our results give an alternative proof. For if $B(T)$ satisfies Corollary 7.1, let $L(T)$ be the lattice-ordered group with the same presentation as that of $B(T)$, but now in the category of lattice-ordered groups. So the generation can use the lattice operations as well as the group ones. If Ω is the totally ordered set of $B(T)$, then we can group embed $B(T)$ in $A(\Omega)$ via the right regular representation. This provides a (group and lattice) homomorphism of $L(T)$ into the lattice-ordered group $A(\Omega)$. Since the right regular representation of $B(T)$ in $A(\Omega)$ is injective, so is the homomorphism of $B(T)$ into $L(T)$. Hence $L(T)$ has insoluble group word problem.

If we take the free product of the two-sided orderable free groups $F(a, b)$ on free generators a, b and $F(c, d)$ on free generators c, d amalgamating $\langle b^{-n}ab^n \mid n \in E \rangle$ and $\langle d^{-n}cd^n \mid n \in E \rangle$ in the obvious way, then the resulting finitely generated group has a recursively enumerable set of defining relations if E is a recursively enumerable subset of $\mathbb{Z}_+$. It is right orderable by Corollary 5.9 and has insoluble word problem if E is not recursive. By Theorem D, it can be embedded in a finitely presented right orderable group (which necessarily has insoluble word problem in this case), though the proof does not lead to a two-sided orderable group.

Finally, we mention that S. Lemieux has proved that Novikov's finitely presented groups (which have soluble word problem and insoluble conjugacy problem) are right orderable [19].

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