Vanishing cycles and non-classical parabolic cohomology

A. J. Scholl[1]

0. Introduction

The work described here began as an attempt to understand the structure of the cohomology groups associated to a subgroup Γ of finite index of $SL_2(\mathbb{Z})$ which are not congruence subgroups. One knows that to the space of cusp forms of weight $w > 2$ on Γ (whose dimension we denote by d) one can **[15]** attach a motive M, which is pure of weight $w-1$ and of rank $2d$, defined over some number field; it is a direct factor of the motive of a suitable compact model of the $(w-2)$-fold fibre product of a family of elliptic curves over the modular curve, as in the classical case of a congruence subgroup. The realisations of M are the parabolic cohomology groups. By looking instead at the uncompactified fibre variety (that is, with the divisor over the cusps removed) one gets a mixed motive M', which is a direct factor of the "motive with compact supports" of the noncompact variety; M' is an extension of M by an Artin motive of rank equal to the number of cusps of Γ. The cohomology classes associated to this Artin motive arise from Eisenstein series.

In the case of a congruence subgroup this extension is trivial—it is split by the action of the Hecke algebra (an example of the "Manin-Drinfeld principle"). Already for weight 2 it is known that these extensions can be nontrivial in general, and for rather simple reasons. Namely, Belyi's theorem implies that any connected smooth curve C over $\overline{\mathbb{Q}}$ (not necessarily projective) has a Zariski open subset U whose complex points are isomorphic to the quotient of the upper-half plane by some finite index subgroup Γ. In this situation M is the H^1-motive of the compactification of C, and the mixed motive M' is $h^1_c(U)$, the H^1 of U with compact supports. If C is the complement in a projective curve of, say, two points whose difference is a divisor of infinite order, then the Abel-Jacobi theorem implies that M' is a nontrivial extension of motives.

The search for algebraic invariants to classify the extension M' in general leads naturally to the study of the motivic cohomology of the fibre varieties. Beilinson's conjectures give a conjectural description of these groups. In this case the regulator map, which plays an essential role in the Beilinson conjectures, turns out to be zero—in fact the target space is zero. Assuming the truth of Beilinson's conjectures, this would imply that the element of motivic cohomology which classify the extension M' can only be non-zero if it is non-integral, which for our purposes may be taken to mean that its image under the ℓ-adic regulator map is not locally trivial. In other words, we have to study the action of inertia at a prime of bad reduction on the ℓ-adic realisation of M'. This is the main problem addressed in the present paper, independentally of any conjectures on motivic cohomology. We refer the reader to **[16]** for a discussion of the interpretation of the results as evidence for an "S-integral" version of Beilinson's conjectures.

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The main tool is, predictably, the theory of vanishing cycles in ℓ-adic cohomology. The situation we consider is the following: let S be the spectrum of a strictly local discrete valuation ring, with generic point η, and $\pi: X \to S$ a curve. Let $D \subset X$ be an effective relative Cartier divisor, contained in the smooth part of X/S, and let $\mathcal{F}$ be a lisse $\mathbb{Q}_\ell$-sheaf on $U = X - D$. Write $g : U \hookrightarrow X$ be the inclusion, and assume that the residue characteristic of S is different from ℓ. We try to compute the action of $\mathrm{Gal}(\bar{\eta}/\eta)$ on the nearby cycle sheaves $R^q\Psi(g_!\mathcal{F})$ at the closed points of D. If D is étale over S then there are no vanishing cycles, by standard results. The case in which we obtain non-trivial results is when some connected component of D is the union of two sections of X/S which meet in the special fibre—here the vanishing cycle groups at the point of intersection were calculated by Deligne in [**7**]. This is described in §1 of the paper, and may be of independent interest.

In the second section we apply these results when X is the base of an elliptic surface $\pi: E \to X$, and D is the divisor over which π fails to be smooth. The sheaf $\mathcal{F}$ is a symmetric power of the restriction of $R^1\pi_*\mathbb{Q}_\ell$ to U. This is the setting of parabolic cohomology. In the case of a classical elliptic modular surface it is known [**8**] that after a finite base-change D is étale over S (the cusps of the modular curves do not meet in characteristic p). But for an elliptic surface attached to a general subgroup of $SL_2(\mathbb{Z})$ of finite index this appears to be a common phenomenon. Some examples of this are described in §4. These were chosen for ease of computation rather than for any intrinsic properties, and from the (admittedly sparse) evidence it appears that the this type of bad reduction is the norm rather than the exception for a noncongruence subgroup.

In §3 we explain how the vanishing cycle results show that certain parabolic elements of motivic cohomology are non-trivial, and interpret them in terms of extensions of motives. As indicated above, the construction gives a family of elements of infinite order in higher K-groups of algebraic varieties which have zero image under the archimedean regulator.

In a companion paper [**18**] we will show how these results can be used to given information about the image of the Galois group in the ℓ-adic parabolic cohomology groups. For the examples considered in §4, we can prove that the "primitive part" of parabolic cohomology is an irreducible symplectic representation of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, and that the image is an open subgroup of the group of symplectic similitudes.

A further application of our method, described in detail in [**19**], is to show that the Hecke algebra for an arbitrary finite index subgroup of $SL_2(\mathbb{Z})$ can act on modular forms in a rather trivial way, supplementing a conjecture of Atkin (partially proved by Serre [**20**] and Thompson [**22**]).

The author would like to thank P. Deligne for interesting discussions about vanishing cycles. Deligne pointed out to us that our proof of the main theorem of §1 is very close to his proof (unpublished) of the theorem of Thom-Sebastiani. He would also like to thank L. Illusie and G. Laumon for valuable discussions, and D. Blasius, who explained how to simplify our original proof of Proposition 5.1.

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1. Vanishing cycles

1.1. For the foundations of the theory of vanishing cycles, the reader should consult [**SGA7**]. We first recall some standard notation. Let $X = \operatorname{Spec} A$ be normal, and ℓ a prime invertible in A. Let ξ be the generic point of X. For any non-zero $t \in A$ there is a canonical character

$$\begin{aligned} \epsilon_t : \mathrm{Gal}(\bar{\xi}/\xi) &\to \mathbb{Z}_\ell(1) \\ \sigma &\mapsto \left(\sqrt[\ell^n]{t}^{\,\sigma-1}\right)_n \end{aligned}$$

It factors through the fundamental group $\pi_1(X - \{t = 0\}, \bar{\xi})$ of the complement of the zero-locus of t. Moreover if X is strictly local and t is a parameter then ϵ_t gives an isomorphism

$$\pi_1^{(\ell)}(X - \{t = 0\}, \bar{\eta}) \xrightarrow{\sim} \mathbb{Z}_\ell(1) \tag{1.1.1}$$

depending only on the ideal $tR \subset R$.

1.2. For the rest of this section, S will denote the spectrum of a strictly henselian discrete valuation ring, with generic point η and closed point s. We write $I = \mathrm{Gal}(\bar{\eta}/\eta)$. Let ℓ be a prime different from the residue characteristic of S.

1.3. Let $f : X \to S$ be a morphism of finite type, and $\mathcal{G}$ a constructible $\mathbb{Q}_\ell$-sheaf on X. There is a commutative diagram:—

$$\begin{array}{ccccccc}
X_{\bar{\eta}} & & & \xrightarrow{\quad \bar{j} \quad} & & & \\
\downarrow & \searrow & & & & & \\
\bar{\eta} & & X_\eta & \xhookrightarrow{\ j\ } & X & \xhookleftarrow{\ i\ } & X_s \\
& \searrow & \downarrow & & \downarrow f & & \downarrow \\
& & \eta & \longrightarrow & S & \longleftarrow & s
\end{array}$$

The sheaves of nearby, resp. vanishing cycles are the complexes of $\mathbb{Q}_\ell$-sheaves on X_s

$$\begin{aligned} R\Psi\mathcal{G} &= i^* R\bar{j}_* \bar{j}^* \mathcal{G} \\ R\Phi\mathcal{G} &= \mathrm{Cone}[i^*\mathcal{G} \to R\Psi\mathcal{G}] \end{aligned}$$

1.4. There is a long exact sequence

$$H^q(X_s, i^*\mathcal{G}) \to H^q(X_s, R\Psi\mathcal{G}) \to H^q(X_s, R\Phi\mathcal{G}) \to \dots \tag{1.4.1}$$

and for f proper,

$$H^q(X_s, R\Psi\mathcal{G}) = H^q(X, R\bar{j}_*\bar{j}^*\mathcal{G}) = H^q(X_{\bar{\eta}}, \bar{j}^*\mathcal{G})$$

is the cohomology of the generic geometric fibre. There is a natural action of I on $R\Psi\mathcal{G}$ and $R\Phi\mathcal{G}$, compatible with the sequence (1.4.1).

1.5. If $x \in X_s$ is a smooth point of the fibre such that $\mathcal{G}$ is smooth in some neighbourhod of x, then $(R\Phi\mathcal{G})_x = 0$, by the acyclicity of a smooth morphism. Suppose that Σ, Y are disjoint closed subschemes of X_s, with $\dim\Sigma = 0$, such that if $x \in X_s - \Sigma \cup Y$ then f is smooth at x and $\mathcal{G}$ is smooth in a neighbourhood of x. Then $R\Phi\mathcal{G}$ is acyclic off $\Sigma \cup Y$ and

$$H^q(X_s, R\Phi\mathcal{G}) = H^q(Y, R\Phi\mathcal{G}|_Y) \oplus \bigoplus_{x\in\Sigma} (R^q\Phi\mathcal{G})_x \tag{1.5.1}$$

1.6. We will calculate $(R\Phi\mathcal{G})_x$ and the action of I on it in the following situation:

(i) $f : X \to S$ is a smooth curve. Let ξ be its generic point.

(ii) There are two distinct sections $z_i : S \to Z$ $(i = 1, 2)$ of f, with images Z_1, $Z_2 \subset X$. We assume that Z_1, Z_2 intersect at $x \in X_s$, with intersection multiplicity d. Write $Z = Z_1 \cup Z_2$, and let $i\colon Z \hookrightarrow X$ be the inclusion, and

$$g : U = X - Z \hookrightarrow X$$

the inclusion of the complement.

(iii) There is a $\mathbb{Q}_\ell$-sheaf $\mathcal{F}$ on U which is smooth in a neighbourhood of x, such that $\mathcal{G} = g_!\mathcal{F}$.

(iv) The local monodromy representation of $\pi_1(\tilde U, \bar\xi)$ on $\mathcal{F}_{\bar\xi}$ is unipotent, and has a factorisation:—

$$\pi_1(\tilde U, \bar\xi) \to \bigoplus_{i=1,2} \pi_1^{(\ell)}(\tilde X_x - \tilde Z_i, \bar\xi) \xrightarrow{\sim} \mathbb{Z}_\ell(1)^2 \xrightarrow{\chi} \operatorname{Aut}\mathcal{F}_{\bar\xi} \tag{1.6.1}$$

for some homomorphism χ.

Here $\tilde X_x$ is the strict henselisation of X at x, $\tilde U = \tilde X_x \times_X U$ and $\tilde Z_i = \tilde X_x \times_X Z_i$. Also write $\tilde{\mathcal{F}} = \mathcal{F}|_{\tilde U}$. The middle arrow in (1.6.1) comes from the isomorphisms (1.1.1) above.

1.7. The determination of the groups of vanishing cycles in essentially this situation is done in [**7**], §3.1. Briefly, as the local monodromy is unipotent there exists a filtration on $\tilde{\mathcal{F}}$ such that $\operatorname{Gr}\tilde{\mathcal{F}}$ extends to a smooth sheaf on $\tilde X_x$. Denoting the fibre of this extension at x by $(\operatorname{Gr}\mathcal{F})_x$, we then have:

$$(R^q\Psi g_!\mathcal{F})_x = (R^q\Phi g_!\mathcal{F})_x = \begin{cases} 0 & \text{for } q \neq 1 \\ (\operatorname{Gr}\mathcal{F})_x & \text{for } q = 1 \end{cases}$$

and there is an exact sequence

$$0 \to (R^0\Phi i_* i^* g_*\mathcal{F})_x \to (R^1\Phi g_!\mathcal{F})_x \to (R^1\Phi g_*\mathcal{F})_x \to 0 \tag{1.7.1}$$

where

$$(R^0\Phi i_* i^* g_*\mathcal{F})_x = \operatorname{coker}\big[(g_*\mathcal{F})_x \to \bigoplus_{i=1,2} H^0(Z_{\bar\eta}, i^* g_*\mathcal{F})\big]. \tag{1.7.2}$$

1.8. Define an action of I on $\mathcal{F}_{\bar{\xi}}$ by

$$I \xrightarrow{\epsilon_\pi} \mathbb{Z}_\ell(1) \xhookrightarrow{d\times\mathrm{diag}} \mathbb{Z}_\ell(1)^2 \xrightarrow{\chi} \mathrm{Aut}\, \mathcal{F}_{\bar{\xi}}$$

Theorem 1.9. *There is an I-equivariant isomorphism*

$$(R^1\Phi g_!\mathcal{F})_x \xrightarrow{\sim} \mathcal{F}_{\bar{\xi}}.$$

The proof will be in four steps.

Proposition 1.10. *(First step) Theorem 1.9 holds in characteristic zero.*

1.11. The transcendental analogue of 1.6 is given by the following:

(i) $f : X = D \times D \to S = D$ is the morphism $f(z,t) = t$, where D is the unit disc. Write $D^* = D - \{0\}$, and let $t_0 \in D^*$ be a base point.

(ii) $z = z_i(t)$ $(i = 1, 2)$ are distinct holomorphic sections of f with $z_1(0) = z_2(0) = 0$. Assume $z_1(t) \neq z_2(t)$ if $t \neq 0$, and that $z_1 - z_2$ vanishes to order d at $t = 0$. Write $g : U \hookrightarrow X$ for the inclusion of the complement of z_1 and z_2.

(iii) $\mathcal{F}$ is a local system of $\mathbb{Q}$-vector spaces on U with unipotent monodromy.

1.12. Let $m : D \to X$ be the section $m = \frac{1}{2}(z_1 + z_2)$. As $m(D^*) \subset U$ we can pull $\mathcal{F}$ back to a local system $m^*\mathcal{F}$ on D^*, giving a natural action of $\pi_1(D^*, t_0) = \mathbb{Z}(1)$ on $(m^*\mathcal{F})_{t_0} = \mathcal{F}_{(m(t_0),t_0)}$.

1.13. Since each fibre X_t is a disc, the group of vanishing cycles $\left(R^1\Phi g_!\mathcal{F}\right)_0$ can be identified with the cohomology of the fibre

$$H^1(X_{t_0}, g_!\mathcal{F})$$

together with its action of $\pi_1(D^*, t_0)$.

Proposition 1.14. *There is a $\pi_1(D^*, t_0)$-equivariant isomorphism*

$$H^1(X_{t_0}, g_!\mathcal{F}) \xrightarrow{\sim} (m^*\mathcal{F})_{t_0}.$$

Proof. Let $V \subset X$ be the subset

$$V = \left\{(\lambda z_1(t) + \mu z_2(t), t) \;\middle|\; \lambda,\ \mu \geq 0,\ \lambda + \mu = 1,\ t \in D^*\right\};$$

thus for each $t \in D^*$, V_t is the line segment joining $z_1(t)$ and $z_2(t)$. We have for each $t \in D^*$ the inclusions

$$X_t - V_t \xhookrightarrow{h} U_t \xhookrightarrow{g} X_t$$

and an associated exact sequence of sheaves on X_t

$$0 \to g_! h_! h^* \mathcal{F}_t \to g_! \mathcal{F}_t \to g_! \mathcal{F}_t\big|_{V_t} \to 0 \tag{1.14.1}$$

(where $\mathcal{F}_t$ is the restriction of $\mathcal{F}$ to the fibre U_t). If $\mathcal{F} = A$ is constant on U then

$$H^*(X_t, g_! h_! h^* F_t) = H^*(X_t \operatorname{rel} V_t, \mathbb{Q}) \otimes A = 0$$

so by choosing a filtration on $\mathcal{F}$ for which $\operatorname{Gr}\mathcal{F}$ is constant, one has in general

$$H^*(X_t, g_! h_! h^* F_t) = 0$$

So by (1.14.1) we have

$$H^1(X_t, g_! F_t) \xrightarrow{\sim} H^1(V_t, g_! \mathcal{F}_t\big|_{V_t}); \tag{1.14.2}$$

and since V_t is an interval

$$H^1(V_t, g_! \mathcal{F}_t\big|_{V_t}) = H^1(V_t \operatorname{rel} \partial V_t, \mathbb{Q}) \otimes \mathcal{F}_{m(t)}. \tag{1.14.3}$$

Fixing an ordering of the sections z_1, z_2 determines an isomorphism

$$H^1(V_t \operatorname{rel} \partial V_t, \mathbb{Q}) \xrightarrow{\sim} \mathbb{Q}$$

which by (1.14.2) and (1.14.3) gives an isomorphism of local systems on D^*:

$$R^1 f_* g_! \mathcal{F}\big|_{D^*} \xrightarrow{\sim} m^* \mathcal{F}.$$

■

Proposition 1.15. *The notation being as in 1.11–1.14, the following diagram commutes:—*

$$\begin{array}{ccccc}
\pi_1(D^*, t_0) & \xrightarrow{m_*} & \pi_1(U, (m(t_0), t_0)) & \longrightarrow & \bigoplus_i \pi_1(X - Z_i, (m(t_0), 0)) \\
\| & & & \searrow & \| \\
\mathbb{Z}(1) & & \xrightarrow{d \times \mathrm{diag}} & & \mathbb{Z}(1)^2
\end{array}$$

Proof. We can assume that $z_1(t) = -z_2(t) = t^d$, so that $m(t) = 0$. Then there are isomorphisms $X - Z_i \xrightarrow{\sim} D^* \times D$ given by $(z, t) \mapsto (z + (-1)^i t^d, t)$. From this is it easy to check that the generator of $\pi_1(D^*, t_0)$ is mapped by m_* to d times the generator of $\pi(X - Z_i, (m(t_0), t_0))$. ■

Combining this with 1.14 and the Lefschetz principle, one obtains 1.10.

Lemma 1.16. *Let the assumptions be as in 1.6 above, but assume in addition that $X \simeq \mathbb{A}^1_S$, that $\mathcal{F}$ is smooth on U, and that the* **global** *monodromy of $\mathcal{F}$ is unipotent. Then*

$$H^q(X_{\bar\eta}, g_!\mathcal{F}) \xrightarrow{\sim} (R^q\Phi g_!\mathcal{F})_x$$

for all q.

Proof. By (1.4.1) it is enough to show that $H^*(X_s, g_!\mathcal{F}) = 0$. First assume that $\mathcal{F} = A$ is constant. Then $H^*(X_s, g_!\mathcal{F}) = h^*(\mathbb{A}^1 \operatorname{rel} 0, \mathbb{Q}_\ell) \otimes A = 0$. The general case reduces to this by choosing a filtration on $\mathcal{F}$ such that $\operatorname{Gr}\mathcal{F}$ is constant. ■

1.17. (Second step) We now imitate the proof of the Picard-Lefschetz formula ([**SGA7**], exposé XIV) to pass from characteristic zero to a particularly simple mixed characteristic situation (over a base of dimension 2). Let p be a prime, and let $\mathbf{S}$ be the strict henselisation of $\operatorname{Spec}\mathbb{Z}[y]$ at (p, y). Write $\mathbf{X} = \mathbb{A}^1 \times \mathbf{S} = \mathbf{S}[z] \xrightarrow{\mathbf{f}} \mathbf{S}$. Let $\boldsymbol{\eta}$, $\boldsymbol{\xi}$ be the generic points of $\mathbf{S}$ and $\mathbf{X}$.

Write $\mathbf{Z}_1$, $\mathbf{Z}_2$ for the sections $z = 0$, $z = y$ of $\mathbf{f}$, and set $\mathbf{Z} = \mathbf{Z}_1 \cup \mathbf{Z}_2$, $\mathbf{g} : \mathfrak{U} = \mathbf{X} - \mathbf{Z} \hookrightarrow \mathbf{X}$.

Finally, let $\mathfrak{F}$ be a smooth $\mathbb{Q}_\ell$-sheaf on $\mathfrak{U}$ (with $l \neq p$) whose **global** monodromy is unipotent, and admits a factorisation

$$\pi_1(\mathfrak{U}, \bar{\boldsymbol{\xi}}) \xrightarrow{(\epsilon_z, \epsilon_{z-y})} \mathbb{Z}_\ell(1)^2 \xrightarrow{\theta} \operatorname{Aut}\mathfrak{F}_{\bar{\boldsymbol{\xi}}}$$

Proposition 1.18.

(i) $R^1\mathbf{f}_*\mathbf{g}_!\mathfrak{F}$ is smooth on $\mathbf{S}^0 = \mathbf{S} - \{y = 0\}$, and its formation commutes with basechange.

(ii) Define an action of $\operatorname{Gal}(\bar{\boldsymbol{\eta}}/\boldsymbol{\eta})$ on $\mathfrak{F}_{\bar{\boldsymbol{\xi}}}$ by the composite $\theta \circ \operatorname{diag} \circ \epsilon_y$. Then there is a $\operatorname{Gal}(\bar{\boldsymbol{\eta}}/\boldsymbol{\eta})$-equivariant isomorphism

$$H^1(\mathbf{X}_{\bar{\boldsymbol{\eta}}}, \mathbf{g}_!\mathfrak{F}) \xrightarrow{\sim} \mathfrak{F}_{\bar{\boldsymbol{\xi}}}.$$

Proof. (i) We have $\mathfrak{U} \times_{\mathbf{S}} \mathbf{S}^0 \simeq \mathbb{A}^1 - \{0, 1\}$, and in this system of coordinates the sheaf $\mathfrak{F}$ is tamely ramified along 0, 1 and ∞. Therefore the dimension of $H^1(\mathbf{X}_{\bar t}, \mathbf{g}_!\mathfrak{F})$ is constant for all geometric points $\bar t$ of $\mathbf{S}^0$, and (i) follows.

(ii) By part (i), $R^1\mathbf{f}_*\mathbf{g}_!\mathfrak{F}$ is tamely ramified along $\{y = 0\}$, and so the action of $\operatorname{Gal}(\bar{\boldsymbol{\eta}}/\boldsymbol{\eta})$ on $H^1(\mathbf{X}_{\bar{\boldsymbol{\xi}}}, \mathbf{g}_!\mathfrak{F})$ factors through the character ϵ_y. Let S be the strict henselisation of $\operatorname{Spec}\mathbb{Q}[y]$ at (y), and $S \to \mathbf{S}$ the obvious morphism. Then (ii) can be verified after pullback to S, where it is a consequence of 1.10 and 1.16 above. ■

1.19. (Third step) Let S be as in 1.2 and $\mathbf{S}$ as in 1.17, and let $\phi : S \to \mathbf{S}$ a morphism. Set $d = \operatorname{ord}_\pi \phi^* y$, and assume that $0 < d < \infty$. Define:—

$X' = \mathbf{X} \underset{\mathbf{S},\phi}{\times} S = \mathbb{A}^1_S$;

$Z' = \operatorname{pr}_1^{-1}(\mathbf{Z}) = Z_1' \cup Z_2$, $g' : U' = X' - Z' \hookrightarrow X'$,

$\mathcal{F}' = \operatorname{pr}_1^* \mathfrak{F}$, a smooth $\mathbb{Q}_\ell$-sheaf on U';

x' = the point $(z = 0)$ on the fibre X_s'.

Define an action of I on $\mathcal{F}_{\bar{\boldsymbol{\xi}}}$ by the composite map $\theta \circ (d \times \operatorname{diag}) \circ \epsilon_\pi : I \to \operatorname{Aut}\mathfrak{F}_{\bar{\boldsymbol{\xi}}}$.

Proposition 1.20. *There is an I-equivariant isomorphism*

$$(R^1\Phi g_!^! \mathcal{F}')_{x'} \xrightarrow{\sim} \mathfrak{F}_{\bar{\xi}}.$$

Proof. By 1.16 above,

$$(R^1\Phi g_!^! \mathcal{F}')_{x'} \xrightarrow{\sim} H^1(\mathbf{X}_{\bar{\eta}}, \mathbf{g}_! \mathfrak{F})$$

and so the proposition follows from 1.18 and the commutativity of the square

$$\begin{array}{ccc} \mathrm{Gal}(\bar{\eta}/\eta) & \xrightarrow{\phi_*} & \pi_1(\mathbf{S}^0, \bar{\boldsymbol{\eta}}) \\ \big\downarrow \wr\, \epsilon_\pi & & \big\downarrow \wr\, \epsilon_y \\ \mathbb{Z}_\ell(1) & \xrightarrow{\times d} & \mathbb{Z}_\ell(1) \end{array}$$

■

Proposition 1.21. *(Fourth step) Given X/S, Z and $\mathcal{F}$ as in 1.6, there exists $\phi : S \to \mathbf{S}$ as in 1.20 and a sheaf $\mathfrak{F}$ on $\mathbf{U}$ as in 1.17 such that the triples*

$$(\tilde{X}_x, \tilde{Z}, \tilde{\mathcal{F}}), \quad (\tilde{X}'_{x'}, \tilde{Z}', \tilde{\mathcal{F}}')$$

are isomorphic.

Proof. The pair (X, Z) is locally isomorphic (for the étale topology) to $\big(\mathrm{Spec}\, \mathcal{O}_S[z], \{z(z - \pi^d) = 0\}\big)$. Therefore if we take $\phi : S \to \mathbf{S}$ by $\phi^*(y) = \pi^d$, the pairs $(\tilde{X}_x, \tilde{Z})$, $(\tilde{X}'_{x'}, \tilde{Z}')$ are isomorphic. It remains to construct $\mathfrak{F}$. There is a commutative diagram

$$\begin{array}{ccccc} \pi_1(\tilde{U}', \bar{\xi}') & \longrightarrow & \pi_1(\mathbf{U}, \bar{\boldsymbol{\xi}}) & & \\ \big\downarrow \wr & \searrow^{(\epsilon_z, \epsilon_{z-\pi^d})} & \big\downarrow (\epsilon_z, \epsilon_{z-y}) & \searrow & \\ \pi_1(\tilde{U}, \bar{\xi}) & \xrightarrow{(1.6.1)} & \mathbb{Z}_\ell(1)^2 & \xrightarrow{\chi} & \mathrm{Aut}\, \mathcal{F}_{\bar{\xi}} \end{array}$$

Therefore the dotted arrow defines a sheaf $\mathfrak{F}$ on $\mathbf{U}$ which satisfies the hypotheses 1.17, and which is equipped with an isomorphism $\mathfrak{F}_{\bar{\xi}} \xrightarrow{\sim} \mathcal{F}_{\bar{\xi}}$ intertwining θ and χ. Therefore $(\tilde{X}_x, \tilde{Z}, \tilde{\mathcal{F}}) \simeq (\tilde{X}'_{x'}, \tilde{Z}', \tilde{\mathcal{F}}')$ as required.

Combining 1.20 and 1.21 gives 1.9. ■

2. Application to parabolic cohomology

2.1. We will apply the results of §1 to parabolic cohomology groups associated to (not necessarily modular) families of elliptic curves. We fix a prime ℓ; cohomology groups will understood to have coefficients in $\mathbb{Q}_\ell$ unless otherwise indicated. To begin with we review some of the results of **[15]**; however in the present context we will have to adopt a different notation.

2.2. We consider a smooth projective curve X over a number field K; it is assumed to be geometrically connected. Let

$$\pi\colon E \to X$$

be an elliptic surface equipped with a section $e\colon X \to E$. We assume:

(i) π is semistable and strictly non-constant; and

(ii) there exists a finite K-group scheme G of sections of π, meeting every irreducible component of every geometric fibre.

Explicitly, the first part of (ii) means that there is a monomorphism of group schemes over X

$$G \times X \to E^{\mathrm{smooth}}$$

where $E^{\mathrm{smooth}} \subset E$ is the open set of points at which π is smooth. By strictly nonconstant we mean that π does not become isomorphic to a constant family after any finite flat basechange $X' \to X$.

2.3. Let $Z \subset X$ be the finite set of closed points over which π is not smooth, and let $g\colon U = X - Z \hookrightarrow X$ be the inclusion of the complement. Write

$$\mathring{\pi} : \mathring{E} = E \times_X U \to U$$

for the restriction of π. Let $\mathcal{F}$ be the rank 2 smooth $\mathbb{Q}_\ell$-sheaf $R^1\mathring{\pi}_*\mathbb{Q}_\ell$ on U, and let $k > 0$ be an integer. The cohomology groups to be considered are

$$H^1(\overline{U}, \mathrm{Sym}^k \mathcal{F}), \quad H^1_c(\overline{U}, \mathrm{Sym}^k \mathcal{F})$$

and the parabolic cohomology

$$H^1(\overline{X}, g_* \mathrm{Sym}^k \mathcal{F}) \simeq \mathrm{Im}\big[H^1_c(\overline{U}, \mathrm{Sym}^k \mathcal{F}) \to H^1(\overline{U}, \mathrm{Sym}^k \mathcal{F})\big].$$

These groups occur in the cohomology of the Kuga-Sato fibre varieties, as we now recall.

2.4. Consider the k-fold fibre product

(2.4.1)
$$\overbrace{E \times_X \ldots \times_X E}^{k}.$$

For $k > 1$ this has singularities at points of the form $(P_1, \ldots, P_k)$ such that $P_i \notin E^{\mathrm{smooth}}$ for at least two distinct i. In **[6]**, **[15]** it is explained how to resolve the singularities of

(2.4.1) by successively blowing up strata of increasing dimension in the singular locus. We write $E^{(k)}$ for the resulting desingularisation and $Y^{(k)}$ for the inverse image of Z in $E^{(k)}$.

2.5. Let $\Gamma_k \subset \underline{\mathrm{Aut}}(E^{(k)}/K)$ be the subgroup scheme generated by:

- $\mathfrak{S}_k$, the symmetric group, acting by permuting the factors of the fibre product;
- μ_2^k, acting as multiplication by ± 1 on each factor; and
- G^k, acting by translation.

Then Γ_k is actually the semidirect product $(G \rtimes \mu_2)^k \rtimes \mathfrak{S}_k$, and there is a unique character $\epsilon\colon \Gamma_k \to \mu_2$ whose restrictions to the subgroups are

the trivial character of G^k;

the character $(x_i) \mapsto \prod x_i$ of μ_2^k;

the sign character of $\mathfrak{S}_k$.

2.6. To avoid risk of confusion we recall some elementary notions about group scheme actions. If V is a representation of $\mathrm{Gal}(\bar{K}/K)$ over (say) $\mathbb{Q}_\ell$, an action of a discrete K-group scheme H is the same as an abstract group action

$$H(\bar{K}) \times V \to V$$

which is $\mathrm{Gal}(\bar{K}/K)$-equivariant. For a character $\phi\colon H \to \mathbb{G}_m$ of H, the eigenspace

$$V(\phi) = \{v \in V \mid gv = \phi(g)v \quad \text{for all } g \in H(\bar{K})\}$$

is then Galois invariant.

In the present setting the action of Γ_k on $E^{(k)}$ induces an action on its ℓ-adic cohomology, as well as that of $Y^{(k)}$ and $E^{(k)} - Y^{(k)}$ (with and without compact supports).

2.7. We first recall the basically standard isomorphisms

$$H^1(\overline{U}, \mathrm{Sym}^k \mathcal{F}) \xrightarrow{\sim} H^{k+1}(\overline{E^{(k)}} - \overline{Y^{(k)}})(\epsilon)$$
$$H^1_c(\overline{U}, \mathrm{Sym}^k \mathcal{F}) \xrightarrow{\sim} H^{k+1}_c(\overline{E^{(k)}} - \overline{Y^{(k)}})(\epsilon).$$

which are compatible with the natural transformation $H_c \to H$. Let $\mathring{\pi}^{(k)}$ be the k-fold fibre product of $\mathring{\pi}$. Then by the Künneth formula $R^*\mathring{\pi}^{(k)}_*\mathbb{Q}_\ell = \otimes^k R^*\mathring{\pi}_*\mathbb{Q}_\ell$, one has $R^*\mathring{\pi}^{(k)}_*\mathbb{Q}_\ell(\epsilon) = \mathrm{Sym}^k \mathcal{F}$, and for $k > 0$ the sheaf $\mathrm{Sym}^k \mathcal{F}$ has no H^0 or H^2 (with compact supports or without). So the isomorphisms follow from the Leray spectral sequence.

2.8. From Theorem 1.2.1 of **[15]** there is also an isomorphism

$$H^1(\overline{X}, g_* \mathrm{Sym}^k \mathcal{F}) \xrightarrow{\sim} H^{k+1}(\overline{E^{(k)}})(\epsilon)$$

which is compatible with the previous two isomorphisms. Moreover there are commutative ladders:

(2.8.1)

$$\begin{array}{ccccccccc}
0 & \to & H^1(\overline{X}, g_* \mathrm{Sym}^k \mathcal{F}) & \to & H^1(\overline{U}, \mathrm{Sym}^k \mathcal{F}) & \to & H^0(\overline{Z})(-k-1) & \to & 0 \\
 & & \downarrow\wr & & \downarrow\wr & & \downarrow\wr & & \\
0 & \to & H^{k+1}(\overline{E^{(k)}})(\epsilon) & \to & H^{k+1}(\overline{E^{(k)}} - \overline{Y^{(k)}})(\epsilon) & \to & H^{k+2}_{\overline{Y^{(k)}}}(\overline{E^{(k)}})(\epsilon) & \to & 0
\end{array}$$

(2.8.2)

$$\begin{array}{ccccccccc} 0 & \to & H^0(\overline{Z}) & \to & H^1_c(\overline{U}, \operatorname{Sym}^k \mathcal{F}) & \to & H^1(\overline{X}, g_* \operatorname{Sym}^k \mathcal{F}) & \to & 0 \\ & & \downarrow\wr & & \downarrow\wr & & \downarrow\wr & & \\ 0 & \to & H^k(\overline{Y^{(k)}})(\epsilon) & \to & H^{k+1}_c(\overline{E^{(k)}} - \overline{Y^{(k)}})(\epsilon) & \to & H^{k+1}(\overline{E^{(k)}})(\epsilon) & \to & 0 \end{array}$$

in which the arrows are equivariant with respect to an open subgroup of $\mathrm{Gal}(\bar{K}/K)$.

We shall indicate how the arguments of **[15]** must be modified to obtain these results in the present more general setting. For convenience we will write references to that paper in the form MF1.2.3.

2.9. The first modification arises in MF1.3.2. The singular fibres of π need not be standard Néron polygons. This depends on whether for $z \in Z$ the connected component of identity of E_z^{smooth} is isomorphic to $\mathbb{G}_m$ or to the norm 1 subgroup of $R_{F/\kappa(z)}\mathbb{G}_m$, where $F/\kappa(z)$ is a quadratic extension. In ℓ-adic cohomology this does not alter the groups obtained, but it means that the maps in (2.8.1), (2.8.2) need not be Galois equivariant.

2.10. The construction of the desingularisation (MF2.1.1, MF3.1.0(i)) requires only that the singularities in the fibres of π be ordinary double points, locally isomorphic (for the étale topology) to $\operatorname{Spec} R[x, y]/(xy - t)$. This holds in the present setting because of hypothesis 2.2(i).

2.11. For the calculation of the cohomology we can replace K by any finite extension, so may assume that G is constant. The group Γ_k then replaces the group $((\mathbb{Z}/n)^2 \rtimes \boldsymbol{\mu}_2)^k \rtimes \mathfrak{S}_k$ of MF1.1.1. The only place where the translations by sections of finite order intervene is MF3.1.0(iii), in the proof of which one passes from invariants under Γ_k to invariants (of a different module) under $\boldsymbol{\mu}_2^k \rtimes \mathfrak{S}_k$. This relies on the fact that Γ_k permutes the components of a singular stratum transitively. This uses only the fact that $(\mathbb{Z}/n)^2$ meets every irreducible component of every geometric fibre of the universal elliptic curve. Hypothesis 2.2(ii) therefore ensures that MF1.3.3 and its corollary MF1.2.1 remain valid.

2.12. The first ladder (2.8.1) is obtained by combining the exact sequences of MF1.2.0 and MF1.3.4. The left hand square commutes by functoriality, and we simply choose the third vertical arrow to make the second square commutative as well (this depends on the choices made earlier, cf. 2.9 above and MF1.3.2). The second ladder (2.8.2) is obtained from (2.8.1) by Poincaré duality, since the two long exact cohomology sequences for the inclusion of a closed subscheme—one for cohomology with supports, the other for cohomology with compact supports—are dual up to sign. (A proof is given in **[17]**, 0.2.)

2.13. For later use we recall a consequence of the Shimura isomorphism. For these we require a further hypothesis:

(iii) The classifying map from U to the modular stack $\mathcal{M}$ of elliptic curves is étale.

Let $\boldsymbol{\omega}$ be the dual of the Lie algebra sheaf of (the smooth part of) E over X. Then (iii) implies that the Kodaira-Spencer map $\boldsymbol{\omega} \to \Omega(\log Z) \otimes \boldsymbol{\omega}^{\otimes -1}$ is an isomorphism (**[11]** A1.3.17). From this it follows that the parabolic cohomology has Hodge type $(k+1, 0) + (0, k+1)$ (see e.g. **[13]**, 2.13(ii)). This gives the classical Shimura isomorphism, and in particular the formulae

$$
\begin{aligned}
&\dim_{\mathbb{Q}_\ell} H^1(\overline{X}, g_* \operatorname{Sym}^k \mathcal{F}) = 2\dim_K H^0(X, \boldsymbol{\omega}^{\otimes k} \otimes \Omega^1_X), \\
&\dim_{\mathbb{Q}_\ell} H^1_c(\overline{U}, g_* \operatorname{Sym}^k \mathcal{F}) = \dim_K H^0(X, \boldsymbol{\omega}^{\otimes k} \otimes \Omega^1_X) \\
&\qquad\qquad + \dim_K H^0(X, \boldsymbol{\omega}^{\otimes k} \otimes \Omega^1_X(\log Z)).
\end{aligned} \tag{2.13.1}
$$

2.14. Let $\mathbf{j}: X \to \mathbb{P}^1_{\mathbb{Q}}$ be the modular invariant of the family π. Then $Z = \mathbf{j}^{-1}(\infty)$ since π is semistable. Write $\mathcal{X}$ for the normalisation of $\mathbb{P}^1_{\mathbb{Z}}$ in the field of functions of X; then $\mathbf{j}$ extends to a finite morphism $\mathbf{j}_{\mathbb{Z}}: \mathcal{X} \to \mathbb{P}^1_{\mathbb{Z}}$. Write $\mathcal{Z} \subset \mathcal{X}$ for the closure of Z (or equivalently the inverse image $\mathbf{j}_{\mathbb{Z}}^{-1}(\infty)$). Set $\mathcal{U} = \mathcal{X} - \mathcal{Z}$.

Theorem 2.15. *Suppose there exists a finite prime v of K, of residue characteristic $p \neq \ell$, and a closed point x of the fibre $\mathcal{X}_v = \mathcal{X} \otimes \kappa(v)$, such that:*

- *$\mathcal{X}_v$ is smooth at x;*
- *The set of irreducible components of $\mathcal{Z}$ which contain x consists of two distinct section z_1, z_2 of $\mathcal{Z}$ over $\operatorname{Spec} \mathfrak{o}_K$.*

Assume also that if $p = 2$ then k is even. Then there is an open subgroup $I' \subset I_v$ of index at most 2 and a commutative diagram of I'-modules

$$
\begin{array}{ccccccccc}
0 & \to & H^0(\overline{Z}) & \to & H^1_c(\overline{U}, \operatorname{Sym}^k \mathcal{F}) & \to & H^1(\overline{X}, g_* \operatorname{Sym}^k \mathcal{F}) & \to & 0 \\
 & & \wr\downarrow \alpha & & \wr\downarrow & & \wr\downarrow & & \\
0 & \to & A & \to & B & \to & C & \to & 0
\end{array}
$$

where $\dim A = 1$, $\dim C = k$ and B is an indecomposable unipotent I'-module, and the top row is given by (2.8.2).

Corollary 2.16. *Under the hypotheses of 2.15, the exact sequences of (2.8.1), (2.8.2) are non-trivial extensions when restricted to any open subgroup of* $\operatorname{Gal}(\bar{K}/K)$.

In fact, since the action of I' on B is unipotent, it remains indecomposable when restricted to any subgroup of finite index.

2.17. *Proof of 2.15.* We will show that we can place ourselves in the situation of 1.6, if necessary after a finite extension of ground field. The base scheme will be $S = \operatorname{Spec} \tilde{\mathfrak{o}}_{K,v}$, and for the smooth curve over S we will take any sufficiently small Zariski open subscheme of $X \otimes \tilde{\mathfrak{o}}_{K,v}$ containing x.

2.18. We first must show that $\operatorname{Sym}^k \mathcal{F}$ extends to a sheaf on $\mathcal{U}$ which is smooth in a neighbourhood of x. We know that $\pi: E \to X$ is "almost" the pullback of an elliptic curve over $\mathbb{P}^1$. To be precise, let

$$\psi: \mathcal{E} \to \mathbb{A}^1_{\mathbb{Z}} - \{0, 1728\}$$

be the elliptic curve with affine equation

$$u^2 + tu = t^3 - (36x + 1)/(j - 1728)$$

(j being the coordinate on $\mathbb{A}^1$). Then the sheaf $R^1\psi_*\mathbb{Q}_\ell$ is smooth of rank 2 with unipotent monodromy along $j=\infty$, and its pullback to $U-\{j=0,1728\}$ is isomorphic to $\mathcal{F}\otimes\mathcal{L}$ for some rank 1 sheaf $\mathcal{L}$ on U with $\mathcal{L}^{\otimes 2}\simeq\mathbb{Q}_\ell$. Therefore if k is even $\operatorname{Sym}^k\mathcal{F}$ is actually isomorphic to the pullback of $\operatorname{Sym}^k R^1\psi_*\mathbb{Q}_\ell$ and so extends to a smooth sheaf on a neighbourhood of x in $\mathcal{U}$. In the case when k is odd $(\operatorname{Sym}^k\mathcal{F})\otimes\mathcal{L}$ is isomorphic to the pullback of $\operatorname{Sym}^k R^1\psi_*\mathbb{Q}_\ell$; but if p is odd, $\mathcal{L}$ is at most tamely ramified along $\mathcal{U}_v$, and so by Abhyankar's lemma extends over $\mathcal{U}_v$ after a quadratic base extension K'/K. We will assume that this basechange has been made in what follows, and I' will then be the inertia subgroup of $\operatorname{Gal}(\bar{K}/K')$ at v.

2.19. Finally we need to check hypothesis 1.6(iv) concerning monodromy. Let e_1, e_2 be the ramification degrees of $\mathbf{j}\colon X\to\mathbb{P}^1_K$ at z_1, z_2. Since E/X is semistable the local monodromy of $\mathcal{F}$ at z_1 and z_2 is unipotent and nontrivial. Now write

$$\tilde{\mathcal{U}}=\tilde{\mathcal{X}}_{\bar{x}}\times_{\mathcal{X}}\mathcal{U}\quad\text{and}\quad\tilde{\mathcal{V}}=\widetilde{\mathbb{P}^1_{S,\infty}}\times_{\mathbb{P}^1}\mathbb{A}^1.$$

The generic point ξ of X lifts to a geometric point $\bar\xi$ of $\tilde{\mathcal{U}}$, and $\mathbf{j}_{\mathbb{Z}}$ induces a morphism $\tilde{\mathcal{U}}\to\tilde{\mathcal{V}}$; by abuse of notation denote the image of $\bar\xi$ under this map by the same symbol. Then there is a commutative diagram:

$$\begin{array}{ccc}
 & \pi_1^{(\ell)}(\tilde{\mathcal{U}},\bar\xi) & \\
\swarrow & & \searrow \\
\bigoplus_{i=1,2}\pi_1^{(\ell)}(\tilde{\mathcal{X}}_{\bar{x}}-\tilde{\mathcal{Z}}_i,\bar\xi) & & \pi_1^{(\ell)}(\tilde{\mathcal{V}},\bar\xi) \\
\downarrow\wr & & \downarrow\wr \\
\mathbb{Z}_\ell(1)^2 & \xrightarrow{(\gamma_1,\gamma_2)\mapsto\Sigma e_i\gamma_i} & \mathbb{Z}_\ell(1)
\end{array}$$

Since the monodromy representation of $\pi_1^{(\ell)}(\tilde{\mathcal{U}},\bar\xi)$ on $\operatorname{Sym}^k\mathcal{F}_{\bar\xi}$ factors through $\pi_1^{(\ell)}(\tilde{\mathcal{V}},\bar\xi)$, the above diagram gives condition (iv).

2.20. We can therefore apply Theorem 1.9. We have a commutative diagram

$$\begin{array}{ccccccccc}
 & & H^0(\overline{Z}) & & & & & & \\
 & & \| & & & & & & \\
0 & \to & H^0(\overline{Z},g_*\operatorname{Sym}^k\mathcal{F}|_{\overline{Z}}) & \to & H^1(\overline{X},g_!\operatorname{Sym}^k\mathcal{F}) & \to & H^1(\overline{X},g_*\operatorname{Sym}^k\mathcal{F}) & \to & 0 \\
 & & \downarrow\alpha & & \downarrow\beta & & \downarrow & & \\
0 & \to & (R^0\Phi i_*i^*g_*\operatorname{Sym}^k\mathcal{F})_{\bar{x}} & \to & (R^1\Phi g_!\operatorname{Sym}^k\mathcal{F})_{\bar{x}} & \to & (R^1\Phi g_*\operatorname{Sym}^k\mathcal{F})_{\bar{x}} & \to & 0 \\
 & & \| & & \| & & \| & & \\
 & & A & & B & & C & &
\end{array}$$

Here the top line is the exact sequence (2.8.1), and the bottom is (1.7.1). The vertical map α is evidently surjective. From (1.7.2), A is 1-dimensional.

Lemma 2.21. *β is surjective.*

Consider the boundary map

$$\partial\colon (R^1\Phi g_! \operatorname{Sym}^k \mathcal{F})_{\bar{x}} \to H^2_c(\mathcal{U}_{\bar{v}}, \operatorname{Sym}^k \mathcal{F})$$

obtained from (1.4.1) and (1.5.1). It factors through $H^2_c(C, \operatorname{Sym}^k \mathcal{F})$, where $C \subset \mathcal{U}_{\bar{v}}$ is the unique irreducible component whose closure contains x. Now since $k > 0$ the sheaf $(\operatorname{Sym}^k \mathcal{F})^\vee = \operatorname{Sym}^k \mathcal{F} \otimes \mathbb{Q}_\ell(k)$ has no global sections over C, so this H^2_c vanishes.

2.22. Now by 1.9 the group B is isomorphic, as I'-module, to $\operatorname{Sym}^k \mathcal{F}_{\bar{\xi}}$ with the action described. By 2.21 above this is isomorphic to the standard $(k+1)$-dimensional representation

$$\operatorname{Sym}^k \begin{pmatrix} 1 & \epsilon_p \\ 0 & 1 \end{pmatrix}$$

and theorem 2.15 is proved.

3. Parabolic elements in motivic cohomology and extensions of motives

3.1. Assume that we are in the situation of 2.15 above. Then the corollary shows that a certain extension of "motives" is non-trivial. In fact, let $\mathcal{MM}$ be Jannsen's category of mixed motives ([**10**], §4) over K. Then the diagram (2.8.1) is the ℓ-adic realisation of an exact sequence of motives

$$0 \to h^{k+1}(E^{(k)})(\epsilon) \to h^{k+1}(E^{(k)} - Y^{(k)})(\epsilon) \to h^{k+2}_{Y^{(k)}}(E^{(k)})(\epsilon) \to 0$$

and that after possibly a finite extension of K we have an isomorphism

$$h^{k+2}_{Y^{(k)}}(E^{(k)})(\epsilon) \xrightarrow{\sim} h^0(Z)(-k-1).$$

Assume henceforth either that the smooth part of the fibre $\pi^{-1}(z_1)$ is the product of $\mathbb{G}_m$ (untwisted) with a finite group scheme, or that k is even. Then z_1 determines a map

$$\mathbb{Q}(-k-1) \xrightarrow{cl(z_1)} h^{k+2}_{Y^{(k)}}(E^{(k)})(\epsilon)$$

and therefore by pullback we obtain an extension of motives

$$0 \to h^{k+1}(E^{(k)})(\epsilon) \to (*) \to \mathbb{Q}(-k-1) \to 0. \tag{3.1.1}$$

The principles of motivic cohomology [**2**] indicate that (3.1.1) is classified by an element of the motivic cohomology group

$$H^{k+2}_{\mathcal{M}}(E^{(k)}, \mathbb{Q}(k+1)) = K^{(k+1)}_k(E^{(k)}).$$

3.2. We will use the calculations of the motivic homology of the boundary $Y^{(k)}$ from [**15**] to identify such an element. We first state a rather general result, which in some cases has been written down by Jannsen ([**10**], §9). Let V be a smooth quasi-projective scheme over K, and let $W \subset V$ be a closed subscheme. The motivic cohomology with supports is defined by

$$H^a_{\mathcal{M},W}(V,b) = \mathrm{Gr}^b\, K^W_{2b-a}(V) \otimes \mathbb{Q}$$

where $K^W_*(V)$ is K-theory of V with supports in W, and the graded part is with respect to the γ-filtration. In **[21]** it is proved that the long exact sequence of K-theory with supports respects the λ-ring structure, so there is a long exact sequence

$$H^a_{\mathcal{M},W}(V,b) \to H^a_{\mathcal{M}}(V,b) \to H^a_{\mathcal{M}}(V-W,b) \to H^{a+1}_{\mathcal{M},W}(V,b)$$

Soulé also shows that motivic cohomology forms part of what is almost a Bloch-Ogus Poincaré duality theory **[5]**—the only restriction being that in the cohomology groups the ambient schemes must be taken to be smooth.

3.3. Let H_{cont} be continuous étale cohomology **[9]**. There are Chern class maps

$$cl_W \colon H^a_{\mathcal{M},W}(V,\mathbb{Q}(b)) \to H^a_{\mathrm{cont},W}(V,\mathbb{Q}_\ell(b))$$

and we now define $H^a_{\mathcal{M},W}(V,\mathbb{Q}(b))^0$ to be the kernel of the composite map

$$H^a_{\mathcal{M},W}(V,\mathbb{Q}(b)) \xrightarrow{cl_W} H^a_{\mathrm{cont},W}(V,\mathbb{Q}_\ell(b)) \to H^a_{\mathrm{cont}}(V,\mathbb{Q}_\ell(b)) \to H^0(K,H^a(\overline{V},\mathbb{Q}_\ell(b))).$$

The edge homomorphism in the Hochschild-Serre spectral sequence therefore yields a map

$$H^a_{\mathcal{M},W}(V,\mathbb{Q}(b))^0 \to H^1(K,H^{a-1}(\overline{V},\mathbb{Q}_\ell(b)))$$

If $\omega \in H^a_{\mathcal{M},W}(V,\mathbb{Q}(b))^0$ we write ω_ℓ for its image in $H^1(K,H^{a-1}(\overline{V},\mathbb{Q}_\ell(b)))$ by the above map.

3.4. If $\omega \in H^a_{\mathcal{M},W}(V,\mathbb{Q}(b))^0$ we can also form an extension of ℓ-adic Galois modules as follows: the image of ω in $H^a(\overline{V},\mathbb{Q}_\ell(b))$ vanishes. Hence by pullback from the exact sequence:

$$\begin{array}{ccccccccc} H^{a-1}_{\overline{W}}(\overline{V},\mathbb{Q}_\ell(b)) & \to & H^{a-1}(\overline{V},\mathbb{Q}_\ell(b)) & \to & H^{a-1}(\overline{V}-\overline{W},\mathbb{Q}_\ell(b)) & \to & H^a_{\overline{W}}(\overline{V},\mathbb{Q}_\ell(b)) & \to & H^a(\overline{V},\mathbb{Q}_\ell(b)) \\ & & & & & & \uparrow{\scriptstyle cl_{\overline{W}}(\omega)} & \nearrow{\scriptstyle 0} & \\ & & & & & & \mathbb{Q}_\ell(0) & & \end{array}$$

we obtain an extension

$$0 \to A \to B \to \mathbb{Q}_\ell(0) \to 0$$

of $\mathbb{Q}_\ell(0)$ with

$$A = \mathrm{coker}\big[H^{a-1}_{\overline{W}}(\overline{V},\mathbb{Q}_\ell(b)) \to H^{a-1}(\bar{V},\mathbb{Q}_\ell(b))\big].$$

Theorem 3.5. *The extension B is classified by the image of ω_ℓ in $H^1(\mathrm{Gal}(\bar{K}/K),A)$.*

Proof. This is very similar to Theorem 9.4 of **[10]**. Namely, let us define analogously

$$\begin{aligned} H^a_{\mathrm{cont},W}(V,\mathbb{Q}_\ell(b))^0 &= \ker[H^a_{\mathrm{cont},W}(V,\mathbb{Q}_\ell(b)) \to H^0(K,H^a(\overline{V},\mathbb{Q}_\ell(b)))] \\ H^a_{\mathrm{cont}}(V,\mathbb{Q}_\ell(b))^0 &= \ker[H^a_{\mathrm{cont}}(V,\mathbb{Q}_\ell(b)) \to H^0(K,H^a(\overline{V},\mathbb{Q}_\ell(b)))]. \end{aligned}$$

Then there is a diagram

$$
\begin{array}{ccc}
H^a_{\mathrm{cont},W}(V,\mathbb{Q}_\ell(b))^0 & \longrightarrow & H^a_{\mathrm{cont}}(V,\mathbb{Q}_\ell(b))^0 \\
\wr\downarrow & & \\
\ker[H^a_{\overline{W}}(\overline{V},\mathbb{Q}_\ell(b))^{\mathrm{Gal}(\bar{K}/K)} \to H^a(\overline{V},\mathbb{Q}_\ell(b))^{\mathrm{Gal}(\bar{K}/K)}] & & \downarrow \sigma \\
\| & & \\
\ker[H^a_{\overline{W}}(\overline{V},\mathbb{Q}_\ell(b))) \to H^a(\overline{V},\mathbb{Q}_\ell(b))]^{\mathrm{Gal}(\bar{K}/K)} & & \\
\wr\downarrow \tau & & H^1(K,H^{a-1}(\overline{V},\mathbb{Q}_\ell(b))) \\
H^1(K,\mathrm{coker}[H^{a-1}_{\overline{W}}(\overline{V},\mathbb{Q}_\ell(b))) \to H^{a-1}(\overline{V},\mathbb{Q}_\ell(b))]) & \longleftarrow &
\end{array}
$$

Here the arrow labelled σ is the edge homomorphism in the Hochschild-Serre spectral sequence, and τ is the boundary map in the long exact cohomology sequence attached to the short exact sequence

$$
\begin{aligned}
0 \longrightarrow & \ \mathrm{coker}[H^{a-1}_{\overline{W}}(\overline{V},\mathbb{Q}_\ell(b)) \to H^{a-1}(\overline{V},\mathbb{Q}_\ell(b))] \\
& \longrightarrow H^{a-1}(\overline{V}-\overline{W},\mathbb{Q}_\ell(b)) \longrightarrow \ker[H^a_{\overline{W}}(\overline{V},\mathbb{Q}_\ell(b)) \to H^a(\overline{V},\mathbb{Q}_\ell(b))] \longrightarrow 0.
\end{aligned}
$$

The commutativity of this diagram is a consequence of Proposition 9.4 of **[10]**, and from this the desired compatibility follows.

3.6. We next write down an element of $H^{k+2}_{\mathcal{M}}(E^{(k)},\mathbb{Q}(k+1))$, which by Poincaré duality is the same as $H^{\mathcal{M}}_k(E^{(k)},\mathbb{Q}(0))$. For this we shall use the results of **[15]**. Let $\mathring{Y}^{(k)} \subset Y^{(k)}$ be the connected component of the identity of the smooth part of $Y^{(k)}$. The proof of MF3.1.0(ii) shows that the inclusion $\mathring{Y}^{(k)} \hookrightarrow Y^{(k)}$ gives isomorphisms

$$
H^{k+2}_{\mathcal{M},Y^{(k)}}(E^{(k)},\mathbb{Q}(k+1)) = H^{\mathcal{M}}_k(Y^{(k)},\mathbb{Q}(0))(\epsilon) \xrightarrow{\sim} H^{\mathcal{M}}_k(\mathring{Y}^{(k)},\mathbb{Q}(0))(\epsilon).
$$

Taking z_1 as in 3.1, one has that $H^{\mathcal{M}}_k(\pi^{-1}(z_1),\mathbb{Q}(0))(\epsilon)$ is isomorphic to $H^{\mathcal{M}}_k(\mathbb{G}^k_{m/\kappa(z_1)},\mathbb{Q}(0))(\epsilon)$, which is 1-dimensional and spanned by the cup-product $t_1 \cup \ldots \cup t_k$, where t_i is the standard coordinate on the i^{th} copy of $\mathbb{G}_m$. Picking such a generator, let ω denote its image in $H^{k+2}_{\mathcal{M},Y^{(k)}}(E^{(k)},\mathbb{Q}(k+1))$. From 2.15 and 3.5 we obtain the following result.

Corollary 3.7. *The ℓ-adic realisation of the extension of motives (3.1.1) is classified by a non-zero multiple of the image of ω_ℓ in $H^1(K,H^{k+1}(E^{(k)}_{/\overline{K}},\mathbb{Q}_\ell(k+1))(\epsilon))$. In particular, if the hypotheses of 2.15 hold, then the image of ω in $H^{k+2}_{\mathcal{M}}(E^{(k)},\mathbb{Q}(k+1)) = K^{(k+1)}_k(E^{(k)})$ is nonzero.*

3.8. The Deligne-Beilinson (or "absolute Hodge", see **[3]**) cohomology group $H^{k+2}_{\mathcal{H}}(E^{(k)}_{/\mathbb{R}},\mathbb{R}(k+1))$ is zero for $k>0$, since $E^{(k)}$ is smooth and proper. Therefore there is no interpretation for ω in terms of archimedean regulators. In fact, Beilinson's conjectures **[2]** predict that

$H^{k+2}_{\mathcal{M}/\mathbb{Z}}(E^{(k)}, \mathbb{Q}(k+1))$, the motivic cohomology group "over $\mathbb{Z}$", vanishes. The existence of ω does not contradict this, since the method we have used to show that ω is non-zero is to show that (modulo certain compatibilities) that it does not map to an element of $H^{k+2}_{\mathcal{M}/\mathbb{Z}}(E^{(k)}, \mathbb{Q}(k+1)) \subset H^{k+2}_{\mathcal{M}}(E^{(k)}, \mathbb{Q}(k+1))$. Its existence should be accounted for by the vanishing of an *incomplete* L-function at $s=1$; and the analysis of the vanishing cycles certainly shows that the local L-factor at p of the parabolic cohomology $H^1(\overline{X}, g_* \operatorname{Sym}^k \mathcal{F})$ has a pole at $s=1$. For further discussion of the relation of these elements to an (as yet unformulated) S-integral version of Beilinson's conjectures, see **[16]**.

4. Examples

4.1. We will give three examples to which the results of §§1–2 apply. In each case the elliptic surface E is constructed as follows. Let $\Gamma \subset SL_2(\mathbb{Z})$ be a subgroup of finite index. The quotient of the upper half-plane by Γ is the set of complex points of a connected smooth curve over some number field K, whose smooth compactification X_Γ is equipped with a finite morphism $\mathbf{j}: X_\Gamma \to \mathbb{P}^1_K$. We let E_Γ be the minimal model of the fibre product (cf. 2.18 above)

$$\mathcal{E} \times_{\mathbb{P}^1} X_\Gamma.$$

4.2. It is not necessarily the case that E_Γ / X_Γ satisfies the hypotheses of 2.2. To get around this we follow a standard procedure, which was used in an almost identical way in **[13]** and **[14]**, and consider the intersection Γ' of Γ by its with the congruence subgroup $\pm\Gamma(N)$, when $N \geq 3$ is chosen to be divisible by the widths of all the cusps of Γ. Then $X_{\Gamma'}$ may be taken to be a component of the normalisation of the fibre product

$$X_\Gamma \times_{\mathbb{P}^1} X(N)$$

where $X(N)$ is the modular curve over $\mathbb{Q}$ of level N, parameterising (generalised) elliptic curves A with an isomorphism $\boldsymbol{\mu}_N \times \mathbb{Z}/N \xrightarrow{\sim} A_N$ of determinant 1.

4.3. By the choice of N the covering $X_{\Gamma'} \to X(N)$ is actually étale, and thus the elliptic surface $\pi': E_{\Gamma'} \to X_{\Gamma'}$ is the pullback to $X_{\Gamma'}$ of the standard universal generalised elliptic curve E^{univ} with level N structure on $X(N)$. In particular it is semistable. Moreover there is a finite group scheme of section of $E_{\Gamma'}/X_{\Gamma'}$ isomorphic to $\boldsymbol{\mu}_N \times \mathbb{Z}/N$ which meets each irreducible component of each degenerate fibre, namely the pullbacks of the tautological sections of the universal elliptic curve. Hence the conditions of 2.2 are satisfied by $E_{\Gamma'}$.

4.4. To recover the parabolic cohomology groups for Γ from those of Γ' it is necessary only to pass to invariants under a suitable subgroup scheme H of $SL(\boldsymbol{\mu}_N \times \mathbb{Z}/N)$. Let $k: X_{\Gamma'} \to X_\Gamma$ be the covering; it is a torsor for some such subgroup scheme H. Denote all objects corresponding to Γ' by adding the symbol $'$. Then define

$$_\Gamma\mathcal{W}^k_\ell \stackrel{\mathrm{def}}{=} H^1(\bar{X}_{\Gamma'}, g'_* \operatorname{Sym}^k R^1 \mathring{\pi}'_* \mathbb{Q}_\ell)^H = H^1(\bar{X}_\Gamma, k_*(g'_* \operatorname{Sym}^k R^1 \mathring{\pi}_* \mathbb{Q}_\ell)^H)$$
$$_\Gamma\mathcal{W}^k_{\ell !} \stackrel{\mathrm{def}}{=} H^1(\bar{X}_{\Gamma'}, g'_! \operatorname{Sym}^k R^1 \mathring{\pi}'_* \mathbb{Q}_\ell)^H = H^1(\bar{X}_\Gamma, k_*(g'_! \operatorname{Sym}^k R^1 \mathring{\pi}_* \mathbb{Q}_\ell)^H)$$

Since the elliptic curve $\mathcal{E}$ (2.18) already has semistable reduction at $j=\infty$, the monodromy of the sheaf $R^1\psi_*\mathbb{Q}_\ell$ on $\mathbb{A}^1 - \{0, 1728\}$ is unipotent at $j=\infty$. So the sheaves

$k_*(g'_* \operatorname{Sym}^k R^1\mathring{\pi}_*\mathbb{Q}_\ell)^H$ and $g_* \operatorname{Sym}^k R^1\mathring{\pi}_*\mathbb{Q}_\ell$ on X_γ are isomorphic away from the points where $j = 0$ or 1728. The same is true for $k_*(g'_! \operatorname{Sym}^k R^1\mathring{\pi}_*\mathbb{Q}_\ell)^H$ and $g_! \operatorname{Sym}^k R^1\mathring{\pi}'_*\mathbb{Q}_\ell$. In other words, for the calculations of vanishing cycles we can work on X_Γ.

4.5. Assume that X_Γ has genus 0, and that the field K is $\mathbb{Q}$. Let t be any generator of the function field of X over $\mathbb{Q}$ which satisfies an equation of the form

$$P(t) + jQ(t) = 0 \tag{4.5.1}$$

where we assume P, $Q \in \mathfrak{o}_K$, $\deg P = d > \deg Q$ and P monic. Then it is shown in Proposition 2.7 of **[14]** that the representations ${}_\Gamma\mathcal{W}^k_\ell$ of $\mathrm{Gal}(\overline{K}/K)$ are unramified at all primes $\mathfrak{p}$ of residue characteristic different from ℓ for which:

(i) $P(t)$, $Q(t)$ are $\mathfrak{p}$-integral, and their reductions $\tilde{P}(t)$, $\tilde{Q}(t)$ modulo $\mathfrak{p}$ are relatively prime; and

(ii) at least one of $\tilde{P}'(t)$, $\tilde{Q}'(t)$ is non-zero.

The same argument applies also to the representations ${}_\Gamma\mathcal{W}^k_{\ell!}$. (In fact the result is not stated exactly in this form in **[14]**, but it is easily checked that the proof yields this result.)

Proposition 4.6. *Suppose that $\mathfrak{p}$ is a prime of K for which condition 4.5(i) holds. Suppose also that*

(ii)' There exists exactly one root $\beta \in \mathbb{Q}_p$ of $Q(t)$ (of arbitrary multiplicity) such that $\mathrm{ord}_p(\beta) < 0$.

Then the hypotheses of Theorem 2.15 are satisfied.

Proof. Let $\mathfrak{o}_\mathfrak{p}$ be the localisation of $\mathfrak{o}_K$ at $\mathfrak{p}$. As in the proof of 4.5, we observe that by condition (i), the defining equation (4.5.1) gives a finite morphism $\mathbb{P}^1 \otimes \mathfrak{o}_\mathfrak{p} \to \mathbb{P}^1 \otimes \mathfrak{o}_\mathfrak{p}$ whose generic fibre is $\mathbf{j}: X_\Gamma \to \mathbb{P}^1_K$. Hence in the notation of 2.15 the morphism $\mathbf{j}_\mathbb{Z}: \mathcal{X} \to \mathbb{P}^1_\mathbb{Z}$ is given by (4.5.1) in a neighbourhood of the fibre at $\mathfrak{p}$. Replacing K by $K(\beta)$, and taking for z_i the sections $t = \infty$, $t = \beta$ lying over $j = \infty$ the conditions of 2.15 are satisfied.

4.7. We will consider the following three cases:

(i) The subgroup $\Gamma_{4,3}$ of index 7, generated by

$$\begin{pmatrix} 1 & 4 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & -1 \\ 2 & -1 \end{pmatrix}.$$

(ii) The subgroup $\Gamma_{5,2}$, also of index 7, generated by

$$\begin{pmatrix} 1 & 5 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix}.$$

(iii) The subgroup Γ_{711} of index 9 generated by

$$\begin{pmatrix} 1 & 7 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 3 & -4 \\ 1 & -1 \end{pmatrix}, \quad \begin{pmatrix} -1 & -4 \\ 1 & 3 \end{pmatrix}$$

which was considered in **[14]** (and before that by Atkin and Swinnerton-Dyer **[1]**).

4.8. In each case the curve X_Γ has genus zero, so the equations (4.5.1) can be found by methods going back to Klein and Fricke, and systemised by Atkin and Swinnerton-Dyer. The relation (4.5.1) can, for suitable t, be put in the form

$$j = \frac{E_3(t)F_3(t)^3}{Q(t)} = 1728 + \frac{E_2(t)F_2(t)^2}{Q(t)}$$

for polynomials $E_\alpha(t)$, $F_\alpha(t)$ with coefficients in K, which may be computed by the method of undetermined coefficients.

4.9. For the three examples the field K is in fact $\mathbb{Q}$, and the j-equations are as follows:

(i) For $\Gamma_{4,3}$:

$$\begin{aligned} j &= -7^{-7}\frac{(t+432)(t^2+80t-3888)^3}{t^3} \\ &= -7^{-7}\frac{(t-16)(t^3+344t^2+1944t+108^3)^2}{t^3} + 1728. \end{aligned}$$

(ii) For $\Gamma_{5,2}$:

$$\begin{aligned} j &= 7^{-7}\frac{(t+125)(t^2+5t-1280)^3}{t^2} \\ &= 7^{-7}\frac{(t-64)(t^3+102t^2+381t+64000)^2}{t^2} + 1728. \end{aligned}$$

(iii) For Γ_{711}:

$$\begin{aligned} j &= 2^6\frac{(t^3+4t^2+10t+6)^3}{t^2+13t/4+8} \\ &= 2^6\frac{t(t^4+6t^3+21t^2+35t+63/2)^2}{t^2+13t/4+8} + 1728. \end{aligned}$$

To put these in a form to which 4.6 applies, we need to make a change of variables, given in the three cases respectively by

$$t = 7t'+2; \quad t = 7t'+1; \quad t = t'/2$$

giving the following result:

Proposition 4.10. *Let Γ be one of $\Gamma_{4,3}$, $\Gamma_{5,2}$, $\Gamma_{7,1,1}$. Let p equal 7, 7 or 2 respectively. Then the hypotheses of 2.15 are satisfied. In particular ${}_\Gamma\mathcal{W}^k_{\ell!}$ is a non-trivial extension of ${}_\Gamma\mathcal{W}^k_\ell$ (for $\ell \neq p$) and the image of ω in $H^{k+2}_{\mathcal{M}}(E^{(k)}, \mathbb{Q}(k+1))$ (cf. 3.7) is nonzero.*

4.11. Return to the general case, and let $\Delta \subset SL_2(\mathbb{Z})$ be the smallest congruence group containing Γ. We choose K and models for X_Γ, X_Δ such that the obvious transition morphism $\phi : X_\Gamma \to X_\Delta$ is defined over K. This determines direct and inverse image maps

$$\phi^*\colon {}_\Delta\mathcal{W}^k_\ell \to {}_\Gamma\mathcal{W}^k_\ell, \quad \phi_*\colon {}_\Gamma\mathcal{W}^k_\ell \to {}_\Delta\mathcal{W}^k_\ell$$

whose composite is multiplication by the degree of ϕ. We define ${}_\Gamma\mathcal{W}_\ell^{k,\mathrm{prim}} \subset {}_\Gamma\mathcal{W}_\ell^k$ to be $\ker\phi_*$. Then ${}_\Gamma\mathcal{W}_\ell^k$ is the direct sum of ${}_\Gamma\mathcal{W}_\ell^{k,\mathrm{prim}}$ and ${}_\Delta\mathcal{W}_\ell^k$. There is a similar decomposition

(4.11.1) $$ {}_\Gamma\mathcal{W}_{\ell\,!}^k = {}_\Delta\mathcal{W}_{\ell\,!}^k \oplus {}_\Gamma\mathcal{W}_{\ell\,!}^{k,\mathrm{prim}}. $$

4.12. Suppose that $\mathfrak{p}$ is a prime of K for which the hypotheses of 2.15 hold. Then the unipotent element $u \in \mathrm{Aut}\,{}_\Gamma\mathcal{W}_{\ell\,!}^k$ preserves the decomposition (4.11.1). The following remark will be used in **[18,19]** in the study of the image of Galois in $\mathrm{Aut}\,{}_\Gamma\mathcal{W}_\ell^{k,\mathrm{prim}}$.

Lemma 4.13. *Let* $u' \in \mathrm{Aut}\,{}_\Gamma\mathcal{W}_{\ell\,!}^{k,\mathrm{prim}}$, $u'' \in \mathrm{Aut}\,{}_\Gamma\mathcal{W}_\ell^{k,\mathrm{prim}}$ *be the images of* u. *Then* $(u'-1)^k \neq 0$ *and* $(u''-1)^{k-1} \neq 0$.

Proof. We have to go back to the construction of u; it comes from the homomorphism

$$ {}_\Gamma\mathcal{W}_{\ell\,!}^k \longrightarrow B = (R^1\Phi_{g!}\,\mathrm{Sym}^k\,\mathcal{F})_x $$

into the group of vanishing cycles at the cusp x in characteristic $\mathfrak{p}$. Now for the congruence subgroup Δ we know (cf. **[12]** 10.8.3) that (for K sufficiently large) the subscheme of cusps is étale over $\mathfrak{o}_K$. Therefore on the curve X_Δ there are no vanishing cycles, so the image of ${}_\Delta\mathcal{W}_{\ell\,!}^k$ in B is zero, which is what we need. ■

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Department of Mathematical Sciences
Science Laboratories
University of Durham
Durham DH1 3LE
England
e-mail: a.j.scholl@durham.ac.uk