

# Hypersurfaces and the Weil conjectures

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## Abstract

We give a proof that the Riemann hypothesis for hypersurfaces over finite fields implies the result for all smooth proper varieties, by a deformation argument which does not use the theory of Lefschetz pencils or the  $\ell$ -adic Fourier transform.

## Introduction

Suppose that  $X \subset \mathbb{P}^{d+1}$  is a smooth geometrically irreducible hypersurface over a finite field  $\mathbb{F}_q$ . It is well-known [6, 9] that in this case the Riemann hypothesis for the zeta function of  $X$  is equivalent to the point-counting estimate:  $|\#X(\mathbb{F}_{q^n}) - \#\mathbb{P}^d(\mathbb{F}_{q^n})| = O(q^{nd/2})$ . It would therefore be extremely interesting to have an “elementary” proof of this Diophantine estimate (see for example [9, p.299]), in the spirit of Stepanov’s method for curves [2, 14]. We hasten to say that we have no idea how to do this. What we show in this paper is a curious, albeit entirely useless, related result (Theorem 3.3 below): one can deduce the Riemann hypothesis for *all* proper smooth  $\mathbb{F}_q$ -schemes from the Riemann hypothesis for smooth hypersurfaces, without using monodromy of Lefschetz pencils or Fourier transform. It is tempting to conclude from this that an elementary proof of the Riemann Hypothesis for hypersurfaces is unlikely to exist.

The ingredients of the proof are: the existence of the Rapoport-Zink vanishing cycles spectral sequence, de Jong’s alterations results, and Deligne’s theorem on the local monodromy of pure sheaves on curves [7, §1.8]. (It is worth noting that the proof of the latter theorem, although ingenious, is very short.) Of course the rules of the game do not permit the use of the hard Lefschetz theorem or of any consequences of the theory of weights.

The idea of deforming to hypersurfaces comes from Ayoub’s work on the conservation conjecture for motives [1, Lemma 5.8]. An outline of the proof is as follows. Suppose  $X$  is a smooth proper  $\mathbb{F}_q$ -scheme of dimension  $d$ . Then  $X$  is birational to a (generally singular) hypersurface  $X' \subset \mathbb{P}^{d+1}$ . Deform  $X'$  to a family of hypersurfaces  $H \subset \mathbb{P}^{d+1} \times \mathbb{A}^1$  whose general member is smooth. Suppose there is a smooth connected curve  $T/\mathbb{F}_q$  and

a finite morphism  $T \rightarrow \mathbb{A}^1$  such that the basechange of  $H$  to  $T$  has a semistable model,  $f: E \rightarrow T$  say. Then almost all closed fibres  $E_t$ ,  $t \in T$  are smooth hypersurfaces, so they satisfy the Riemann hypothesis. Let  $E_{\bar{K}}$  be the geometric generic fibre of  $f$ . Then Deligne's local monodromy theorem can be applied to the sheaves  $R^i f_* \mathbb{Q}_\ell$  to show that at the points of degeneration of  $f$ , the monodromy and weight filtrations on  $H^i(E \otimes \bar{K}, \mathbb{Q}_\ell)$  agree.

By construction, there will be some degenerate fibre  $E_t$  which is a normal crossings divisor, and one of whose components admits a dominant rational map to  $X$ . Some piece of the cohomology of  $X$  therefore contributes, via the Rapoport-Zink spectral sequence, to  $H^*(E \otimes \bar{K}, \mathbb{Q}_\ell)$ . Using the equality of monodromy and weight filtrations, a calculation using the spectral sequence then shows that the piece of  $H^d(X \otimes \bar{\mathbb{F}}_q, \mathbb{Q}_\ell)$  which does contribute is pure of weight  $d$ , and that the remaining piece comes from varieties of lower dimension, hence is pure by induction on dimension.

Since the existence of semi-stable models in finite characteristic is unknown, to make a proof one has to work with alterations [3, 4] instead; while complicating the details somewhat, this does not add any essential further difficulty.

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*Notations and terminology:*  $p$  and  $\ell$  are prime numbers with  $\ell \neq p$ . We write  $\mathbb{F}$  for an algebraic closure of  $\mathbb{F}_p$ . If  $X$  is a scheme of finite type over some separably closed field of characteristic different from  $\ell$ , then  $H^*(X)$  denotes its  $\ell$ -adic cohomology, and  $H_c^*(X)$  its  $\ell$ -adic cohomology with proper support. A *variety* over an algebraically closed field  $k$  will mean an integral separated scheme of finite type over  $k$ . A morphism of varieties is an *alteration* [3] if it is proper, dominant and generically finite.

## 1 Weights and monodromy

In this section we review basic and well-known facts concerning the Riemann hypothesis and the weight-monodromy conjecture.

Let  $X/\mathbb{F}$  be a scheme of finite type. Then  $X$  comes by basechange from a scheme  $X_0/\mathbb{F}_q$  for some  $q = p^r$ , and so  $H^*(X) = H^*(X_0 \times \text{Spec } \mathbb{F})$  carries an action of  $\text{Gal}(\mathbb{F}/\mathbb{F}_q)$ . Let  $F_q \in \text{Gal}(\mathbb{F}/\mathbb{F}_q)$  be the geometric Frobenius, which is the inverse of the Frobenius substitution  $x \mapsto x^q$ . Recall that a finite-dimensional  $\mathbb{Q}_\ell$ -representation  $V$  of  $\text{Gal}(\mathbb{F}/\mathbb{F}_q)$  is *mixed of weights*  $\geq w$  (resp. *mixed of weights*  $\leq w$ , *pure of weight*  $w$ ) if for every eigenvalue  $\alpha \in \bar{\mathbb{Q}}_\ell$  of  $F_q$  on  $V$  there exists an integer  $j \geq 0$  (resp.  $j \leq 0$ ,  $j = 0$ ) such that, for every isomorphism  $\iota: \bar{\mathbb{Q}}_\ell \xrightarrow{\sim} \mathbb{C}$ , one has  $|\iota\alpha| = q^{(w+j)/2}$ . (This

implies in particular that  $\alpha$  is algebraic.) The Riemann hypothesis for a smooth proper  $\mathbb{F}$ -scheme  $X$  is the statement:

RH( $X$ ): For all  $i \in [0, 2 \dim X]$ ,  $H^i(X)$  is pure of weight  $i$ .

Since a representation of  $\text{Gal}(\mathbb{F}/\mathbb{F}_q)$  is pure (or mixed) if and only if for some  $r \geq 1$  its restriction to  $\text{Gal}(\mathbb{F}/\mathbb{F}_{q^r})$  is (with the same weights), this depends only on  $X$ , not on the model  $X_0/\mathbb{F}_q$ . For any  $d \in \mathbb{N}$  let RH( $d$ ) be the statement

RH( $d$ ): For all proper smooth  $X$  of dimension  $\leq d$ , RH( $X$ ) holds.

The following reductions are completely standard arguments, but we include the proofs just to be clear we are not smuggling in forbidden ingredients.

**Lemma 1.1.** *Suppose RH( $X$ ) holds for all smooth projective  $\mathbb{F}$ -varieties  $X$  of dimension  $\leq d$ . Then RH( $d$ ) holds.*

*Proof.* Suppose  $X$  is a smooth proper  $\mathbb{F}$ -scheme of dimension  $\leq d$ . We may assume  $X$  is irreducible. By [3, Thm. 4.1] there exists an alteration  $f: X' \rightarrow X$  where  $X'$  is a smooth and projective variety, inducing an injection  $f^*: H^*(X) \hookrightarrow H^*(X')$ .  $\square$

**Lemma 1.2.** *Suppose  $X$  is smooth and projective of dimension  $d$ , and that RH( $d - 1$ ) holds.*

- (i) *If  $H^d(X)$  is pure of weight  $d$  then RH( $X$ ) holds.*
- (ii) *If  $X'$  is a variety birationally equivalent to  $X$ , then RH( $X$ ) holds if and only if, for every  $i$ ,  $H_c^i(X')$  is mixed of weights  $\leq i$ .*

*Proof.* (i) Let  $Y \subset X$  be a smooth hyperplane section. Then if  $i < d$ , by weak Lefschetz  $H^i(X)$  injects into  $H^i(Y)$ , which is pure of weight  $i$  by hypothesis. Poincaré duality then gives purity for  $H^i(X)$  if  $i > d$ .

(ii) There exist nonempty open subschemes  $U \subset X$ ,  $U' \subset X'$  with  $U \simeq U'$ . Let  $Y = X \setminus U$ ,  $Y' = X' \setminus U'$ . Consider the long exact sequences of cohomology with proper support

$$\begin{array}{ccccccc} H^{i-1}(Y) & \longrightarrow & H_c^i(U) & \longrightarrow & H^i(X) & \longrightarrow & H^i(Y) \\ & & \simeq \downarrow & & & & \\ H^{i-1}(Y') & \longrightarrow & H_c^i(U') & \longrightarrow & H_c^i(X') & \longrightarrow & H_c^i(Y') \end{array}$$

By induction on dimension and RH( $d - 1$ ), we may assume that  $H_c^i(Y')$  and  $H^i(Y)$  are mixed of weights  $\leq i$  for every  $i$ . Therefore

$$H_c^i(X') \text{ is mixed of weights } \leq i \iff H^i(X) \text{ is mixed of weights } \leq i.$$

By Poincaré duality this is equivalent to the condition:  $H^{2d-i}(X)$  is mixed of weights  $\geq 2d - i$ . Combining these for all  $i$  gives the result.  $\square$

**Lemma 1.3.** *Let  $X, X'$ , be smooth projective varieties, and let  $f: X' \rightarrow X$  be an alteration. Let  $\Gamma \subset \text{Aut}(X')$  be a finite subgroup such that  $f$  factors as  $X' \rightarrow X'/\Gamma \xrightarrow{g} X$ , where  $g$  is a radicial alteration. Assume  $\text{RH}(X)$  and  $\text{RH}(d-1)$ . Then for every  $i$ , the invariant subspace  $H^i(X')^\Gamma$  is pure of weight  $i$ .*

*Proof.* There exists a nonempty open  $\Gamma$ -invariant subscheme  $U' \subset X'$  such that  $U' \rightarrow U'/\Gamma$  is étale and  $g: U'/\Gamma \rightarrow U = f(U')$  is finite, flat and radicial. Then  $H_c^i(U) \simeq H_c^i(U')^\Gamma$ , and  $H_c^i(U)$  is mixed of weights  $\leq i$  by Lemma 1.2(ii). The obvious equivariant generalisation of 1.2(ii) then implies that  $H^i(X')^\Gamma$  is pure of weight  $i$ .  $\square$

**Lemma 1.4.** *Let  $X, X'$  be smooth projective varieties of dimension  $d$ , let  $\Gamma$  be a finite group acting on  $X'$ , and let  $f: X'/\Gamma \dashrightarrow X$  be a dominant rational map. Assume  $\text{RH}(d-1)$  holds. Then if  $H^d(X')^\Gamma$  is pure of weight  $d$ ,  $\text{RH}(X)$  holds.*

*Proof.* One can find a  $\Gamma$ -invariant nonempty open subscheme  $U' \subset X'$  and an open subscheme  $U \subset X$  such that  $f$  induces a finite flat morphism  $U' \rightarrow U$ , and then  $f^*: H_c^*(U) \hookrightarrow H_c^*(U')^\Gamma$ . Then one can reverse the argument of the previous lemma.  $\square$

Now let  $R$  be a Henselian discrete valuation ring, with finite residue field of characteristic  $p$ , and field of fractions  $K$ . Let  $G_K$  denote the absolute Galois group of  $K$  and  $I_K$  its inertia subgroup. Let  $t_\ell: I_K \rightarrow \mathbb{Z}_\ell(1)$  be the tame Kummer character, given by

$$t_\ell(\sigma) = \left( \ell^n \sqrt[\ell]{\pi}^{\sigma-1} \right)_{n \in \mathbb{N}}$$

for any uniformiser  $\pi$  of  $R$ .

Let  $\rho: G_K \rightarrow \text{Aut}(V)$  be a continuous finite-dimensional  $\mathbb{Q}_\ell$ -representation. Then one knows that there exists a finite extension  $K'/K$  such that  $\rho(I_{K'})$  is unipotent. Moreover the restriction of  $\rho$  to  $I_{K'}$  factors as  $\chi \circ t_\ell$  for some homomorphism  $\chi: \mathbb{Z}_\ell(1) \rightarrow GL(V)$ . One defines the *logarithm of monodromy* to be  $N = \log(\chi): V \rightarrow V(-1)$ , which can be regarded as a nilpotent endomorphism of  $V$ , with twisted Galois action. There is then an associated *monodromy filtration*  $(M_n)_{n \in \mathbb{Z}}$  on  $V$ , exhaustive and separated, which is uniquely characterised by the properties:

- $N(M_n(V)) \subset M_{n-2}(V)(-1)$
- For every  $j \in \mathbb{N}$ ,  $N^j$  induces an isomorphism  $\text{gr}_j^M(V) \xrightarrow{\sim} \text{gr}_{-j}^M(V)(-j)$ .

The monodromy filtration is stable under  $G_K$ , and the action of  $I_{K'}$  on  $\text{gr}_*^M(V)$  is trivial. Write  $q$  for the order of the residue field of  $K'$ , so that  $\text{Gal}(\mathbb{F}/\mathbb{F}_q)$  acts on  $\text{gr}_*^M(V)$ . Say that  $V$  is *monodromy-pure of weight*  $w \in \mathbb{Z}$  if for every  $n \in \mathbb{Z}$ ,  $\text{gr}_n^M(V)$  is pure of weight  $n + w$ . Deligne's weight-monodromy conjecture [5] is then the statement:

**Weight-Monodromy Conjecture 1.5.** Let  $X/K$  be smooth and proper. Then for every  $i$ ,  $H^i(X \otimes \bar{K})$  is monodromy-pure of weight  $i$ .

**Lemma 1.6.** Let  $V$  be monodromy-pure of weight  $w$ , and let  $G_\bullet$  be a filtration on  $V$  by  $G_K$ -invariant subspaces such that

- (a) for all  $n \in \mathbb{Z}$ ,  $N(G_n V) \subset (G_{n-2} V)(-1)$ ;
- (b) for every  $n \neq 0$ ,  $\text{gr}_n^G V$  is pure of weight  $w + n$ .

Then  $G_\bullet = M_\bullet$ .

*Proof.* Note that (a) implies that  $I_K$  acts on  $\text{gr}_\bullet^G V$  by a finite quotient, so (b) makes sense. By assumption,  $\text{gr}_n^M V$  is pure of weight  $w + n$ . Assumption (b) therefore implies that on  $G_{-1} V$  the filtrations induced by  $G_\bullet$  and  $M_\bullet$  are equal, so that

$$n < 0 \implies \text{gr}_n^G V = \text{gr}_n^G G_{-1} V = \text{gr}_n^M G_{-1} V \subset \text{gr}_n^M V \quad (1.7)$$

Dually, on  $V/G_0 V$  the filtrations induced by  $G_\bullet$  and  $M_\bullet$  are equal, and

$$n > 0 \implies \text{gr}_n^M V \longrightarrow \text{gr}_n^M V/G_0 V = \text{gr}_n^G V/G_0 V = \text{gr}_n^G V. \quad (1.8)$$

Then for every  $j > 0$  we have a commutative diagram

$$\begin{array}{ccc} \text{gr}_j^G(V) & \longleftarrow & \text{gr}_j^M(V) \\ N^j \downarrow & & \simeq \downarrow N^j \\ \text{gr}_{-j}^G(V)(-j) & \hookrightarrow & \text{gr}_{-j}^M(V)(-j) \end{array}$$

and therefore  $N^j: \text{gr}_j^G(V) \xrightarrow{\simeq} \text{gr}_{-j}^G(V)(-j)$ . By the uniqueness of the monodromy filtration this implies  $G_\bullet = M_\bullet$ .  $\square$

We finally recall Deligne's theorem on the monodromy of pure sheaves on curves. Let  $T/\mathbb{F}_q$  be a smooth curve,  $U \subset T$  the complement of a closed point  $t \in T$ . Let  $\hat{\mathcal{O}}_{T,t}$  be the henselised local ring,  $K$  its field of fractions, and  $\bar{\eta}: \text{Spec } \bar{K} \rightarrow U$  the associated geometric point.

**Theorem 1.9.** [7, Thm 1.8.4] Let  $\mathcal{F}$  be a lisse  $\mathbb{Q}_\ell$ -sheaf on  $U$  which is punctually pure of weight  $w$  (i.e. for every closed point  $s \in U$ ,  $\mathcal{F}_{\bar{s}}$  is pure of weight  $w$ ). Then the representation  $\mathcal{F}_{\bar{\eta}}$  of  $\text{Gal}(\bar{K}/K)$  is monodromy-pure of weight  $w$ .

## 2 Vanishing cycles

We first review Rapoport-Zink's weight spectral sequence [11], which is the  $\ell$ -adic version of Steenbrink's spectral sequence in Hodge theory [13]. A detailed account of all of this theory can be found in [8].

Let  $S$  be a regular scheme of dimension 1, and  $f: X \rightarrow S$  a proper morphism. Recall that  $f$  is *strictly semistable* if  $f$  is flat and generically smooth,  $X$  is regular, and for each closed point  $s \in S$  the fibre  $X_s$  is a reduced divisor with strict normal crossings (i.e., the irreducible components of  $X_s$  are smooth over  $s$  and intersect transversally).

Let  $R$  be a Henselian DVR, and  $K$  its field of fractions. Assume that its residue field  $k$  has characteristic different from  $\ell$ . Let  $\mathcal{Y} \rightarrow \text{Spec } R$  be proper and strictly semistable, with  $\mathcal{Y}$  integral. Let  $d$  be the relative dimension of  $f$ , and write  $Y = \mathcal{Y} \otimes \bar{k}$  for the geometric special fibre. Thus  $Y = \bigcup_{0 \leq i \leq N} Y_i$  where each  $Y_i$  is smooth and proper, of dimension  $d$ . As usual we write

$$Y_I = \bigcap_{i \in I} Y_i, \quad \emptyset \neq I \subset \{0, \dots, N\}$$

$$Y_{\langle m \rangle} = \bigsqcup_{|I|=m+1} Y_I, \quad 0 \leq m \leq d$$

(our numbering differs from that of Rapoport and Zink, for whom this would be  $Y_{\langle m+1 \rangle}$ ). The scheme  $Y_{\langle m \rangle}$ , if nonempty, is proper and smooth of dimension  $d - m$ . For  $a, b, n \in \mathbb{Z}$  set

$${}^n C^{a,b} = \begin{cases} H^{n+2b}(Y_{\langle a-b \rangle})(b) & \text{if } a \geq 0 \geq b \\ 0 & \text{otherwise} \end{cases}$$

Let

$$\rho = \rho^{a,b,n}: {}^n C^{a,b} \rightarrow {}^n C^{a+1,b}$$

be the alternating sum of the restriction maps, and

$$\gamma = \gamma^{a,b,n}: {}^n C^{a,b} \rightarrow {}^n C^{a,b+1}$$

the alternating sum of the Gysin maps. Then these maps make  ${}^n C^{a,b}$  into a cohomological double complex; let  $({}^m C^n, d_C)$  be the associated simple complex. The Rapoport-Zink spectral sequence is a spectral sequence

$$E_1^{mn} = {}^m C^n \implies H^*(\mathcal{Y}_{\bar{K}}).$$

There is a mapping  $N: E \rightarrow E(-1)$  on the entire spectral sequence, given in degree 1 by the tautological map (the identity map when both source and target are nonzero)

$$N: {}^n C^{a,b} \rightarrow {}^{n-2} C^{a+1,b+1}(-1)$$

and which on the abutment is the logarithm of monodromy operator. In particular, the abutment filtration<sup>1</sup>  $G_\bullet$  on  $E_\infty^*$  satisfies  $N(G_m) \subset G_{m-2}(-1)$ .

Poincaré duality induces isomorphisms  $(E_1^{m,n})^\vee \simeq E_1^{-m,2d-n}(d)$  and (up to sign)  $d_1^{-m,2d-n}$  is the transpose of  $d_1^{m-r,n+r-1}$ , hence also  $(E_2^{m,n})^\vee \simeq E_2^{-m,2d-n}(d)$ . (In fact the whole spectral sequence is compatible with Poincaré duality on the generic and special fibres, but we will not use this deeper fact — see for example [12].)

For the rest of this section we assume that  $k = \mathbb{F}$ . In this case, the Weil conjectures for the varieties  $Y_{\langle m \rangle}$  imply that the spectral sequence degenerates at  $E_2$ , and that its abutment filtration is the weight filtration. The weight-monodromy conjecture for  $H^i(\mathcal{Y}_K)$  is then equivalent to the statement: for every  $j \geq 0$  the map  $N^j: E_\infty^{-j,i+j} \rightarrow E_\infty^{j,i-j}(-j)$  is an isomorphism. Conversely, from the weight-monodromy conjecture one can deduce parts of the Weil conjectures:

**Proposition 2.1.** *Suppose that  $\mathcal{Y}$  is projective over  $\text{Spec } R$ , and let  $\mathcal{Z} \subset \mathcal{Y}$  be a hyperplane section in general position. Assume:*

- (a)  $\text{RH}(d-1)$  holds;
- (b)  $\mathcal{Y}_K$  and  $\mathcal{Z}_K$  satisfy the weight-monodromy conjecture.

*Then the Rapoport-Zink spectral sequence degenerates at  $E_2$ , and  $\text{RH}(Y_i)$  holds for each component  $Y_i$  of  $Y$ .*

(By “hyperplane in general position”, we mean one that meets each  $Y_{\langle m \rangle}$  transversally.)

*Proof.* Assumption (a) and the proof of Lemma 1.2(i) imply that if  $(m, i) \neq (0, d)$  then  $H^i(Y_{\langle m \rangle})$  is pure of weight  $i$ . Therefore  $E_1^{m,n}$  is pure of weight  $n$  except possibly for  $(m, n) = (0, d)$ , and so for every  $i \neq d$  the abutment filtration on  $E_\infty^i$  equals the monodromy filtration shifted by  $i$ . Moreover, for  $r \geq 2$  the only differentials which can possibly be non-zero are those with source or target  $E_r^{0,d}$ , namely

$$E_2^{-r,d+r-1} = E_r^{-r,d+r-1} \xrightarrow{d_r^{-r,d+r-1}} E_r^{0,d} \xrightarrow{d_r^{0,d}} E_r^{r,d-r+1} = E_2^{r,d-r+1} \quad (2.2)$$

We have  $E_2^{r,d-r-1} = E_\infty^{r,d-r-1}$ , since this group is not a source or target of any of the differentials (2.2). Also, since the abutment and weight filtrations on  $E_\infty^{d-1}$  are equal, by hypothesis (b) the map

$$N^r: E_\infty^{-r,d+r-1} \rightarrow E_\infty^{r,d-r-1}(-r).$$

is an isomorphism. So

$$\dim E_2^{r,d-r-1} = \dim E_\infty^{r,d-r-1} = \dim E_\infty^{-r,d+r-1} \leq \dim E_2^{-r,d+r-1}$$

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<sup>1</sup>The increasing filtration, normalised so that  $\text{gr}_n E_\infty^k = E_\infty^{k-n,n}$ .

with equality if and only if  $d_r^{-r, d+r-1} = 0$ . By 2.3(i) below we have equality, hence for each  $r \geq 2$ , the differentials  $d_r^{-r, d+r-1}$  vanish. The dual argument using 2.3(ii) shows that the differentials  $d_r^{d, 0}$  also vanish. Therefore the spectral sequence degenerates at  $E_2$ .

For the second assertion, it is enough to show that  $H^d(Y_{(0)})$  is pure of weight  $d$ . Consider  $V = H^d(\mathcal{Y}_{\bar{K}})$ . By hypothesis  $V$  satisfies the weight-monodromy conjecture, hence  $N^j: \text{gr}_j^W(V) \xrightarrow{\sim} \text{gr}_{-j}^W(V)(-j)$  for every  $j \geq 0$ . The abutment filtration  $G_\bullet$  on  $V$  then satisfies the hypotheses of Lemma 1.6 with  $w = d$ , and so  $\text{gr}_0^G V$  is also pure of weight  $d$ . Therefore  $\text{gr}_0^G(V) = E_2^{0, d}$  is pure of weight  $d$ . Then since  $E_2^{0, d}$  is the middle homology of the complex

$$\begin{aligned} \bigoplus_{a \geq 0} H^{d-2a-2}(Y_{(2a+1)})(-a-1) &\rightarrow \bigoplus_{a \geq 0} H^{d-2a}(Y_{(2a)})(-a) \\ &\rightarrow \bigoplus_{a \geq 1} H^{d-2a+2}(Y_{(2a-1)})(-a+1) \end{aligned}$$

by hypothesis (a) we see that  $H^d(Y_{(0)})$  is also pure of weight  $d$ .  $\square$

**Lemma 2.3.** (i) For every  $m > 0$  the map

$$N^m: E_2^{-m, d+m-1} \rightarrow E_2^{m, d-m-1}(-m)$$

is injective.

(ii) For every  $m > 0$  the map

$$N^m: E_2^{-m, d+m+1} \rightarrow E_2^{m, d-m+1}(-m).$$

is surjective.

*Proof.* The second assertion follows from the first by Poincaré duality. Let  $(Z E_r^{m, n})$  be the Rapoport-Zink spectral sequence associated to  $\mathcal{Z}$ . It degenerates at  $E_2$ , since we are assuming the Weil conjectures in dimension  $< d$ . In the commutative square:

$$\begin{array}{ccc} E_2^{-m, d-1+m} & \xrightarrow{N^m} & E_2^{m, d-1-m} \\ \alpha \downarrow & & \downarrow \\ Z E_2^{-m, d-1+m} & \xrightarrow[N^m]{\sim} & Z E_2^{m, d-1-m} \end{array}$$

the bottom map is an isomorphism by hypothesis (b). It is therefore enough to show that the map  $\alpha$  is an injection. But  $\alpha$  is induced by the vertical maps  $\beta_j$  in the diagram

$$\begin{array}{ccccc} E_1^{-m-1, d-1+m} & \longrightarrow & E_1^{-m, d-1+m} & \longrightarrow & E_1^{-m+1, d-1+m} \\ \beta_{-1} \downarrow & & \beta_0 \downarrow & & \beta_1 \downarrow \\ Z E_1^{-m-1, d-1+m} & \longrightarrow & Z E_1^{-m, d-1+m} & \longrightarrow & Z E_1^{-m+1, d-1+m} \end{array}$$



We have for  $j = 0$  or  $-1$

$$E_1^{-m+j, d-1+m} = \bigoplus_{a \geq 0} H^{d-m-2a+2j-1}(Y_{(2a+m-j)})(-a-m+j)$$

and  $\dim Y_{(2a+m-j)} = d - m - 2a + j$ , so by weak Lefschetz  $\beta_j$  is an isomorphism for  $j = -1$  and an injection for  $j = 0$ . Therefore  $\alpha$  is an injection.  $\square$

We need a variant of Proposition 2.1, in which we are given a finite subgroup  $\Gamma \subset \text{Aut}(\mathcal{Y}/R)$ . Then  $\Gamma$  acts on the Rapoport-Zink spectral sequence, and going through all the steps of the previous proof, one obtains:

**Proposition 2.4.** *Let  $\mathcal{Y}/R$  be projective and strictly semistable, and  $\Gamma \subset \text{Aut}(\mathcal{Y}/R)$  a finite subgroup. Let  $\mathcal{Z} \subset \mathcal{Y}$  be a hyperplane section in general position. Assume the following hold:*

- (a)  $\text{RH}(d-1)$  holds.
- (b)  $H^*(\mathcal{Y}_{\bar{K}})^\Gamma$  and  $H^*(\mathcal{Z}_{\bar{K}})$  satisfy the weight-monodromy conjecture.

*Then the  $\Gamma$ -invariant part of the Rapoport-Zink spectral sequence for  $H^*(\mathcal{Y}_{\bar{K}})$  degenerates at  $E_2$ , and for every  $i$ ,  $H^i(Y_{(0)})^\Gamma$  is pure of weight  $i$ .  $\square$*

### 3 Deformation to hypersurfaces

In this section we work over an algebraically closed field  $k$  (ultimately  $k = \mathbb{F}$ ), and all morphisms will be  $k$ -morphisms.

**Lemma 3.1.** *Let  $X$  be a smooth variety of dimension  $d$ . Then there exists:*

- a projective strictly semistable morphism  $g: E \rightarrow T$ , where  $T$  is a smooth curve;
- a finite subgroup  $\Gamma \subset \text{Aut}(E/T)$ ;
- a nonempty open subscheme  $U \subset T$  and a family of smooth hypersurfaces  $Z \subset \mathbb{P}^{d+1} \times U$ , together with a dominant  $U$ -morphism  $E \times_T U \rightarrow Z$ , inducing a purely inseparable inclusion of function fields  $\kappa(Z) \subset \kappa(E)^\Gamma$ ;
- a point  $t \in T(k)$  and a dominant rational map from  $E_t/\Gamma$  to  $X$ .

*Proof.* There exists a birational morphism  $p: X \rightarrow H_0$  where  $H_0 \subset \mathbb{P}^n$  is an integral hypersurface (in general singular). Choose a pencil  $f: H \rightarrow \mathbb{P}^1$  of hypersurfaces whose generic fibre is smooth, and with  $f^{-1}(0) = H_0$ . We then apply the procedure of the last paragraph of the proof of [4, 5.13] to

$f$ , taking the group  $G$  in *loc. cit.* to be trivial. This gives a commutative diagram of varieties and dominant projective morphisms

$$\begin{array}{ccc} E & \xrightarrow{g} & T \\ \downarrow & & \downarrow \pi \\ H & \xrightarrow{f} & \mathbb{P}^1 \end{array}$$

in which  $T$  is a smooth curve and  $E$  is regular. The unlabelled vertical morphism is an alteration, and  $g$  (which is obtained by repeated blowups from a “ $G'$ -pluri nodal fibration”, in the terminology of *loc. cit.*) is strictly semistable. Finally there is a finite group  $\Gamma$  (which is  $N$  in de Jong’s notation) acting on  $E$  covering the trivial actions on  $H$  and  $T$ , such the extension  $k(E)^\Gamma/k(H \times_{\mathbb{P}^1} T)$  is purely inseparable. We take  $U \subset T$  sufficiently small such that  $Z = H \times_{\mathbb{P}^1} U$  is smooth over  $U$ . Finally, for any point  $t \in \pi^{-1}(0)$  we get a dominant rational map  $E_t/\Gamma \rightarrow H_0 \dashrightarrow X$ .  $\square$

*Remark 3.2.* Ayoub’s lemma [1, Lemma 5.8] is similar but stronger, since an alteration is not required in characteristic zero, as one may in that case appeal to semi-stable reduction [10].

We now suppose that  $k = \mathbb{F}$ .

**Theorem 3.3.** *Assume that  $\text{RH}(d-1)$  holds, and that  $\text{RH}(V)$  holds for every smooth hypersurface  $V \subset \mathbb{P}_{\mathbb{F}}^{d+1}$ . Then  $\text{RH}(d)$  holds.*

*Proof.* Let  $X/\mathbb{F}$  be smooth and proper of dimension  $d$ . Choose  $E \rightarrow T$  be as in the lemma. Fix a projective embedding  $E \hookrightarrow \mathbb{P}^D \times T$ , and choose a hyperplane  $L \subset \mathbb{P}^D$  which meets  $E_t$  in general position. Let  $W = E \cap (L \times T) \xrightarrow{g} T$  be the hyperplane section given by  $L$ . Replacing  $T$  by an open subscheme containing  $t$ , and  $E$  by its inverse image by  $f$ , we may assume that for every  $s \in U = T - \{t\}$  the fibres  $H_{\pi(s)}$ ,  $E_s$  and  $W_s$  are smooth, and that the morphism  $E_s/\Gamma \rightarrow H_{\pi(s)}$  is the composition of a modification and a radicial morphism. The sheaves

$$\mathcal{F}^i = R^i f_* \mathbb{Q}_\ell|_U, \quad \mathcal{G}^i = R^i g_* \mathbb{Q}_\ell|_U$$

and  $(\mathcal{F}^i)^G$  are therefore lisse  $\mathbb{Q}_\ell$ -sheaves on  $U$ . Let  $R = \tilde{\mathcal{O}}_{T,t}$ ,  $K = \text{Frac}(R)$  and  $\bar{\eta}: \text{Spec } \bar{K} \rightarrow U$  as in §1.

**Proposition 3.4.**  *$H^i(E_{\bar{\eta}})^\Gamma$  and  $H^i(W_{\bar{\eta}})$  are monodromy-pure of weight  $i$ .*

*Proof.* Let  $s \in U$  be a closed point. By hypothesis  $\text{RH}(W_s)$  holds, and by 1.3 and the hypotheses,  $H^i(E_s)^\Gamma$  is pure of weight  $i$ . Thus the sheaves  $(\mathcal{F}^i)^\Gamma$  and  $\mathcal{G}^i$  are punctually pure of weight  $i$ , and Theorem 1.9 then implies that  $H^i(E_{\bar{\eta}})^\Gamma = (\mathcal{F}_{\bar{\eta}}^i)^\Gamma$  and  $H^i(W_{\bar{\eta}}) = \mathcal{G}_{\bar{\eta}}^i$  are monodromy-pure of weight  $i$ .  $\square$

Let  $\tilde{E}_t$  be the normalisation of  $E_t$ , and apply Proposition 2.4 with  $\mathcal{Y} = E \times_T \text{Spec } R$ ,  $\mathcal{Z} = W \times_T \text{Spec } R$ . We conclude that  $H^i(\tilde{E}_t)^\Gamma$  is pure of weight  $i$ . Now let  $X' \subset E_t$  be any component that dominates  $X$  under the rational map  $E_t/\Gamma \rightarrow X$ , and let  $\Delta \subset G$  be its stabiliser. Then  $H^i(X')^\Delta \subset H^i(\tilde{E}_t)^\Gamma$ . So  $H^i(X')^\Delta$  is pure of weight  $i$ , and so by Lemma 1.4, the Riemann hypothesis holds for  $X$ .  $\square$

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