Correction to "Vanishing cycles and nonclassical parabolic cohomology"

Stefan Wevers has kindly pointed out to me that the deduction of Corollary 2.16 from Theorem 2.15 is unjustified. Here is an argument which fills this gap. (Wevers has also found a proof of a much more general result than this one.)

I claim that in the diagram in the statement of Theorem 2.15

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H^0(\overline{Z}, \mathbb{Q}_\ell) & \longrightarrow & H^1_c(\overline{U}, \operatorname{Sym}^k \mathcal{F}) & \longrightarrow & H^1(\overline{X}, g_* \operatorname{Sym}^k \mathcal{F}) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow \beta & & \downarrow \gamma & & \\ 0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0 \end{array}$$

the right-hand surjection γ admits an inertia-equivariant splitting $\delta\colon C \to H^1(\overline{X}, g_*\mathcal{G})$. Granted this, the corollary is immediate: if $\sigma\colon H^1(\overline{X}, g_*\mathcal{G}) \to H^1_c(\overline{U}, \mathcal{G})$ is a splitting of the middle row, then $\beta \circ \sigma \circ \delta$ splits the map $B \to C$, contradicting the fact that B is indecomposable.

To prove the claim, we first replace K by its completion at v. Put $\mathcal{X}_{\bar{v}} = \mathcal{X} \otimes \overline{k(v)}$ and $\mathcal{G} = \operatorname{Sym}^k \mathcal{F}$. Consider the maps (with notation as in the proof of 2.15)

$$H^1_x(\mathcal{X}_{\bar{v}}, R\Psi g_*\mathcal{G}) \to H^1(\mathcal{X}_{\bar{v}}, R\Psi g_*\mathcal{G}) = H^1(\overline{X}, g_*\mathcal{G}) \to (R^1\Phi g_*\mathcal{G})_{\bar{x}} = C$$

which are evidently $\operatorname{Gal}(\bar{K}_v/K_v)$-equivariant. To obtain the desired splitting, I shall show that their composite is an isomorphism. As this is local around x we can replace $\mathcal{X}$ by a neighbourhood of x, and therefore assume that $\mathcal{X}$ is smooth and that $\mathcal{F}$ is lisse on $\mathcal{U}$.

From 1.7 and 1.9 we know $R^1\Phi g_*\mathcal{G} \simeq x_*(\mathcal{G}_{\bar{\xi}})$. We need to compute $H^q_{\bar{x}}(R^0\Psi g_*\mathcal{G})$. From the definition of the nearby cycles functor and the assumptions just made, it is the subsheaf of $g_*(\mathcal{G}_v)$ whose restriction to $\mathcal{U}_v$ is $\mathcal{G}_v$ and whose fibre at x is the space of invariants of $\mathbb{Z}_\ell(1)^2$ acting on $\mathcal{G}_{\bar{\xi}}$. But as in 2.19 this is the same as the space of local invariants $g_*(\mathcal{G}_v)_x$; thus $R^0\Psi g_*\mathcal{G} = g_*(\mathcal{G}_v)$. Now recall that for any constructible $\mathbb{Q}_\ell$-sheaf F on a strictly henselian trait S of residue characteristic prime to ℓ we have (in the usual notation) $H^1_s(S, F) = \operatorname{coker}[F_s \to F^I_{\bar{\eta}}]$ and $H^2_s(S, F) = F_{\bar{\eta}}(-1)_I$. In the present case this gives $H^1_{\bar{x}}(R^0\Psi g_*\mathcal{G}) = 0$ and $H^2_{\bar{x}}(R^0\Psi g_*\mathcal{G}) = \mathcal{G}_{\bar{\xi}}(-1)_{\mathbb{Z}_\ell(1)}$.

The "Leray" spectral sequence $E_2^{ab} = H^a_x(R^b\Psi) \Rightarrow H^*_x(R\Psi)$ then becomes a short exact sequence

$$0 \to H^1_{\bar{x}}(\mathcal{X}_{\bar{v}}, R\Psi g_*\mathcal{G}) \to (R^1\Psi g_*\mathcal{G})_{\bar{x}} \xrightarrow{\tau} \mathcal{G}_{\bar{\xi}}(-1)_{\mathbb{Z}_\ell(1)}$$

Now by the theory of the Tate curve, the representation $\mathcal{G}_{\bar{\xi}} = \operatorname{Sym}^k \mathcal{F}_{\bar{\xi}}$ of the local Galois group at x is indecomposable, and its monodromy filtration has successive quotients $\mathbb{Q}_\ell(-i)$, $i \in \{0, 1, \ldots, k\}$. Therefore $\mathcal{G}_{\bar{\xi}}(-1)_{\mathbb{Z}_\ell(1)} \simeq \mathbb{Q}_\ell(-k-1)$. Therefore τ must vanish, as required.