ON MODULAR UNITS

A. J. Scholl

Introduction. In [**6**], Kubert and Lang describe the group of integral modular units on $\Gamma(n)$ ("units over $\mathbf{Z}$" in their terminology), and in particular determine its rank. Their method is based on finding explicit generators for the group of all modular units, and then by calculating their q-expansions to determine which are integral. In §5 of , Beilinson suggests another approach, based on representation theory and the geometry of the moduli schemes M_n. In this note I shall carry out this programme and give a representation-theoretic description of the group of integral modular units tensored with $\mathbf{Q}$. From this it will be a simple exercise to calculate the rank for any reasonable congruence subgroup; we give an example at the end of §2. The proof uses the adelic language, and exploits in an essential way the action of Hecke operators at primes dividing the level. This approach reduces the problem to showing that the modular units $\Delta(qz)/\Delta(z)$ (for a prime q) are not integral; we give a proof of this fact by "pure thought" at the very end of the paper.

I am very grateful to several people for discussions on this topic, especially to R. Weissauer and N. Schappacher. Indeed, the main part of this note was originally to have formed part of our joint paper [**7**] on Beilinson's conjecture for modular curves.

Notations. We denote by G the algebraic group GL_2 over $\mathbf{Z}$. We use the following notations for subgroups of G:

$$B = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix}; N = \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}; D = \begin{pmatrix} * & 0 \\ 0 & 1 \end{pmatrix}; A = \begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix}.$$

Write Z for the centre of G. If H is any one of A, $B\dots$ write H_R for the group of R-valued points of H, and H_f, H_p for $H_{\mathbf{A}_f}$, $H_{\mathbf{Q}_p}$ respectively ($\mathbf{A}_f$ being the ring of finite adèles of $\mathbf{Q}$). Set

$$H_{\mathbf{Q}}^+ = \{h \in H_{\mathbf{Q}} \mid \det(h) > 0\}.$$

1. Recall the standard description of the modular curves M_n. Write K_n for the compact open subgroup

$$K_n = \ker\{p_n : G_{\hat{\mathbf{Z}}} \to G_{\mathbf{Z}/n\mathbf{Z}}\}$$

of G_f. For any integer $n \geq 3$, there exists a moduli scheme M_n for elliptic curves E with level n structure $(\mathbf{Z}/n\mathbf{Z})^2 \xrightarrow{\sim} E[n]$. It is a smooth curve over $\mathbf{Q}$, and its complex points can be described as

$$M_n(\mathbf{C}) = G_{\mathbf{Q}} \backslash \mathcal{H}^{\pm} \times G_f/K_n$$

where $\mathcal{H}^{\pm} = \mathbf{C} - \mathbf{R}$. The e_n-pairing defines a morphism $M_n \longrightarrow \operatorname{Spec} \mathbf{Q}(\boldsymbol{\mu}_n)$, whose geometric fibre is connected.

More generally, for any compact open subgroup K of G_f, there is a modular curve M_K defined over $\mathbf{Q}$ with

$$M_K(\mathbf{C}) = G_{\mathbf{Q}} \backslash \mathcal{H}^{\pm} \times G_f/K.$$

We have $M_n = M_{K_n}$, and the curves M_n form a cofinal family in the inverse system $(M_K)_K$. We consider the inverse limit

$$\begin{aligned} M &= \varprojlim_n M_n \\ &= \varprojlim_K M_K \end{aligned}$$

which is an affine scheme over $\mathbf{Q}(\boldsymbol{\mu}_\infty)$ and is regular. The second description shows that G_f acts on M. The group of modular units is by definition $\mathcal{O}^*(M)$; it contains $\mathbf{Q}(\boldsymbol{\mu}_\infty)^*$.

Write $\overline{\mathbf{Q}}$ for the algebraic closure of $\mathbf{Q}$ in $\mathbf{C}$. Write $|\cdot|_f$ for the finite idèle modulus. For an even Dirichlet character χ of $\mathbf{Q}$, define $\mathcal{S}(\chi)$ to be the space of all locally constant functions $\phi : G_f \to \overline{\mathbf{Q}}$ satisfying

$$\phi(hg) = \chi(d)\left|\frac{a}{d}\right|_f \phi(g),$$

for every $h = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \in G_f$. The group G_f acts on $\mathcal{S}(\chi)$ by right translation.

It is clear that any $\phi \in \mathcal{S}(\chi)$ is determined by its restriction to $G_{\hat{\mathbf{Z}}}$; letting μ denote a non-zero invariant measure on this subgroup, define $\mathcal{S}(\chi)^0$ to be the subspace comprising all $\phi \in \mathcal{S}(\chi)$ for which

$$\int_{G_{\hat{\mathbf{Z}}}} \phi \, d\mu = 0.$$

We have $\mathcal{S}(\chi)^0 = \mathcal{S}(\chi)$ if $\chi \neq 1$, and $\mathcal{S}(1)^0$ is an invariant subspace of $\mathcal{S}(1)$ of codimension one.

In §3 we will sketch a proof of the following fact (which should be well known).

Proposition 1. *There is a G_f-equivariant isomorphism*

$$(\mathcal{O}^*(M)/\mathbf{Q}(\boldsymbol{\mu}_\infty)^*) \otimes_{\mathbf{Z}} \overline{\mathbf{Q}} \xrightarrow{\sim} \bigoplus_{\chi \neq 1} \mathcal{S}(\chi) \oplus \mathcal{S}(1)^0,$$

the sum being taken over all nontrivial even Dirichlet characters χ of $\mathbf{Q}$.

To define the integral units we denote by $M_{n/\mathbf{Z}}$ the regular model of M_n constructed in [**3**], [**5**]. This can be most simply described as the normalisation of the affine j-line in the function field of M_n (to avoid problems of representability we should assume that n is the product of two coprime integers, each greater than 2). Let $M_{\mathbf{Z}} = \varprojlim_n M_{n/\mathbf{Z}}$. The group of integral modular units is then by definition $\mathcal{O}^*(M_{\mathbf{Z}})$. (This is equivalent to the definition given in [**6**].) We have $\mathbf{Z}\,[\boldsymbol{\mu}_\infty]^* \subset \mathcal{O}^*(M_{\mathbf{Z}})$.

To describe $\mathcal{O}^*(M_{\mathbf{Z}})$ in representation-theoretic terms, we must recall that the representations $\mathcal{S}(\chi)$ of G_f are restricted tensor products of admissable representations of G_p. (For the elementary facts about representations of G_p we use here and later, see for example [**1**] or [**4**], §3.) More precisely, $\mathcal{S}(\chi) = \bigotimes_p' \mathcal{S}_p(\chi)$, where $\mathcal{S}_p(\chi)$ denotes the space of locally constant functions $\phi : G_p \to \overline{\mathbf{Q}}$ satisfying

$$\phi(hg) = \chi_p(d) \left| \frac{a}{d} \right|_p \phi(g),$$

for every $h = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \in G_p$. If we denote by $\mathcal{S}_p(\chi)^0$ the subspace comprising all $\phi \in \mathcal{S}_p(\chi)$ such that

$$\int_{SL_2(\mathbf{Z}_p)} \phi \, d\mu_p = 0$$

then $\mathcal{S}_p(\chi)$ is irreducible whenever $\chi_p \neq 1$. If $\chi_p = 1$ then $\mathcal{S}_p(\chi)^0 = \mathcal{S}_p(1)^0$ is the unique nontrivial invariant subspace of $\mathcal{S}_p(\chi) = \mathcal{S}_p(1)$; it has codimension one. We have

$$\mathcal{S}(1)^0 = \left\{ \phi = \otimes_p' \phi_p \in \mathcal{S}(1) \mid \text{for at least one } p,\ \phi_p \in \mathcal{S}_p(1)^0 \right\}.$$

Define accordingly

$$\mathcal{S}(1)^{00} = \left\{ \phi = \otimes_p' \phi_p \in \mathcal{S}(1) \mid \text{for at least two distinct } p,\ \phi_p \in \mathcal{S}_p(1)^0 \right\}.$$

We can now state our main result.

Theorem 2. *The isomorphism of Proposition 1 gives an identification*

$$(\mathcal{O}^*(M_{\mathbf{Z}})/\mathbf{Z}[\boldsymbol{\mu}_\infty]^*) \otimes_{\mathbf{Z}} \overline{\mathbf{Q}} = \bigoplus_{\chi\neq 1} \mathcal{S}(\chi) \oplus \mathcal{S}(1)^{00}.$$

2. In this section we describe the purely algebraic part of the proof of Theorem 2. In §4 below we shall construct, for each prime q, a G_f-morphism

$$\Phi_q : \mathcal{O}^*(M) \otimes \overline{\mathbf{Q}} \longrightarrow \mathcal{S}_q(1)^0 \otimes \mathcal{V}_q$$

with the following properties:

(a) $\mathcal{V}_q$ is a direct sum of one-dimensional representations of $\prod'_{p\neq q} G_p$;

(b) Φ_q is nonzero;

(c) $\bigcap_q \ker \Phi_q = (\mathcal{O}^*(M_{\mathbf{Z}}) \cdot \mathbf{Q}(\boldsymbol{\mu}_\infty)^*) \otimes \overline{\mathbf{Q}}$.

Since Φ_q is trivial on $\mathbf{Q}(\boldsymbol{\mu}_\infty)^*$ (by property (c) above) we see that Φ_q factors:

$$\begin{array}{ccc} \mathcal{O}^*(M) \otimes \overline{\mathbf{Q}} & \xrightarrow{\Phi_q} & \mathcal{S}_q(1)^0 \otimes \mathcal{V}_q \\ \downarrow & & \uparrow \oplus\Phi_{q,\chi} \\ \left(\mathcal{O}^*(M)/\mathbf{Q}\left(\boldsymbol{\mu}_\infty\right)^*\right) \otimes \overline{\mathbf{Q}} & \xrightarrow{\sim} & \bigoplus_{\chi\neq 1} \mathcal{S}(\chi) \oplus \mathcal{S}(1)^0 \end{array}$$

If $\chi \neq 1$ then choose a prime $p \neq q$ for which $\chi_p \neq 1$. The irreducibility of $\mathcal{S}_p(\chi)$ for such a prime p in conjunction with a) implies that $\Phi_{q,\chi} = 0$. Consider therefore the component $\Phi_{q,1}$. The representation $\mathcal{S}(1)^0$ is the sum of the subspaces

$$\mathcal{U}_p = \mathcal{S}_p(1)^0 \otimes {\bigotimes_{l\neq p}}' \mathcal{S}_l(1)$$

for all p (including $p = q$). If $p \neq q$ then the same argument as used in the case $\chi \neq 1$ shows that $\Phi_{q,1}(\mathcal{U}_p) = 0$. So by property b) we have $\Phi_{q,1}(\mathcal{U}_q) \neq 0$. Let

$$\lambda_l : \mathcal{S}_l(1) \to \overline{\mathbf{Q}}$$

denote the unique G_l-invariant linear form on $\mathcal{S}_l(1)$ which takes the value 1 on the spherical vector. It is then clear that for some non-zero invariant $v \in \mathcal{V}_q$ the mapping $\Phi_{q,1}$ is given by

$$\Phi_{q,1} : x = {\bigotimes_l}' x_l \mapsto {\prod_{l\neq q}}' \lambda_l(x_l) \cdot x_q \otimes v \in \mathcal{S}_q(1)^0 \otimes \mathcal{V}_q$$

for any $x \in \mathcal{S}(1)^0$. (Note that the formula makes sense since firstly, $\lambda_l x_l = 1$ for all but finitely many l, and secondly $x_l \in \mathcal{S}_l(1)^0$ for at least one l.) Therefore $\bigcap_q \ker \Phi_{q,1} = \mathcal{S}(1)^{00}$, and from property c) Theorem 2 follows.

To illustrate the result, let $K \subset G_f$ be an open compact subgroup. We have an exact sequence

$$0 \to \left(\mathcal{S}(1)^{00}\right)^K \to \left(\mathcal{S}(1)\right)^K \xrightarrow{\alpha} \left(\bigoplus_q \mathcal{S}_q(1)^0\right)^K \to 0$$

where α is the sum of the maps $\Phi_{q,1}$ in the previous paragraph. In particular, the corank of the image of $\mathcal{O}^*(M_{\mathbf{Z}})^K$ in $(\mathcal{O}^*(M)/\mathbf{Q}(\boldsymbol{\mu}_\infty)^*)^K$ is

$$\dim_{\overline{\mathbf{Q}}} \left(\bigoplus_q \mathcal{S}_q(1)^0\right)^K.$$

For example, let $K = K_n$, with $n = \sum q_i^{e_i}$. The corank is then

$$\sum_i \left(\#\mathbf{P}^1(\mathbf{Z}/q_i^{e_i}) - 1\right)$$

in agreement with [**6**].

3. For the proof of proposition 1 we need to recall the standard compactification of the modular curves. M_K is the complement in a smooth and proper curve $\overline{M}_K$ over $\mathbf{Q}$ of a finite set of reduced points (cusps) M_K^∞. Define $M^\infty = \varprojlim_K M_K^\infty$, a profinite $\mathbf{Q}$-scheme on which G_f acts.

The set of complex points $M^\infty(\mathbf{C})$ can be identified with the separated quotient of $\mathbf{P}^1(\mathbf{Q}) \times G_f$ by $G_\mathbf{Q}$; fixing the standard cusp $(\infty, 1) \in \mathbf{P}^1(\mathbf{Q}) \times G_f$ gives a G_f-bijection

$$M^\infty(\mathbf{C}) \xrightarrow{\sim} A^+_\mathbf{Q} N_f \backslash G_f.$$

Let $t : \mathrm{Aut} \longrightarrow \hat{\mathbf{Z}}^*$ denote the cyclotomic character, giving the action of Aut on $\exp(2\pi i \cdot \mathbf{Q})$. Then $\alpha \in \mathrm{Aut}$ acts on the cusps by left multiplication by $\begin{pmatrix} t(\alpha) & 0 \\ 0 & 1 \end{pmatrix} \in D_{\hat{\mathbf{Z}}}$, and thus we get for the set $|M^\infty|$ of closed points of M^∞

$$|M^\infty| \xrightarrow{\sim} Z_\mathbf{Q} D_f N_f \backslash G_f.$$

Denote by V^∞ the set of all nontrivial valuations $v : \mathbf{Q}(M)^*/\mathbf{Q}^* \longrightarrow \mathbf{Q}$ whose center belongs to M^∞. There is a canonical valuation v_∞, centred at the standard cusp $(\infty, 1)$, corresponding to the choice of uniformiser $q = e^{2\pi i z}$ on the upper half-plane. By considering its stabiliser we see that

$$V^\infty \xrightarrow{\sim} Z_\mathbf{Q} D_{\hat{\mathbf{Z}}} N_f \backslash G_f$$

and that if $a \in \mathbf{Q}^*$, $a > 0$ and $v \in V^\infty$ then $\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} \cdot v$ is $a^{-1}v$.

The natural G_f-equivariant projection $\pi : V^\infty \longrightarrow |M^\infty|$ has a section σ, defined as follows. By the Iwasawa decomposition $G_f = B^+_\mathbf{Q} G_{\hat{\mathbf{Z}}}$ any element of $|M^\infty|$ may be represented by some $k \in G_{\hat{\mathbf{Z}}}$, and then

$$\sigma : Z_\mathbf{Q} D_f N_f k \mapsto Z_\mathbf{Q} D_{\hat{\mathbf{Z}}} N_f k$$

is well-defined and $G_{\hat{\mathbf{Z}}}$-equivariant (but not G_f-equivariant).

Now let $K = K_n$ for some n, and for $x \in |M^\infty|$ let $\mathrm{ord}_{x,K}$ denote the normalised valuation of $\mathbf{Q}(M_K)$ centred at the image of x in $\overline{M}_K$ (taking value 1 on a uniformiser). Since $G_{\hat{\mathbf{Z}}}$ acts transitively on $|M_K^\infty|$, there exists a positive constant $c_n \in \mathbf{Q}$ such that for every $x \in |M^\infty|$,

$$\sigma(x)|_{\mathbf{Q}(M_K)} = c_n \mathrm{ord}_{x,K}. \tag{1}$$

With these preliminaries we can prove Proposition 1. For $u \in \mathcal{O}^*(M)$ define the function

$$\begin{aligned} \mathrm{div}(u) : & V^\infty \longrightarrow \mathbf{Q} \\ & v \mapsto v(u). \end{aligned}$$

Then:

(i) $\mathrm{div}(u)$ is locally constant, and vanishes if and only if u is constant;

(ii) $(\mathrm{div}\, u) \cdot \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} = a^{-1} \mathrm{div}\, u$ for all $a \in \mathbf{Q}$, $a > 0$;

(iii) If μ is a $G_{\hat{\mathbf{Z}}}$-invariant measure on $|M^\infty|$,

$$\int_{|M^\infty|} (\mathrm{div}\, u)(\sigma(x))\, d\mu(x) = 0.$$

Indeed, by (1) above (iii) is simply the assertion that the divisor of a function on $\overline{M}_n$ has degree zero.

The Manin-Drinfeld theorem states that any divisor of degree zero on $\overline{M}_n$ supported at the cusps is the divisor of some element of $\mathcal{O}^*(M_n)$. This, in conjunction with the above, implies that $u \mapsto \operatorname{div} u$ defines an isomorphism of G_f-modules

$$(\mathcal{O}^*(M)/\mathbf{Q}(\boldsymbol{\mu}_\infty)^*) \otimes_{\mathbf{Z}} \mathbf{Q} \xrightarrow{\sim} \mathcal{S}^0$$

where $\mathcal{S}^0$ is the space of all locally constant functions $\phi : G_f \to \mathbf{Q}$ such that

$$\phi\left(\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} g\right) = \left|\frac{a}{d}\right|_f \phi(g)$$

for every $a \in \mathbf{A}_f^*$, $b \in \mathbf{A}_f$, $d \in \mathbf{Q}^*$, and

$$\int_{G_{\hat{\mathbf{Z}}}} \phi \, d\mu = 0.$$

Now decomposing $\mathcal{S}^0 \otimes \overline{\mathbf{Q}}$ under the action of Z_f gives proposition 1.

Remark. An alternative route to the proposition is to identify $\mathcal{S}(\chi)^0$ with the space of holomorphic weight 2 Eisenstein series with central character χ; the isomorphism is then given by the map $u \mapsto d \log u$.

4. To construct the maps Φ_q we first must recall ([**3**] V.4, [**5**] §13.7) the structure of the reduction of $\overline{M}_{\mathbf{Z}}$ modulo a prime q. Suppose that $r \geq 1$, $(n, q) = 1$, and n, $q^r \geq 3$. Then by [**5**] Theorem 13.7.6 the irreducible components of $\overline{M}_{nq^r/\mathbf{Z}} \otimes_{\mathbf{Z}} \mathbf{F}_q$ are indexed by $\mathbf{P}^1(\mathbf{Z}/q^r\mathbf{Z}) \times |\operatorname{Spec} \mathbf{Z}[\boldsymbol{\mu}_n] \otimes \mathbf{F}_q|$. The action of $G_{\mathbf{Z}/nq^r\mathbf{Z}} = G_{\mathbf{Z}/n\mathbf{Z}} \times G_{\mathbf{Z}/q^r\mathbf{Z}}$ is given as follows:-

- the first factor acts on $\mathbf{P}^1(\mathbf{Z}/q^r\mathbf{Z})$ by linear fractional transformations, and acts trivially on $|\operatorname{Spec} \mathbf{Z}[\boldsymbol{\mu}_n] \otimes \mathbf{F}_q|$;
- the second factor acts trivially on $\mathbf{P}^1(\mathbf{Z}/q^r\mathbf{Z})$, and has Galois action on $|\operatorname{Spec} \mathbf{Z}[\boldsymbol{\mu}_n] \otimes \mathbf{F}_q|$ given by

$$G_{\mathbf{Z}/n\mathbf{Z}} \xrightarrow{\det} (\mathbf{Z}/n\mathbf{Z})^* \xrightarrow{\sim} \operatorname{Gal}(\mathbf{Q}(\boldsymbol{\mu}_n)/\mathbf{Q}).$$

Write $\mathcal{X}_q$ for the set of irreducible components of $\overline{M}_{\mathbf{Z}} \otimes \mathbf{F}_q$, and $\mathcal{Y}_q$ for the set of primes of $\mathbf{Z}[\boldsymbol{\mu}_\infty]$ over q. It then follows by passage to the limit that $\mathcal{X}_q \xrightarrow{\sim} \mathbf{P}^1(\mathbf{Q}_q) \times \mathcal{Y}_q$.

Claim. The action of G_f on $\mathcal{X}_q$ is the product of:

- the action of G_q on $\mathbf{P}^1(\mathbf{Q}_q)$ by linear fractional transformations;
- the Galois action of ${\prod}'_{p \neq q} G_p$ on $\mathcal{Y}_q$ by determinant followed by the Artin map.

Indeed, we know that the action of G_f on $\overline{M}$ induces the Galois action on $\mathbf{Q}(\boldsymbol{\mu}_\infty) = \mathcal{O}(\overline{M})$ given by determinant followed by the Artin map. By the above $G_{\hat{\mathbf{Z}}}$ acts as claimed, so by the Iwasawa decomposition is suffices to consider the action of $B^+_{\mathbf{Q}}$. One way to do this is to consider the point of M^∞ which is the Galois orbit of the standard cusp $(\infty, 1)$; let T_∞ denote its closure in $\overline{M}_{\mathbf{Z}}$. Then from the description in [**5**] of the irreducible components it is easy to see that T_∞ meets precisely those components with labels $(\infty, y) \in \mathbf{P}^1(\mathbf{Q}_q) \times \mathcal{Y}_q$. Since T_∞ is fixed by $B^+_{\mathbf{Q}}$ the claim then follows.

There is an exact sequence

$$0 \to \mathcal{O}^*(M_{\mathbf{Z}}) \otimes \overline{\mathbf{Q}} \to \mathcal{O}^*(M) \otimes \overline{\mathbf{Q}} \xrightarrow{(\Psi_q)_q} \coprod_q H^0(\mathcal{X}_q, \overline{\mathbf{Q}})$$

where $H^0(\mathcal{X}_q, \overline{\mathbf{Q}})$ is the space of locally constant functions from $\mathcal{X}_q$ to $\overline{\mathbf{Q}}$, and Ψ_q is defined as follows: for each $x \in \mathcal{X}_q$ let v_x denote the corresponding $\mathbf{Q}$-valuation of the function field $\mathbf{Q}(M)$, normalised so that $v_x(p) = 1$. Then $\Psi_q(u)$ is the function $x \mapsto v_x(u)$. Clearly Ψ_q is G_f-equivariant, and

$$\Psi_q\left(\mathbf{Q}(\boldsymbol{\mu}_\infty)^* \otimes \overline{\mathbf{Q}}\right) = H^0(\mathcal{Y}_q, \overline{\mathbf{Q}}) \subset H^0(\mathcal{X}_q, \overline{\mathbf{Q}}).$$

The quotient $H^0(\mathcal{X}_q, \overline{\mathbf{Q}})/H^0(\mathcal{Y}_q, \overline{\mathbf{Q}})$ is of the form $\mathcal{S}_q(1)^0 \otimes \mathcal{V}_q$, since in fact $\mathcal{S}_q(1)^0 \xrightarrow{\sim} H^0(\mathbf{P}^1(\mathbf{Q}_q), \overline{\mathbf{Q}})/\overline{\mathbf{Q}}$; we then define Φ_q to be the composite

$$\begin{array}{ccccc} \mathcal{O}^*(M) \otimes \overline{\mathbf{Q}} & \xrightarrow{\Psi_q} & H^0(\mathcal{X}_q, \overline{\mathbf{Q}}) & \longrightarrow & H^0(\mathcal{X}_q, \overline{\mathbf{Q}})/H^0(\mathcal{Y}_q, \overline{\mathbf{Q}}) \\ & & & & \downarrow \wr \\ & & \Phi_q & & \mathcal{S}_q(1)^0 \otimes \mathcal{V}_q \end{array}$$

It remains only to construct a unit $u_q \in \mathcal{O}^*(M)$ such that $\Phi_q(u_q) \neq 0$. To do this recall ([**2**], §1) that the *discriminant* $\Delta(E/S)$ of an elliptic curve $f : E \longrightarrow S$ is a nowhere-vanishing section of $\boldsymbol{\omega}_{E/S}{}^{\otimes 12} = (R^1 f_* \Omega^1_{E/S})^{\otimes 12}$, whose formation is compatible with base-change. Set

$$K_0(q) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G_{\hat{\mathbf{Z}}} \mid c \equiv 0 \pmod{q} \right\}.$$

Then $M_{\mathbf{Z}}/K_0(q)$ is the modular curve $Y_0(q)$ classifying triples $(E/S, E'/S, \alpha)$ where E, E' are elliptic curves over S and $\alpha : E \longrightarrow E'$ is an S-isogeny of degree q ([**3**] V.1.6, [**5**] Theorem 6.6.2).

Associate to $(E/S, E'/S, \alpha)$ the section

$$\alpha^* \Delta(E'/S)/\Delta(E/S) \in \Gamma(S, \mathcal{O}_S).$$

This defines an element $u_q \in \mathcal{O}(Y_0(q)) \subset \mathcal{O}(M_{\mathbf{Z}})$. Now $\alpha^* : \boldsymbol{\omega}_{E'/S} \longrightarrow \boldsymbol{\omega}_{E/S}$ is an isomorphism if α is étale. Thus over $\mathbf{Z}[1/q]$, u_q is nonvanishing, and so in particular $u_q \in \mathcal{O}^*(Y_0(q)_{\mathbf{Q}}) \subset \mathcal{O}^*(M)$.

Now the fibre $Y_0(q) \otimes \mathbf{F}_q$ has two irreducible components (see [**3**], Theorem V.1.16); one of these classifies triples (E, E', α) for which α is the relative Frobenius, and the other classifies triples for which α is the Verschiebung. When α is the Frobenius, the mapping α^* is zero. Therefore u_q vanishes identically on the one component, but not on the other. Now the irreducible components of $M_{\mathbf{Z}} \otimes \mathbf{F}_q$ map surjectively to the components of $Y_0(q) \otimes \mathbf{F}_q$. Therefore $\Phi_q(u_q) \neq 0$ as required.

References.

1. W. Casselman; An assortment of results on representations of $GL_2(k)$. Modular functions of one variable II. Lect. notes in mathematics **349**, 1–54 (Springer, 1973).

2. P. Deligne; Courbes elliptiques: formulaire. Modular functions of one variable IV. Lect. notes in mathematics **476**, 79–88 (Springer, 1975).

3. P. Deligne, M. Rapoport; Les schémas de modules des courbes elliptiques. Modular functions of one variable II. Lect. notes in mathematics **349**, 143–316 (Springer, 1973).

4. H. Jacquet, R. P. Langlands; Automorphic forms on $GL(2)$. Lect. notes in mathematics **114** (Springer, 1970).

5. N. M. Katz, B. Mazur; Arithmetic moduli of elliptic curves. Ann. of Math. Studies **108** (1985).

6. D. Kubert, S.Lang; Units in the modular function field II. Math. Ann. **218** (1975), 175–189.

7. N. Schappacher, A. J. Scholl; Beilinson's theorem on modular curves. Beilinson's conjectures on special values of L-functions, ed. M. Rapoport, N. Schappacher, P. Schneider (Academic Press, 1988).