A note on trilinear forms for reducible representations and Beilinson's conjectures

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Introduction

Let F be a non-Archimedean local field, and π_i $(i = 1, 2, 3)$ irreducible admissible representations of $G = GL_2(F)$, such that the product of their central characters is trivial. In [8], Prasad shows that there exists, up to a scalar factor, at most one G-invariant linear form on $\pi_1 \otimes \pi_2 \otimes \pi_3$, and determines exactly when such a form exists. These results have been used by Harris and Kudla [6] in the study of the triple product L-function attached to three cuspidal automorphic representations of GL_2 of a global field.

In this note we consider the case when π_i is permitted to be a reducible principal series representation, whose unique irreducible subspace is infinite-dimensional. It is relatively trivial to extend Prasad's results to cover these cases. The interest in so doing is global. In [1] Beilinson constructs certain subspaces of the motivic cohomology of the product of two modular curves using modular units. His construction can be interpreted as a certain invariant trilinear form on $\pi \otimes \pi' \otimes \pi''$ taking values in motivic cohomology: here π, π' are weight 2 cuspidal (irreducible) representations of GL_2 of the finite adeles of $\mathbb{Q}$, and π'' is the space of weight 2 holomorphic Eisenstein series (which is highly reducible). The regulators of these elements of motivic cohomology can be computed as special values of Rankin double product L-functions attached to π and π', and Beilinson's calculation of the regulator, together with his general conjectures, predict that these subspaces are one-dimensional. The main aim of the present note is to verify this prediction unconditionally (Theorem 3.1 below).

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1 Local trilinear forms

Throughout this section, F denotes a non-Archimedean local field, $\mathfrak{o}$ its valuation ring, and ϖ a uniformiser. We let $|-| : F^* \to \mathbb{Q}^*$ be the normalised absolute value, so that $|\varpi|^{-1} = \#(\mathfrak{o}/\varpi\mathfrak{o})$. We write $G = GL_2(F)$, and denote by B the standard Borel subgroup of upper triangular matrices, by A the diagonal torus, and by K the maximal compact subgroup $GL_2(\mathfrak{o})$. As usual $\delta\colon B \to \mathbb{Q}^*$ denotes the character

$$\delta \begin{pmatrix} b_1 & * \\ 0 & b_2 \end{pmatrix} = \left| \frac{b_1}{b_2} \right|$$

(which is the inverse of the modular character of B). Fix an algebraically closed field k of characteristic zero (in the applications we will take $k = \overline{\mathbb{Q}}$), and a square root $\sqrt{p}$ of the residue characteristic of F, which determines a square root $\delta^{1/2}$ of the character δ. We work in the category of smooth representations of G over k. As is customary we do not distinguish between a representation and the space on which it is realised.

We recall standard facts about induced representations of G, as can be found in [4, 7] or (in much greater generality) in [2, 3, 5]. Let $\mu = (\mu_1, \mu_2)\colon A \to k^*$ be a character of A, extended to B in the obvious way. Write $\mu^w = (\mu_2, \mu_1)$. The normalised[1] induced representation is then

$$\mathrm{Ind}_{\mathrm{B}}^{\mathrm{G}}(\mu) = \left\{ \begin{matrix} f\colon G \to k \text{ locally constant s.t.} \\ f(bg) = \mu(b)\delta(b)^{1/2} f(g) \text{ for all } b \in B,\ g \in G \end{matrix} \right\}.$$

This is an admissible representation of G which is indecomposable. It is irreducible if and only $\mu_1\mu_2^{-1} \neq |-|^{\pm 1}$, in which case it is also isomorphic to $\mathrm{Ind}_{\mathrm{B}}^{\mathrm{G}}\, \mu^w$. If it is reducible we may assume, twisting by a character of F^* if necessary, that $\mu = \delta^{\pm 1/2} = (\mu^{-1})^w$, and there are then non-split exact sequences of G-modules

$$0 \to k \to \mathrm{Ind}_{\mathrm{B}}^{\mathrm{G}}(\delta^{-1/2}) \to \mathrm{Sp} \to 0 \tag{1.1}$$

$$0 \to \mathrm{Sp} \to \mathrm{Ind}_{\mathrm{B}}^{\mathrm{G}}(\delta^{1/2}) \xrightarrow{\ \ell\ } k \to 0 \tag{1.2}$$

where Sp, the special or Steinberg representation, is the representation of G acting on the space of locally constant functions on $\mathbb{P}^1(F) = B\backslash G$ modulo constant functions. The space of K-invariants of each of the representations

[1] It would be preferable to use unnormalised induction, but we refrain from doing so in order to be able to quote from [8] without confusion.

$\mathrm{Ind}_{\mathrm{B}}^{\mathrm{G}}\,\delta^{\pm 1/2}$ is one-dimensional: for $\mathrm{Ind}_{\mathrm{B}}^{\mathrm{G}}\,\delta^{-1/2}$ it is the G-invariant subspace of constant functions; for $\mathrm{Ind}_{\mathrm{B}}^{\mathrm{G}}\,\delta^{1/2}$ it is the subspace spanned by the function $\phi\colon bk \mapsto \delta(b)$ (for $b \in B$, $k \in K$), and the linear form ℓ in (1.2) can be normalised so that $\ell(\phi) = 1$. Recall also that Sp is its own contragredient, and that $\dim \mathrm{Sp}^{K_0(\varpi)} = 1$, where $K_0(\varpi)$ denotes the Iwahori subgroup (elements of K which are congruent mod ϖ to an element of B). It follows that the G-invariant form $\mathrm{Sp} \otimes \mathrm{Sp} \to k$ is symmetric, because it must be non-zero on $\mathrm{Sp}^{K_0(\varpi)} \otimes \mathrm{Sp}^{K_0(\varpi)}$. (The same holds for any irreducible admissible representation of G with trivial central character by the theory of newvectors, an observation of Prasad and Ramakrishnan).

If π is an irreducible admissible representation of G, its central character will be denoted ω_π.

Write G' for the group of invertible elements of the unique quaternion division algebra over F. If π is a square-integrable (= discrete series) irreducible admissible representation of G, let π' be the irreducible representation of G' associated to π by the Jacquet-Langlands correspondence [7, §12].

Prasad proves [8, Thms 1.1, 1.2, 1.3]

Theorem 1.1. *Let π_i $(1 \le i \le 3)$ be irreducible admissible infinite-dimensional representations of G with $\prod \omega_{\pi_i} = 1$.*

(i) If at least one of π_i is principal series, then

$$\dim \mathrm{Hom}_G(\pi_1 \otimes \pi_2 \otimes \pi_3, k) = 1.$$

(ii) If all of π_i are discrete series, then

$$\dim \mathrm{Hom}_G(\pi_1 \otimes \pi_2 \otimes \pi_3, k) + \dim \mathrm{Hom}_{G'}(\pi_1' \otimes \pi_2' \otimes \pi_3', k) = 1$$

(iii) If all of π_i are unramified, then the restriction of a non-zero G-invariant form on $\pi_1 \otimes \pi_2 \otimes \pi_3$ to $\pi_1^K \otimes \pi_2^K \otimes \pi_3^K$ is non-zero.

As the Jacquet-Langlands correspondence takes the special representation Sp of G to the trivial representation of G', one has:

Corollary 1.2. *If π_1, π_2 are discrete series then*

$$\dim \mathrm{Hom}_G(\pi_1 \otimes \pi_2 \otimes \mathrm{Sp}, k) = 1 \iff \pi_1 \not\simeq \tilde{\pi}_2.$$

For convenience we quote two intermediate results from Prasad's paper which we shall need:

Proposition 1.3. [8, Cors. 5.7 & 5.8] *For any admissible representation π of G and any character χ of B,*

$$\begin{aligned}\mathrm{Ext}^1_G(\mathrm{Ind}_{\mathrm{B}}^{\mathrm{G}}\,\chi,\pi)=0 &\iff \mathrm{Hom}_G(\mathrm{Ind}_{\mathrm{B}}^{\mathrm{G}}\,\chi,\pi)=0\\ \mathrm{Ext}^1_G(\pi,\mathrm{Ind}_{\mathrm{B}}^{\mathrm{G}}\,\chi)=0 &\iff \mathrm{Hom}_G(\pi,\mathrm{Ind}_{\mathrm{B}}^{\mathrm{G}}\,\chi)=0\end{aligned}$$

Proposition 1.4. [8, p.17] *Let μ, μ' be characters of A. Then there is an exact sequence of G-modules:*

$$0\to c\text{-}\mathrm{Ind}_A^G(\mu\mu'^w)\to \mathrm{Ind}_{\mathrm{B}}^{\mathrm{G}}\,\mu\otimes\mathrm{Ind}_{\mathrm{B}}^{\mathrm{G}}\,\mu'\to\mathrm{Ind}_{\mathrm{B}}^{\mathrm{G}}(\mu\mu'\delta^{1/2})\to 0$$

where for a character $\nu\colon A\to k^*$,

$$c\text{-}\mathrm{Ind}_A^G\,\nu=\left\{\begin{matrix}f\colon G\to k \text{ compactly supported mod } A \text{ and locally constant}\\ \text{s.t. } f(ag)=\nu(a)f(g) \text{ for all } a\in A,\ g\in G\end{matrix}\right\}.$$

We now consider the case when π_i are admissible representations which are either irreducible or isomorphic to a twist of $\mathrm{Ind}_{\mathrm{B}}^{\mathrm{G}}\,\delta^{1/2}$.

Proposition 1.5. *Suppose that π, π' are infinite-dimensional irreducible admissible representations of G, with $\omega_\pi\omega_{\pi'}=1$. Then*

$$\dim\mathrm{Hom}_G(\pi\otimes\pi'\otimes\mathrm{Ind}_{\mathrm{B}}^{\mathrm{G}}\,\delta^{1/2},k)=1.$$

Moreover if π and π' are unramified, then the restriction of a non-zero invariant trilinear form to $\pi^K\otimes\pi'^K\otimes(\mathrm{Ind}_{\mathrm{B}}^{\mathrm{G}}\,\delta^{1/2})^K$ is non-zero.

Proof. For the most part we simply adapt the proofs in [8] — note that the hard case (three supercuspidals) doesn't arise.

Case 1: π is supercuspidal.
The analogous case is treated in [8, middle of p.18]. As π is supercuspidal, we have by the theory of the Kirillov model $\pi|_B\simeq c\text{-}\mathrm{Ind}_{ZN}^B\,\psi\omega_\pi$, and therefore by two applications of Frobenius reciprocity

$$\begin{aligned}\mathrm{Hom}_G(\pi\otimes\pi'\otimes\mathrm{Ind}_{\mathrm{B}}^{\mathrm{G}}\,\delta^{1/2}) &= \mathrm{Hom}_G(\pi\otimes\pi',\mathrm{Ind}_{\mathrm{B}}^{\mathrm{G}}\,\delta^{-1/2})\\ &= \mathrm{Hom}_B(c\text{-}\mathrm{Ind}_{ZN}^B(\psi\omega_\pi)\otimes\pi'|_B,k)\\ &= \mathrm{Hom}_{ZN}(\pi'|_{ZN},\psi^{-1}\omega_{\pi'})\end{aligned}$$

and the last group is simply $\mathrm{Hom}_N(\pi'|_N,\psi^{-1})$ which is 1-dimensional by the existence and uniqueness of the Kirillov model.

(It is worth noting that by [4, Theorem 1.6], π is projective in the category of smooth G-modules with central character ω_π, so $\pi \otimes \mathrm{Ind}_{\mathrm{B}}^{\mathrm{G}}\, \delta^{1/2} = \pi \oplus (\pi \otimes \mathrm{Sp})$ and

$$\mathrm{Hom}_G(\pi \otimes \pi' \otimes \mathrm{Ind}_{\mathrm{B}}^{\mathrm{G}}\, \delta^{1/2}, k) = \mathrm{Hom}_G(\pi \otimes \pi', k) \oplus \mathrm{Hom}_G(\pi \otimes \pi' \otimes \mathrm{Sp}, k)$$

which gives a direct proof of 1.2 when at least one of the representations is supercuspidal.)

Case 2: both π and π' are special.
After twisting we can assume that $\pi = \pi' = \mathrm{Sp}$. Then as $\mathrm{Hom}_G(\mathrm{Sp} \otimes \mathrm{Sp} \otimes \mathrm{Sp}, k) = 0$, we get from (1.2)

$$\mathrm{Hom}_G(\mathrm{Sp} \otimes \mathrm{Sp} \otimes \mathrm{Ind}_{\mathrm{B}}^{\mathrm{G}}\, \delta^{1/2}, k) = \mathrm{Hom}_G(\mathrm{Sp} \otimes \mathrm{Sp}, k) \simeq k.$$

Case 3: π principal series, π' principal series or special.
Suppose $\pi = \mathrm{Ind}_{\mathrm{B}}^{\mathrm{G}}\, \mu$ where $\mu_1/\mu_2 \neq |{-}|^{\pm 1}$. If $\pi' \not\simeq \tilde{\pi}$, then by Proposition 1.3

$$\mathrm{Hom}_G(\pi', \tilde{\pi}) = \mathrm{Ext}^1_G(\pi', \tilde{\pi}) = 0$$

and by Theorem 1.1, $\dim \mathrm{Hom}_G(\pi' \otimes \mathrm{Sp}, \tilde{\pi}) = 1$. Now by (1.1) we have a long exact sequence

$$(1.3) \quad 0 \to \mathrm{Hom}_G(\pi', \tilde{\pi}) \to \mathrm{Hom}_G(\pi' \otimes \mathrm{Ind}_{\mathrm{B}}^{\mathrm{G}}\, \delta^{1/2}, \tilde{\pi}) \to \mathrm{Hom}_G(\pi' \otimes \mathrm{Sp}, \tilde{\pi}) \to \mathrm{Ext}^1_G(\pi', \tilde{\pi}).$$

and therefore $\mathrm{Hom}_G(\pi \otimes \pi' \otimes \mathrm{Ind}_{\mathrm{B}}^{\mathrm{G}}\, \delta^{1/2}, k) = \mathrm{Hom}_G(\pi' \otimes \mathrm{Ind}_{\mathrm{B}}^{\mathrm{G}}\, \delta^{1/2}, \tilde{\pi}) \simeq k$.

In the case $\pi' = \tilde{\pi}$, the exact sequence (1.3) shows that there is at least one nonzero trilinear form. To show it is the only one, we proceed as in §5 of [8]; using Proposition 1.4 for $\pi \otimes \mathrm{Ind}_{\mathrm{B}}^{\mathrm{G}}\, \delta^{1/2}$ and then applying the functor $\mathrm{Hom}_G(-, \pi) = \mathrm{Hom}_G(-, \tilde{\pi}')$ we get a long exact sequence:

$$0 \to \mathrm{Hom}_G(\mathrm{Ind}_{\mathrm{B}}^{\mathrm{G}}\, \mu\delta, \pi) \to \mathrm{Hom}_G(\pi \otimes \mathrm{Ind}_{\mathrm{B}}^{\mathrm{G}}\, \delta^{1/2}, \pi) \to \mathrm{Hom}_G(c\text{-}\mathrm{Ind}_A^G\, \mu\delta^{-1/2}, \pi).$$

Since $\pi = \mathrm{Ind}_{\mathrm{B}}^{\mathrm{G}}\, \mu$ is irreducible, $\mathrm{Hom}_G(\mathrm{Ind}_{\mathrm{B}}^{\mathrm{G}}\, \mu\delta, \pi)$ can only be nonzero if $\mathrm{Ind}_{\mathrm{B}}^{\mathrm{G}}\, \mu \simeq \mathrm{Ind}_{\mathrm{B}}^{\mathrm{G}}\, \mu\delta$, which means $\mu\delta = \mu^w$, forcing $\mu_1/\mu_2 = |{-}|^{-1}$ which is not the case. Also

$$\begin{aligned} \mathrm{Hom}_G(c\text{-}\mathrm{Ind}_A^G\, \mu\delta^{-1/2}, \pi) &= \mathrm{Hom}_G(c\text{-}\mathrm{Ind}_A^G\, \mu\delta^{-1/2} \otimes \tilde{\pi}, k) \\ &= \mathrm{Hom}_A(\mu\delta^{-1/2} \otimes \tilde{\pi}|_A, k) \end{aligned}$$

by Frobenius reciprocity, and this last space is one-dimensional by [8, Lemma 5.6(a)]. Therefore $\dim_G(\pi \otimes \mathrm{Ind}_{\mathrm{B}}^{\mathrm{G}} \delta^{1/2}, \pi) \leq 1$, and the dimension is therefore exactly one.

For the final statement about unramified representations, we simply go through word-for-word the proof of [8, Thm. 5.10], taking V_3 (in the notation of *loc. cit.*) to be π. The key point is that in the displayed formula in the middle of page 20, the denominator is non-zero; it vanishes only when one of V_1, V_2 is isomorphic to $\mathrm{Ind}_{\mathrm{B}}^{\mathrm{G}} \delta^{-1/2}$ (possibly twisted by a quadratic character). ☐

Proposition 1.6. *Suppose that π is an infinite-dimensional irreducible admissible representation of G, with $\omega_\pi = 1$. Then*

$$\dim \mathrm{Hom}_G(\pi \otimes \mathrm{Ind}_{\mathrm{B}}^{\mathrm{G}} \delta^{1/2} \otimes \mathrm{Ind}_{\mathrm{B}}^{\mathrm{G}} \delta^{1/2}, k) = 1.$$

If π is unramified then the restriction of any non-zero invariant trilinear form to $\pi^K \otimes (\mathrm{Ind}_{\mathrm{B}}^{\mathrm{G}} \delta^{1/2})^K \otimes (\mathrm{Ind}_{\mathrm{B}}^{\mathrm{G}} \delta^{1/2})^K$ is non-zero.

Proof. We have again the exact sequence (1.3) with $\pi' = \mathrm{Ind}_{\mathrm{B}}^{\mathrm{G}} \delta^{1/2}$, and since π is irreducible and not 1-dimensional, $\mathrm{Hom}_G(\mathrm{Ind}_{\mathrm{B}}^{\mathrm{G}} \delta^{1/2}, \tilde{\pi}) = 0$. By Proposition 1.3 we also have $\mathrm{Ext}^1_G(\mathrm{Ind}_{\mathrm{B}}^{\mathrm{G}} \delta^{1/2}, \tilde{\pi}) = 0$, and by 1.5 we have $\dim \mathrm{Hom}_G(\mathrm{Ind}_{\mathrm{B}}^{\mathrm{G}} \delta^{1/2} \otimes \mathrm{Sp}, \tilde{\pi}) = 1$, giving the result. The proof of the final part is the same as for Proposition 1.5. ☐

For completeness we also show:

Proposition 1.7. $\mathrm{Hom}_G(\mathrm{Ind}_{\mathrm{B}}^{\mathrm{G}} \delta^{1/2} \otimes \mathrm{Ind}_{\mathrm{B}}^{\mathrm{G}} \delta^{1/2} \otimes \mathrm{Ind}_{\mathrm{B}}^{\mathrm{G}} \delta^{1/2}, k)$ *is* 1*-dimensional. It is generated by the form $\ell \otimes \ell \otimes \ell$, which is nonzero on* $(\mathrm{Ind}_{\mathrm{B}}^{\mathrm{G}} \delta^{1/2})^K \otimes (\mathrm{Ind}_{\mathrm{B}}^{\mathrm{G}} \delta^{1/2})^K \otimes (\mathrm{Ind}_{\mathrm{B}}^{\mathrm{G}} \delta^{1/2})^K$.

Proof. Recall (1.2) that ℓ denotes a nonzero invariant linear form on $\mathrm{Ind}_{\mathrm{B}}^{\mathrm{G}} \delta^{1/2}$, and that there is a unique K-fixed vector $\phi \in \mathrm{Ind}_{\mathrm{B}}^{\mathrm{G}} \delta^{1/2}$ with $\ell(\phi) = 1$. Fix a non-zero invariant form $(-,-)\colon \mathrm{Sp} \otimes \mathrm{Sp} \to k$. Let $\beta\colon \mathrm{Ind}_{\mathrm{B}}^{\mathrm{G}} \delta^{1/2} \otimes \mathrm{Ind}_{\mathrm{B}}^{\mathrm{G}} \delta^{1/2} \otimes \mathrm{Ind}_{\mathrm{B}}^{\mathrm{G}} \delta^{1/2} \to k$ be a G-invariant form. Then β vanishes on $\mathrm{Sp} \otimes \mathrm{Sp} \otimes \mathrm{Sp}$ by Corollary 1.2. Therefore there are constants a, b, $c \in k$ such that if v, $v' \in \mathrm{Sp}$ and $w \in \mathrm{Ind}_{\mathrm{B}}^{\mathrm{G}} \delta^{1/2}$, then

$$\beta(w \otimes v \otimes v') = a\,\ell(w)(v, v')$$
$$\beta(v' \otimes w \otimes v) = b\,\ell(w)(v, v')$$
$$\beta(v \otimes v' \otimes w) = c\,\ell(w)(v, v')$$

Since $\mathrm{Sp}^K = 0$ we have

(1.4) $$\beta(v \otimes \phi \otimes \phi) = 0 \quad \text{for all } v \in \mathrm{Sp}.$$

Put $u_g = g\phi - \phi \in \mathrm{Sp}$. Then for any $v \in \mathrm{Sp}$,

$$\begin{aligned} 0 &= \beta(g^{-1}v \otimes \phi \otimes \phi) = \beta(v \otimes g\phi \otimes g\phi) \\ &= \beta(v \otimes u_g \otimes \phi) + \beta(v \otimes \phi \otimes u_g) = c(v, u_g) + b(u_g, v) \end{aligned}$$

hence $b = -c$ since $(-,-)$ is symmetric. Likewise $b = -a = c$ hence $a = b = c = 0$. The vectors $\{u_g \mid g \in G\}$ span Sp over k, since ϕ is a generator for $\mathrm{Ind}_{\mathrm{B}}^{\mathrm{G}}\, \delta^{1/2}$. Therefore β vanishes on all products $u \otimes v \otimes w$ where at least two factors lie in Sp.

It then follows easily from (1.4) that β vanishes on all products where at least one factor lies in Sp, which implies that β is a multiple of $\ell \otimes \ell \otimes \ell$. □

2 Global trilinear forms

In this section, F will denote a global field. The symbols v, w will denote finite places of F. Let $\mathbb{A}_f$ be the ring of finite adeles of F (the restricted direct product of the completions F_v over all finite places v), and $F^*_{>0} \subset F^*$ the subgroup of elements which are positive at every real place. For each v write $G_v = GL_2(k_v)$. We use the same notations for objects associated to G_v as in the previous section, with a subscript v added.

Write G_f for the group $GL_2(\mathbb{A}_f)$ (which is the restricted direct product of the local groups G_v), B_f for the upper triangular subgroup of G_f and $\delta_f = \prod_v \delta_v \colon B_f \to \mathbb{Q}^*$.

We first consider the passage from local to global forms.

Proposition 2.1. *Let $\pi = \otimes'\pi_v$, $\pi' = \otimes'\pi'_v$, $\pi'' = \otimes'\pi''_v$ be factorisable admissible representations of G_f. Assume that each of π_v, π'_v, π''_v is either irreducible or a twist of $\mathrm{Ind}_{B_v}^{G_v}\, \delta_v^{1/2}$. Then*

$$\dim \mathrm{Hom}_{G_f}(\pi \otimes \pi' \otimes \pi'', k) \leq 1$$

with equality if and only if for every v

$$\dim \mathrm{Hom}_{G_v}(\pi_v \otimes \pi'_v \otimes \pi''_v, k) = 1.$$

Proof. Recall first the definition of the restricted tensor product $\pi = \otimes'\pi_v$, which depends on a choice of spherical vector $\phi_v \in \pi_v^{K_v}$ for all v outside some finite set Σ. It is defined to be the inductive limit of finite tensor products $\pi_S = \otimes_{v \in S}\pi_v$, where S runs over finite sets of places containing Σ. If $S \subset T$

then the inclusion mapping $\pi_S \hookrightarrow \pi_T$ is defined by $x \mapsto x \otimes \bigotimes_{v \in T-S} \phi_v$. In particular, if

$$\pi = \bigotimes_{\{\phi_v | v \notin \Sigma\}}{}' \pi_v, \quad \pi' = \bigotimes_{\{\phi'_v | v \notin \Sigma\}}{}' \pi'_v, \quad \pi'' = \bigotimes_{\{\phi''_v | v \notin \Sigma\}}{}' \pi''_v,$$

then their tensor product is

$$\pi \otimes \pi' \otimes \pi'' = \bigotimes_{\{\phi_v \otimes \phi'_v \otimes \phi''_v | v \notin \Sigma\}}{}' \pi_v \otimes \pi'_v \otimes \pi''_v.$$

(Of course it need not be the case that $(\pi_v \otimes \pi'_v \otimes \pi''_v)^{K_v}$ is 1-dimensional, or even finite-dimensional). To give a non-zero invariant form on $\pi \otimes \pi' \otimes \pi''$ is therefore equivalent to giving, for each v, a non-zero invariant form on $\pi_v \otimes \pi'_v \otimes \pi''_v$, which for almost all v takes the value 1 on $\phi_v \otimes \phi'_v \otimes \phi''_v$. Now use Prasad's results (Theorem 1.1) and Propositions 1.5, 1.6 and 1.7. (We have not excluded the possibility that some of the local components of the original representations are one-dimensional, but in that case the local theory is trivial.) □

The representations to which 2.1 applies can be highly reducible. We next restrict to a particular class of such representations which (for $F = \mathbb{Q}$) arise from weight 2 Eisenstein series. Let $\chi \colon \mathbb{A}_f^*/F_{>0}^* \to k^*$ be any character of finite order (in other words, χ is the restriction to $\mathbb{A}_f^*$ of an idele class character of finite order). Set

$$\mathcal{I}(\chi) = \left\{ \begin{matrix} f \colon G_f \to k \text{ locally constant s.t. } f(bg) = \chi(b_1)\delta_f(b)f(g) \\ \text{for all } g \in G_f \text{ and } b = \begin{pmatrix} b_1 & * \\ 0 & b_2 \end{pmatrix} \in B_f \end{matrix} \right\}.$$

Then $\mathcal{I}(\chi)$ is an admissible G_f-module and is isomorphic to the restricted tensor product $\otimes'_v \mathcal{I}_v(\chi_v)$, where

$$\mathcal{I}_v(\chi_v) = \mathrm{Ind}_{B_v}^{G_v}(\chi_v \, |{-}|_v^{1/2}, |{-}|_v^{-1/2})$$

If $\chi_v = 1$ then $\mathcal{I}_v(\chi_v) = \mathcal{I}_v(1) = \mathrm{Ind}_{B_v}^{G_v} \delta_v^{1/2}$, and we have the exact sequence (1.2):

$$0 \to \mathrm{Sp}_v \to \mathcal{I}_v(1) \xrightarrow{\ell_v} k \to 0.$$

We assume that when $\mathcal{I}_v(1)$ occurs in a restricted tensor product, the associated K_v-invariant vector ϕ_v is taken to be the unique one satisfying $\ell_v(\phi_v) = 1$.

If $\chi = 1$ then we have a local linear form ℓ_v for every v, hence their product $\ell_f = \otimes' \ell_v$ is a G_f-invariant linear form $\ell_f \colon \mathcal{I}(1) \to k$; we write $\mathcal{I}(1)^0 = \ker \ell_f \subset \mathcal{I}(1)$. If we set

$$U_w = \mathrm{Sp}_w \otimes \bigotimes_{v \neq w}{}' \mathcal{I}_v(1)$$

then $\mathcal{I}(1)^0$ is the sum of the subspaces U_w.

For arbitrary χ, observe that by Chebotarev $\chi_v = 1$ for infinitely many v, so that the global representation $\mathcal{I}(\chi)$ is an admissible G_f-module of infinite length.

Proposition 2.2. *Let $\pi = \otimes' \pi_v$, $\pi' = \otimes' \pi'_v$ be irreducible admissible representations of G_f, all of whose local components are infinite-dimensional.*

(i) If $\chi \colon \mathbb{A}_f^* / F_{>0}^* \to k^*$ is any character of finite order and $\omega_\pi \omega_{\pi'} \chi = 1$ then

$$\dim \mathrm{Hom}_{G_f}(\pi \otimes \pi' \otimes \mathcal{I}(\chi), k) = 1.$$

(ii) If $\pi' \not\simeq \tilde{\pi}$ and $\omega_\pi \omega_{\pi'} = 1$ then

$$\dim \mathrm{Hom}_{G_f}(\pi \otimes \pi' \otimes \mathcal{I}(1)^0, k) = 1.$$

(iii) If $\pi' \simeq \tilde{\pi}$ then

$$\dim \mathrm{Hom}_{G_f}(\pi \otimes \pi' \otimes \mathcal{I}(1)^0, k) = \infty$$

Proof. (i) This follows immediately from 2.1, 1.1 and 1.5.

(ii) Pick w with $\pi'_w \not\simeq \tilde{\pi}_w$. Observe that on the quotient

$$\mathcal{I}(1)/U_w = \bigotimes_{v \neq w}{}' \mathcal{I}_v(1)$$

the subgroup $G_w \subset G_f$ acts trivially (hence also on $\mathcal{I}(1)^0/U_w$). Therefore $\mathrm{Hom}_{G_f}(\pi \otimes \pi' \otimes \mathcal{I}(1)^0/U_w, k) = \mathrm{Hom}_{G_f}(\pi \otimes \pi' \otimes \mathcal{I}(1)/U_w, k) = 0$, and thus the homomorphisms of restriction

$$\text{(2.1)} \quad \mathrm{Hom}_{G_f}(\pi \otimes \pi' \otimes \mathcal{I}(1), k) \to \mathrm{Hom}_{G_f}(\pi \otimes \pi' \otimes \mathcal{I}(1)^0, k) \to \mathrm{Hom}_{G_f}(\pi \otimes \pi' \otimes U_w, k),$$

are injective. But the proof of (i) shows that

$$\dim \mathrm{Hom}_{G_f}(\pi \otimes \pi' \otimes \mathcal{I}(1), k) = 1 = \dim \mathrm{Hom}_{G_f}(\pi \otimes \pi' \otimes U_w, k),$$

so we are done.

(iii) For each $w \notin S$ there is a G_f-equivariant surjective homomorphism

$$\begin{aligned} \lambda_w \colon \mathcal{I}(1) &\to \mathcal{I}_w(1) \\ \otimes' x_v &\mapsto x_w \prod_{v \neq w} \ell_v(x_v) \end{aligned}$$

where G_f acts on $\mathcal{I}_w$ via the projection $G_f \to G_w$, and whose kernel is

$$\ker \lambda_w = \sum_{w' \neq w} U_{w'}.$$

Observe that $\lambda_w(\mathcal{I}(1)^0) = \mathrm{Sp}_w \subset \mathcal{I}_w(1)$, and that for any $x \in \mathcal{I}(1)^0$, $\lambda_w(x) = 0$ for all but finitely many w. Therefore the sum of these homomorphisms is a G_f-equivariant surjection

$$\lambda = (\lambda_w) \colon \mathcal{I}(1)^0 \to \bigoplus_w \mathrm{Sp}_w$$

whose kernel is the subspace $\sum_{w \neq w'} U_w \cap U_{w'}$. Therefore we have a G_f-equivariant surjection

$$\pi \otimes \pi' \otimes \mathcal{I}(1)^0 \to \bigoplus_w \pi \otimes \pi' \otimes \mathrm{Sp}_w \tag{2.2}$$

Now for all but finitely many w the local components π_w, π'_w are unramified, hence principal series, so there will exist a nonzero trilinear form on $\pi_w \otimes \pi'_w \otimes \mathrm{Sp}_w$. For all $v \neq w$ we have a pairing $\pi_v \otimes \pi'_v \to k$ by hypothesis. Therefore the right-hand side of (2.2) has an infinite-dimensional quotient on which G_f acts trivially. □

We also have an analogous result when two of the representations are of the form $\mathcal{I}(\chi)$ or $\mathcal{I}(1)^0$:

Proposition 2.3. *Let $\pi = \otimes' \pi_v$ be an irreducible admissible representations of G_f whose local components are all infinite-dimensional. Suppose that π' and π'' are representations of the form $\mathcal{I}(\chi)$ or $\mathcal{I}(1)^0$, and that $\omega_\pi \omega_{\pi'} \omega_{\pi''} = 1$. Then*

$$\dim \mathrm{Hom}_{G_f}(\pi \otimes \pi' \otimes \pi'', k) = 1.$$

Proof. If both of π', π'' are of the form $\mathcal{I}(\chi)$, then this follows from 2.1.

If $\pi' = \mathcal{I}(\chi)$ and $\pi'' = \mathcal{I}(1)^0$, then we can choose w such that $\mathrm{Hom}_{G_w}(\pi_w \otimes \mathcal{I}_w(\chi_w), k) = 0$ (it is enough to take w such that $\chi_w = 1$ and π_w is unramified). Then the same argument as in 2.2(ii) applies, using 1.6 in place of 1.5.

Finally suppose that $\pi' = \pi'' = \mathcal{I}(1)^0$. Then consider the inclusions

$$U_w \otimes \mathcal{I}(1)^0 \subset \mathcal{I}(1)^0 \otimes \mathcal{I}(1)^0 \subset \mathcal{I}(1) \otimes \mathcal{I}(1)^0$$

whose successive quotients are $(\mathcal{I}(1)^0/U_w) \otimes \mathcal{I}(1)^0$ and $\mathcal{I}(1)^0$. We have $\mathrm{Hom}_{G_f}(\pi \otimes \mathcal{I}(1)^0, k) = 0$. In fact, as $\mathcal{I}(1)^0 = \sum U_w$ it is enough to show that $\mathrm{Hom}_{G_f}(\pi \otimes U_w, k) = 0$ for every w, which is clear locally. We claim that for w such that π_w is unramified, $\mathrm{Hom}_{G_f}(\pi \otimes (\mathcal{I}(1)^0/U_w) \otimes \mathcal{I}(1)^0, k) = 0$. Again it is enough to show that for every w', $\mathrm{Hom}_{G_f}(\pi \otimes (\mathcal{I}(1)^0/U_w) \otimes U_{w'}, k) = 0$, and this is true locally at w, since $\mathcal{I}(1)^0/U_w$ is trivial at w.

For such w the restriction homomorphisms

$$\begin{aligned}\mathrm{Hom}_{G_f}(\pi \otimes \mathcal{I}(1) \otimes \mathcal{I}(1)^0, k) \to \mathrm{Hom}_{G_f}(\pi \otimes \mathcal{I}(1)^0 \otimes \mathcal{I}(1)^0, k)&\\ \to \mathrm{Hom}_{G_f}(\pi \otimes U_w \otimes \mathcal{I}(1)^0, k)&\end{aligned}$$

are then injective, and Proposition 2.1 and the appropriate local results show that the two outer groups have dimension one. □

3 Beilinson's subspaces

We briefly review here Beilinson's results [1] concerning the L-function of a product of two modular curves at $s = 1$. We use the notation and formulation of [10, §2] where details can be found. For a positive integer n, M_n denotes the modular curve over $\mathbb{Q}$ parameterising elliptic curves with full level n structure, and $\overline{M}_n$ denotes its smooth compactification. Write $M = \varprojlim M_n$, $\overline{M} = \varprojlim \overline{M}_n$ for the modular curves at infinite level. These are schemes over the maximal abelian extension $\mathbb{Q}^{ab}$ of $\mathbb{Q}$.

In the notation of the previous section we take $F = \mathbb{Q}$. Then G_f acts on M and $\overline{M}$. (We assume that our level structures are defined in such a way that this is a right action). If

$$K_n = \ker\Big(GL_2(\hat{\mathbb{Z}}) \to GL_2(\mathbb{Z}/n\mathbb{Z})\Big)$$

is the standard level n open compact subgroup of G_f then M_n is the quotient M/K_n and $\overline{M}_n = \overline{M}/K_n$.

Next recall the decomposition of the motive of a modular curve under the Hecke algebra. We work in the category $\mathcal{M}_{\mathbb{Q}} \otimes \overline{\mathbb{Q}}$ of Chow motives over $\mathbb{Q}$ with coefficients in $\overline{\mathbb{Q}}$. One has a Chow-Künneth decomposition

$$h(\overline{M}_n) = h^0(\overline{M}_n) \oplus h^1(\overline{M}_n) \oplus h^2(\overline{M}_n).$$

The space $\Omega^1(\overline{M}) \otimes \overline{\mathbb{Q}}$ of holomorphic weight 2 cusp forms with coefficients in $\overline{\mathbb{Q}}$ decomposes as a direct sum of irreducible admissible representations π of G_f with multiplicity one. To each such π there is associated a rank 2 motive V_π in $\mathcal{M}_{\mathbb{Q}} \otimes \overline{\mathbb{Q}}$, which is a direct factor of $h^1(\overline{M}_n)$ if $\pi^{K_n} \neq 0$. The motives V_π are simple of rank 2, and V_π, $V_{\pi'}$ are isomorphic if and only if $\pi \simeq \pi'$. One then has

$$h^1(\overline{M}) = \varinjlim h^1(\overline{M}_n) = \bigoplus_{\pi} V_\pi \otimes [\pi].$$

Here $V_\pi \otimes [\pi]$ means simply the direct sum of an infinite number of copies of V_π, indexed by a basis for π. It is an ind-object of $\mathcal{M}_{\mathbb{Q}} \otimes \overline{\mathbb{Q}}$ which carries an action of G_f.

In [1] Beilinson constructs a certain subspace of the motivic cohomology $H^3_{\mathcal{M}}(\overline{M}^2, \mathbb{Q}(2))$ using modular units supported on Hecke correspondences. One has a decomposition

$$h(\overline{M}^2) \supset h^1(\overline{M})^{\otimes 2} = \bigoplus_{\pi,\pi'} V_\pi \otimes_{\overline{\mathbb{Q}}} V_{\pi'} \otimes [\pi \times \pi']$$

where $[\pi \times \pi']$ is the space of the exterior tensor product of π and π'. Applying this one can rewrite Beilinson's construction as giving, for each pair (π, π'), a homomorphism [10, §2.3.3]

$$\text{Б}(\pi \times \pi') \colon \left(\mathcal{O}^*(M) \otimes_{\mathbb{Z}} \tilde{\pi} \otimes_{\overline{\mathbb{Q}}} \tilde{\pi}'\right)_{G_f} \to H^3_{\mathcal{M}}(V_\pi \otimes V_{\pi'}, \mathbb{Q}(2))$$

whose source is the maximal quotient of $\mathcal{O}^*(M) \otimes_{\mathbb{Z}} \tilde{\pi} \otimes_{\overline{\mathbb{Q}}} \tilde{\pi}'$ on which G_f acts trivially.

The G_f-module $\mathcal{O}^*(M) \otimes_{\mathbb{Z}} \overline{\mathbb{Q}}$ can be described almost completely [9]. There is an exact sequence

$$0 \to \mathbb{Q}^{\mathrm{ab}*} \otimes_{\mathbb{Z}} \overline{\mathbb{Q}} \to \mathcal{O}^*(M) \otimes_{\mathbb{Z}} \overline{\mathbb{Q}} \to \mathcal{I}(1)^0 \oplus \bigoplus_{\chi} \mathcal{I}(\chi) \to 0$$

where the direct sum is over all even non-trivial characters $\chi \colon \mathbb{A}_f^*/\mathbb{Q}^* \to \overline{\mathbb{Q}}^*$ of finite order. The action of G_f on the trivial modular units $\mathbb{Q}^{\mathrm{ab}*} \otimes_{\mathbb{Z}} \overline{\mathbb{Q}}$ is the composite of the determinant and the reciprocity law of class field theory.

We now assume that π' is not isomorphic to a twist of π; this implies in particular [10, Lemma 2.5.2] that $Б(\pi \times \pi')$ is trivial on $\mathbb{Q}^{\mathrm{ab}*}$ and [10, Theorem 2.3.4] that its image lies in the integral part of the motivic cohomology, hence factors as

$$Б(\pi \times \pi') \colon \left(\mathcal{I}(\chi)^0 \otimes \tilde{\pi} \otimes \tilde{\pi}'\right)_{G_f} \to H^3_{\mathcal{M}/\mathbb{Z}}(V_\pi \otimes V_{\pi'}, \mathbb{Q}(2)).$$

Here $\chi = \omega_\pi \omega_{\pi'}$, and if $\chi \neq 1$, $\mathcal{I}(\chi)^0 \stackrel{\mathrm{def}}{=} \mathcal{I}(\chi)$. As we shall recall in a moment, one of Beilinson's main results [1, Thm. 6.1.1] shows that $Б(\pi \times \pi')$ is non-zero. We can then apply Proposition 2.2 to the source of the homomorphism to give:

Theorem 3.1. *Assume that π' is not isomorphic to a twist of π. Then the image of $Б(\pi \times \pi')$ has dimension one.* □

There is a regulator homomorphism from motivic cohomology to real Deligne cohomology:

$$r_{\mathcal{H}} \colon H^3_{\mathcal{M}/\mathbb{Z}}(V_\pi \otimes V_{\pi'}, \mathbb{Q}(2)) \to H^3_{\mathcal{H}}(V_\pi \otimes V_{\pi'}, \mathbb{R}(2))$$

whose target is in this case a free $\mathbb{R} \otimes \overline{\mathbb{Q}}$-module of rank one. In [1, §6] Beilinson explains how to compute the composite $r_{\mathcal{H}} \circ Б(\pi \times \pi')$ as a Rankin-Selberg integral; its image is a 1-dimension $\overline{\mathbb{Q}}$-subspace in $H^3_{\mathcal{H}}(V_\pi \otimes V_{\pi'}, \mathbb{R}(2))$, which can be described in terms of the special value $L(V_\pi \otimes V_{\pi'}, 2)$. In particular $Б(\pi \times \pi') \neq 0$, and $\dim_{\overline{\mathbb{Q}}} H^3_{\mathcal{M}/\mathbb{Z}}(V_\pi \otimes V_{\pi'}, \mathbb{Q}(2)) \geq 1$. Beilinson's general conjectures predict that the dimension is one, but at present even finite-dimensionality is unknown.

It would be nice if the same argument worked for Beilinson's construction of elements of $H^2_{\mathcal{M}}(V_\pi, \mathbb{Q}(2))$. However in this case the generating homomorphism is a G_f-invariant linear map

$$Б(\pi) \colon \mathcal{O}^*(M) \otimes \mathcal{O}^*(M) \otimes \tilde{\pi} \to H^2_{\mathcal{M}/\mathbb{Z}}(V_\pi, \mathbb{Q}(2))$$

When constant units are factored out, its source becomes a direct sum of tensor products

$$\bigoplus_{\chi \text{ even}} \mathcal{I}(\chi)^{(0)} \otimes \mathcal{I}(\chi^{-1}\omega_\pi)^{(0)} \otimes \tilde{\pi}$$

(where $\mathcal{I}(\chi)^{(0)}$ denotes $\mathcal{I}(1)^0$ for χ trivial, and $\mathcal{I}(\chi)$ otherwise). The space of G_f-coinvariants of each summand is one-dimensional by Proposition 2.3, but this alone does not suffice to bound the image of $Б(\pi)$.

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