Remarks on special values of L-functions

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Introduction.

This article does not represent precisely a talk given at the symposium, but is complementary to [**DenS**]. Its purpose is to explain a setting in which the various conjectures on special values of L-functions admit a unified formulation. At critical points, Deligne's conjecture [**Del2**] relates the value of an L-function to a certain period, and at non-critical points, the conjectures of Beilinson [**Be1**] give an interpretation in terms of regulators. Finally, at the point of symmetry of the functional equation, there is the conjecture of Birch and Swinnerton-Dyer, generalised by Bloch [**Bl2**] and Beilinson [**Be2**], in which the determinant of the height pairing on cycles appears.

Both the periods and the regulators are constructed globally, and their definitions are in some sense archimedean. The height pairing, on the other hand, is defined as a sum of local terms. Our aim is to show how all of these objects—periods, regulators, and heights—may be interpreted as "periods of mixed motives".

That such a reformulation is possible in the case of regulators is clearly indicated in the letter of Deligne to Soulé [**Del3**]. Perhaps the only novel feature of our account is to regard the mixed motives as primary objects, rather than the Ext groups. It is appropriate to mention in this connection work of Anderson and of Harder [**H**], in which certain particular mixed motives arising in the study of the cohomology of Shimura varieties are investigated. These motives fit directly (and without assuming a grand conjectural framework) into our setting, although their connection with the K-theoretical formulation of Beilinson's conjectures remains obscure.

Section I recalls some of the properties of pure motives, and Deligne's period conjecture [**Del2**]. In section II we state a suitable generalisation of this conjecture to mixed motives. Although there does not as yet exist an entirely satisfactory definition of the category of mixed motives, Deligne [**Del4**] and Jannsen [**J**] have given an unconditional definition, based on absolute Hodge cycles.

In the third section we review the expected relation between extensions of motives and "motivic cohomology" ([**Del3**], [**Be2**], [**J**]). We define a category of " mixed motives over $\mathbf{Z}$", in which the Ext-groups should correspond

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to the integral part of motivic cohomology (coming from the K-theory of regular schemes, projective over Spec $\mathbf{Z}$). In the appendix to [**DenS**] some evidence for this relation is described.

In sections IV and V we describe the consequences of the period conjecture in the case of certain particular mixed motives ("universal extensions"). This shows how the conjecture includes the relevant parts of the conjectures of Beilinson and Birch-Swinnerton-Dyer as special cases. The comparison at the central point is somewhat more complicated than at the other points, and we only give a sketch; more details will appear later [**S**].

In the final section we show, in answer to a question raised by Deligne, that the period conjecture of section II contains no further information on L-values than the existing conjectures. We also repeat the calculations of §5 of [**Del2**] to show that it is compatible with the functional equation. Thus the period conjecture satisfies the most obvious consistency conditions.

An obvious gap in our account is the failure to allow motives with coefficients other than $\mathbf{Q}$. However we hope that it is apparent that the same constructions can carried out be done word-for-word for motives with coefficients. The other restriction we have imposed—that the ground field be always $\mathbf{Q}$—seems in contrast to be essential to our approach, mainly because of the "peculiar" behaviour of the Riemann zeta function at $s=0$ and $s=1$.

As should be clear to the reader, this article is really only a naïve attempt to come to grips with Beilinson's conjectures in a motivic setting, and relies heavily on the ideas of [**Be1**], [**Be2**], [**Del3**] and [**J**]. I would like to thank particularly Peter Schneider and Uwe Jannsen for stimulating discussions and comments.

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I. Pure motives and Deligne's conjecture.

We first recall the notion of a (pure, or homogeneous) motive. For the moment it will not be too important which category of motives we consider–either Grothendieck motives (defined by algebraic correspondences modulo homological equivalence), or Deligne's category of motives defined by absolute Hodge cycles, would do. However in subsequent sections it will be important that the realisation functors should be faithful—this being the case in both categories mentioned, by construction. For simplicity we consider only motives defined over $\mathbf{Q}$, with coefficients in $\mathbf{Q}$.

Associated to a motive M over $\mathbf{Q}$ are its various realisations M_B, M_l, M_{DR} and the comparison isomorphisms

$$I_l : M_B \otimes \mathbf{Q}_l \xrightarrow{\sim} M_l, \qquad I_\infty : M_B \otimes \mathbf{C} \xrightarrow{\sim} M_{DR} \otimes \mathbf{C}.$$

M will be *pure of weight* w if the eigenvalues of an unramified Frobenius Frob_p acting on M_l have absolute value $p^{w/2}$, and if the Hodge filtration induces a Hodge structure on M_B which is pure of weight w. The example to bear in mind is of course $M = h^i(X)(m)$, where X is smooth and projective over $\mathbf{Q}$, and $w = i - 2m$.

The L-function $L(M,s)$ is the Euler product:

$$L(M,s) = \prod_p L_{(p)}(M,s)$$

where

$$L_{(p)}(M,s) = \det\left(1 - p^{-s}\,\mathrm{Frob}_p \,|M_l^{I_p}\right)^{-1}, \quad l \neq p$$

(conjecturally independent of l). We assume the existence of the meromorphic continuation and functional equation whenever necessary.

The period mapping I_∞^+ is defined as the composite

$$M_B^+ \otimes \mathbf{R} \hookrightarrow M_B \otimes \mathbf{C} \xrightarrow{\sim} M_{DR} \otimes \mathbf{C} \twoheadrightarrow M_{DR} \otimes \mathbf{R} \twoheadrightarrow \frac{M_{DR}}{F^0} \otimes \mathbf{R}$$

(this differs slightly from the notations of [**Del2**]). M is *critical* if I_∞^+ is an isomorphism. In this case its determinant is a well-defined element of $\mathbf{R}^*/\mathbf{Q}^*$, denoted $c^+(M)$.

Deligne's conjecture is that, for critical M,

$$L(M,0) \cdot c^+(M)^{-1} \in \mathbf{Q}.$$

Remarks. Deligne also conjectures that $L(M,0) \neq 0$ if $w \neq -1$. We shall return to this point later. Of course, the period c^+ may be defined even when M is not critical, but only satisfies a rather mild restriction (see [**Del2**] §1.7), but for mixed motives this will not turn out to be the case. Finally we remind the reader that the notion of being critical depends only on the vanishing of certain Hodge numbers h^{pq}, $h^{p\pm}$.

II. Mixed motives.

If X is an arbitrary scheme of finite type over $\mathbf{Q}$, then the cohomology groups $H^i(X)$, with respect to either de Rham, Betti or l-adic theory, have a natural filtration $W_{\bullet}$ (the weight filtration), compatible with the comparison isomorphisms and (in the case of l-adic cohomology) with the Galois action. The weight filtration induces a mixed Hodge structure on $H^i_B(X)$, and the Galois modules $\operatorname{Gr}^W_j H^i(X \otimes \overline{\mathbf{Q}}, \mathbf{Q}_l)$ are pure of weight j.

We now require the existence of a category $\mathcal{MM}_{\mathbf{Q}}$ of *mixed motives* over $\mathbf{Q}$. (As mentioned in the introduction, such a category has been constructed by Deligne and Jannsen.) This is to be an abelian category, containing $\mathcal{M}$ as a full subcategory. To each mixed motive E will be associated in a functorial way realisations E_B, E_{DR}, E_l. E_l is to be a finite-dimensional representation of $\operatorname{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$, and E_B, E_{DR} finite-dimensional vector spaces over $\mathbf{Q}$, equipped with an involution Φ_∞ and a decreasing filtration $F^{\bullet}$. There will be comparison isomorphisms

$$I_l : E_B \otimes \mathbf{Q}_l \xrightarrow{\sim} E_l, \qquad I_\infty : E_B \otimes \mathbf{C} \xrightarrow{\sim} E_{DR} \otimes \mathbf{C}.$$

Finally there is to be an increasing filtration $W^{\bullet}$ on E_B, stable under Φ_∞. The filtration induced (via I_l) on E_l is to be stable under the action of $\operatorname{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$, and the graded pieces $\operatorname{Gr}^W_j E_l$ are to be pure Galois representations of weight j. The filtration induced (by I_∞) on $E_{DR} \otimes \mathbf{C}$ is to be defined over $\mathbf{Q}$, and together with $F^{\bullet}$ and Φ_∞ is to define a mixed Hodge structure over $\mathbf{R}$. Lastly, the comparison isomorphisms I_l are to take the involution Φ_∞ to complex conjugation in $\operatorname{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$.

We may define the L-function of a mixed motive E in the same way as for a pure motive—notice that in general $L(E,s)$ and $\prod L(\operatorname{Gr}^W_j E, s)$ will differ by a finite number of Euler factors, as the passage to invariants under inertia is not an exact functor. There is one obvious case in which we have equality.

Definition. E is a mixed motive over $\mathbf{Z}$ if the weight filtration on E_l splits over $\mathbf{Q}_p^{nr}$, for every l, p with $l \neq p$.

Remark. Presumably one should expect that for a given p this condition need only be checked for one l.

The mixed motives over $\mathbf{Z}$ form a full subcategory $\mathcal{MM}_{\mathbf{Z}}$ of $\mathcal{MM}_{\mathbf{Q}}$, containing $\mathcal{M}$. They will play an important part in the next section.

If X is of finite type over $\mathbf{Q}$, then there should be mixed motives $h^i(X)$ in $\mathcal{MM}_{\mathbf{Q}}$. In general they will not be motives over $\mathbf{Z}$ (see the example below).

Returning to the case of an arbitrary mixed motive E, we define the period map

$$I_\infty^+ : E_B^+ \otimes \mathbf{R} \longrightarrow \frac{E_{DR}}{F^0} \otimes \mathbf{R}$$

in the same way as for pure motives.

Definition. A mixed motive E is critical if I_∞^+ is an isomorphism. If this holds, define $c^+(E) = \det I_\infty^+$.

Remark. It is obvious that if the pure motives $\mathrm{Gr}_j^W E$ are critical, then so is E, but the converse is far from true. For mixed motives, the notion of critical does not just depend on the Hodge numbers and the action of Φ_∞.

Conjecture A. If E is critical, then $L(E,0) \cdot c^+(E)^{-1} \in \mathbf{Q}$.

Example. (Trivial.) Let X be the singular curve obtained from $\mathbf{G}_m/\mathbf{Q}$ by identifying the points 1, p for some prime p. Then $E = h^1(X)(1)$ is an extension:

$$0 \longrightarrow \mathbf{Q}(1) \longrightarrow E \longrightarrow \mathbf{Q}(0) \longrightarrow 0$$

Here E is the 1-motive $[\mathbf{Z} \xrightarrow{1 \mapsto p} \mathbf{G}_m]$, in the sense of Deligne [**Del1**]. It is easy to see (using the explicit realisations of 1-motives given in *loc. cit.*) that

$$L(E,s) = \zeta(s)\zeta(s+1)(1-p^{-s})$$

so that $L(E,0) = -\frac{1}{2}\log p$. Moreover E is critical, and

$$c^+(E) = \int_1^p \frac{dt}{t}.$$

So in this case Conjecture A holds. Notice that in this case we have obtained the leading term of an *incomplete* L-function as a period.

III. Extensions of motives.

To discuss conditions under which a motive is critical, and to predict orders of L-functions, we need Ext groups. Write $\mathrm{Ext}_{\mathbf{Q}}$ for the Ext groups in $\mathcal{MM}_{\mathbf{Q}}$, and $\mathrm{Ext}_{\mathbf{Z}}$ for the groups in $\mathcal{MM}_{\mathbf{Z}}$. We should have $\mathrm{Ext}^q_{\mathbf{Q}} = \mathrm{Ext}^q_{\mathbf{Z}} = 0$ unless $q = 0$ or 1. If M, M' are motives over $\mathbf{Z}$, then $\mathrm{Ext}^0_{\mathbf{Q}}(M', M) = \mathrm{Ext}^0_{\mathbf{Z}}(M', M) = \mathrm{Hom}(M', M)$. Moreover $\mathrm{Ext}^1_{\mathbf{Z}}(M', M)$ is the subgroup of $\mathrm{Ext}^1_{\mathbf{Q}}(M', M)$ comprising the classes of those extensions

$$0 \longrightarrow M \longrightarrow E \longrightarrow M' \longrightarrow 0$$

such that for every p and every $l \neq p$, the extension E_l of Galois modules splits over $\mathbf{Q}_p^{nr}$. Conjecturally, the groups $\mathrm{Ext}_{\mathbf{Z}}$ will be finite-dimensional over $\mathbf{Q}$.

Suppose that X is smooth and proper over $\mathbf{Q}$, and that $M = h^i(X)(m)$, $N = M^\vee(1) \xrightarrow{\sim} h^i(X)(n)$ with $n = i+1-m$. Then we should have

$$\begin{aligned}
\mathrm{Ext}^0_{\mathbf{Z}}(M, \mathbf{Q}(1)) = \mathrm{Ext}^0_{\mathbf{Z}}(\mathbf{Q}(0), N) &= \mathrm{Hom}(\mathbf{Q}(-n), h^i(X)) \\
&= \begin{cases} 0 & \text{if } i \neq 2n \\ CH^n(X)/CH^n(X)^0 \otimes \mathbf{Q} & \text{if } i = 2n; \end{cases} \\
\mathrm{Ext}^1_{\mathbf{Z}}(M, \mathbf{Q}(1)) = \mathrm{Ext}^1_{\mathbf{Z}}(\mathbf{Q}(0), N) &= \begin{cases} H^{i+1}_{\mathcal{M}}(X, \mathbf{Q}(n))_{\mathbf{Z}} & \text{if } i+1 \neq 2n \\ CH^n(X)^0 \otimes \mathbf{Q} & \text{if } i+1 = 2n. \end{cases}
\end{aligned}$$

Here $CH^n(X)$ is the Chow group of codimension n cycles on X modulo rational equivalence, and $CH^n(X)^0$ is the subgroup of classes of cycles homologically equivalent to zero. $H_{\mathcal{M}}$ denotes the motivic cohomology:

$$H^i_{\mathcal{M}}(X, \mathbf{Q}(j)) = (K_{2j-i}X \otimes \mathbf{Q})^{(j)}$$

and $H^*_{\mathcal{M}}(X, \cdot)_{\mathbf{Z}}$ is the image in $H^*_{\mathcal{M}}(X, \cdot)$ of the K-theory of a regular model for X, proper and flat over $\mathbf{Z}$. The $\mathrm{Ext}_{\mathbf{Q}}$ groups will be the given by the same rules, but with $H^*_{\mathcal{M}}(X, \cdot)_{\mathbf{Z}}$ replaced by $H^*_{\mathcal{M}}(X, \cdot)$. In the case $i = 2n-1$ the equality of the groups $\mathrm{Ext}^1_{\mathbf{Z}}$ and $\mathrm{Ext}^1_{\mathbf{Q}}$ would be a consequence of the monodromy-weight filtration conjecture, which implies that any extension of $H^{2n-1}(X \otimes \overline{\mathbf{Q}}, \mathbf{Q}_l)$ by $\mathbf{Q}_l(n)$ splits over $\mathbf{Q}_p^{nr}$ for every $p \neq l$.

In the case of the Tate motive, the above would imply that

$$\begin{aligned}
\mathrm{Ext}^1_{\mathbf{Q}}(\mathbf{Q}(0), \mathbf{Q}(1)) &= \mathbf{Q}^* \otimes_{\mathbf{Z}} \mathbf{Q}; \\
\mathrm{Ext}^1_{\mathbf{Z}}(\mathbf{Q}(0), \mathbf{Q}(1)) &= \mathbf{Z}^* \otimes_{\mathbf{Z}} \mathbf{Q} = 0.
\end{aligned}$$

(The extension in the example of the previous section corresponds to $p \otimes 1 \in \mathbf{Q}^* \otimes \mathbf{Q}$.)

Although we are far from having a good category of mixed motives in which the above isomorphisms hold, one can nevertheless unconditionally associate to elements of $H_{\mathcal{M}}$-groups explicit extensions of cohomology arising from the cohomology of non-compact or singular schemes. For example, let X be smooth and projective over $\mathbf{Q}$, and let ξ be a cycle on X of codimension n, homologically equivalent to zero. If $H(-)$ denotes (say) l-adic cohomology, then by pullback from the exact sequence

$$\begin{array}{ccccccc} 0 & \longrightarrow & H^{2n-1}(X)(n) & \longrightarrow & H^{2n-1}(X-|\xi|)(n) & \longrightarrow & H^{2n}_{|\xi|}(X)(n) \\ & & & & & & \uparrow \\ & & & & & & \mathbf{Q}\cdot\xi \end{array}$$

one obtains an extension of $\mathbf{Q}_l(0)$ by $H^{2n-1}(X)(n)$, whose class depends only on the rational equivalence class of ξ—see [**J**], §9. The general case can be treated in a similar way, using Bloch's description of motivic cohomology by means of higher Chow groups [**Bl1**]—see the appendix of [**DenS**] for a sketch.

In this context, the regulator maps arise from the realisation functors. For example, suppose that $n > \frac{i}{2}+1$. Then it is shown in [**Be2**] that there is a canonical isomorphism

$$H^{i+1}_{\mathcal{D}}(X/\mathbf{R},\mathbf{R}(n)) \xrightarrow{\sim} \mathrm{Ext}^1_{\mathbf{R}-\mathrm{Hdg}}(\mathbf{R}, H^i(X)(n))$$

(where the second group is the Ext group in the category of real Hodge structures with an infinite Frobenius) and the regulator should then fit into a commutative square

$$\begin{array}{ccc} \mathrm{Ext}^1_{\mathbf{Z}}(\mathbf{Q}, h^i(X)(n)) & \xrightarrow{\sim} & H^{i+1}_{\mathcal{M}}(X,\mathbf{Q}(n))_{\mathbf{Z}} \\ \big\downarrow \text{realisation} & & \big\downarrow \text{regulator} \\ \mathrm{Ext}^1_{\mathbf{R}-\mathrm{Hdg}}(\mathbf{R}, H^i(X)(n)) & \xrightarrow{\sim} & H^{i+1}_{\mathcal{D}}(X/\mathbf{R},\mathbf{R}(n)) \end{array}$$

The conjectures of Birch–Swinnerton-Dyer, Tate and Beilinson on the orders of L-series at integer points can be simply stated in terms of Ext-groups:

Conjecture B. Let E be a motive over $\mathbf{Z}$. Then

$$\mathrm{ord}_{s=0} L(E,s) = \dim \mathrm{Ext}^1_{\mathbf{Z}}(E,\mathbf{Q}(1)) - \dim \mathrm{Ext}^0_{\mathbf{Z}}(E,\mathbf{Q}(1)).$$

Remarks. (i) In the case of motives over $\mathbf{Z}$ both sides of this conjectural identity should be additive in exact sequences, so the essential case of the conjecture is for a pure motive—in which case it it simply a restatement of the existing conjectures.

(ii) We can also write $\mathrm{Ext}^q_{\mathbf{Z}}(E,\mathbf{Q}(1)) = \mathrm{Ext}^q_{\mathbf{Z}}(\mathbf{Q}(0),E^\vee(1))$ to write the order of $L(E,s)$ in terms of the dual motive $E^\vee(1)$, in accordance with the principles of [**Del3**].

Now we can propose an algebraic criterion for a mixed motive to be critical.

Definition. The mixed motive E over $\mathbf{Z}$ is *highly critical* if

$$\mathrm{Ext}^q_{\mathbf{Z}}(E,\mathbf{Q}(1)) = \mathrm{Ext}^q_{\mathbf{Z}}(\mathbf{Q}(0),E) = 0 \quad \text{for } q=0,\,1.$$

Conjecture C. If E is highly critical, then it is critical.

IV. Motives with two weights.

It should be clear that, starting from an arbitrary motive M, one should be able to construct some kind of "universal extension" by sums of $\mathbf{Q}(0)$ and $\mathbf{Q}(1)$ to create a new motive E over $\mathbf{Z}$ which is highly critical, and whose L-function is of the form

$$L(E,s) = \zeta(s)^?\zeta(s+1)^? L(M,s).$$

Up to a nonzero rational, $L(E,0)$ will equal the leading coefficient of $L(M,s)$ at $s=0$, and conjecture A will then be applicable to E. We now describe the consequences of this, using the conjectural framework outlined above.

We first consider the case of a pure motive of weight $w \le -2$, which we denote N. Since the weight filtration is increasing and $\mathcal{M}$ is supposed to be semisimple we have

$$\mathrm{Ext}^1_{\mathbf{Q}}(N,\mathbf{Q}(1)) = \mathrm{Ext}^0_{\mathbf{Q}}(\mathbf{Q}(0),N) = 0.$$

First assume that $\mathrm{Hom}(N,\mathbf{Q}(1)) = 0$, and let $\rho = \dim \mathrm{Ext}^1_{\mathbf{Z}}(\mathbf{Q}(0),N)$. Then the universal extension

$$0 \longrightarrow N \longrightarrow N^\dagger \longrightarrow \mathrm{Ext}^1_{\mathbf{Z}}(\mathbf{Q}(0),N)\otimes\mathbf{Q}(0) \longrightarrow 0$$

will have $\mathrm{Ext}^q_{\mathbf{Z}}(N^\dagger,\mathbf{Q}(1)) = \mathrm{Ext}^q_{\mathbf{Z}}(\mathbf{Q}(0),N^\dagger) = 0$ for $q=0,\,1$. (We are using the expected vanishing of $\mathrm{Ext}^1_{\mathbf{Z}}(\mathbf{Q}(0),\mathbf{Q}(1))$.) Conjecture C therefore implies that $N^\dagger$ is critical. Let us examine its periods.

The period map $I_\infty^+(N)$ is easily seen to be injective (from consideration of the Hodge numbers) whereas $I_\infty^+(\mathbf{Q}(0))=0$. Therefore $N^\dagger$ will be critical if and only if the connecting homomorphism:

$$\mathrm{Ext}^1_{\mathbf{Z}}(\mathbf{Q}(0),N)\xrightarrow{\partial_N}\frac{N_{DR}\otimes\mathbf{R}}{F^0N_{DR}\otimes\mathbf{R}+N_B^+\otimes\mathbf{R}}=\operatorname{coker} I_\infty^+(N)$$

becomes an isomorphism when tensored with $\mathbf{R}$.

If $N=h^i(X)(n)$ and $n>1+\frac{i}{2}$ then the first group is $H^{i+1}_{\mathcal{M}}(X,\mathbf{Q}(n))_{\mathbf{Z}}$, the second group is $H^{i+1}_{\mathcal{D}}(X_{\mathbf{R}},\mathbf{R}(n))$ and the homomorphism ∂_N is the regulator map. The canonical $\mathbf{Q}$-structure on the target of ∂_N is the $\mathbf{Q}$-structure $\mathcal{D}_{i,n}$ (cf [**DenS**], (2.3.1)). Since $L(N^\dagger,0)=L(N,0)\zeta(0)^\rho$ is a non-zero rational multiple of $L(N,0)$, we see that conjectures A, B, C imply Deligne's reformulation ([**Del3**], [**DenS**] 3.1) of Beilinson's conjectures for $L(h^i(X),n)$.

Now consider a pure motive M of weight ≥ 0 with $\mathrm{Hom}(\mathbf{Q}(0),M)=0$. Dually to the above, there is a universal extension

$$0\longrightarrow\mathbf{Q}(1)^\rho\longrightarrow\widetilde{M}\longrightarrow M\longrightarrow 0$$

where $\rho=\dim\mathrm{Ext}^1_{\mathbf{Z}}(M,\mathbf{Q}(1))$. The period map for $\widetilde{M}$ gives a connecting homomorphism

$$\partial_M\colon\ker I_\infty^+(M)\longrightarrow\mathbf{R}^\rho.$$

If we write $N=M^\vee(1)$ then there is a canonical isomorphism

$$\operatorname{coker} I_\infty^+(N)\xrightarrow{\sim}\big(\ker I_\infty^+(M)\big)^\vee$$

(compare [**Del2**], §5.1) in terms of which ∂_M and ∂_N are adjoint. Observe that the highest exterior powers of both sides have a natural $\mathbf{Q}$-structure. However the isomorphism does not respect these $\mathbf{Q}$-structures. In the case $N=h^i(X)(n)$, $M=h^i(X)(i+1-n)$ the natural $\mathbf{Q}$-structure on the right hand side can be seen to be Beilinson's $\mathbf{Q}$-structure ($\mathcal{B}_{i,n}$ in the notations of [**DenS**] 2.3.1). Therefore $c^+(\widetilde{M})$ is Beilinson's regulator. Since $L(\widetilde{M},s)=\zeta(s+1)^\rho L(M,s)$, the conjectures imply that the leading coefficient of $L(M,s)$ at $s=0$ is a nonzero rational multiple of $c^+(\widetilde{M})$, and we recover the original formulation of Beilinson's conjectures ([**DenS**], 3.1.3).

If $\dim\mathrm{Hom}(\mathbf{Q}(0),M)=\sigma>0$, then we should replace M by the quotient $M/\mathbf{Q}(0)^\sigma$. Since $\mathrm{Ext}^q_{\mathbf{Z}}(\mathbf{Q}(0),\mathbf{Q}(1))=0$ for $q=1$, 2 we have

$$\mathrm{Ext}^1_{\mathbf{Z}}(M,\mathbf{Q}(1))\xrightarrow{\sim}\mathrm{Ext}^1_{\mathbf{Z}}(M/\mathbf{Q}(0)^\sigma,\mathbf{Q}(1))$$

and we take $\widetilde{M}$ to be the universal extension of $M/\mathbf{Q}(0)^\sigma$ by $\mathbf{Q}(1)$. This corresponds precisely to the "thickening" of the regulator which occurs in the Beilinson conjectures at the near-central point (ie., $m=i/2$ and $n=1+i/2$).

V. Motives with three weights.

We now consider possible extensions of a motive M which is pure of weight -1. Let $G = \mathrm{Ext}^1_{\mathbf{Z}}(M, \mathbf{Q}(1))$ and $G' = \mathrm{Ext}^1_{\mathbf{Z}}(\mathbf{Q}(0), M)$. Then we obtain two universal extensions:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & G^* \otimes \mathbf{Q}(1) & \longrightarrow & \widetilde{M} & \longrightarrow & M & \longrightarrow & 0 \\ 0 & \longrightarrow & M & \longrightarrow & M^\dagger & \longrightarrow & G' \otimes \mathbf{Q}(0) & \longrightarrow & 0 \end{array}$$

where $G^* = \mathrm{Hom}(G, \mathbf{Q})$.

Assuming $\mathrm{Ext}^1_{\mathbf{Z}}(\mathbf{Q}(0), \mathbf{Q}(1)) = \mathrm{Ext}^2_{\mathbf{Z}}(\mathbf{Q}(0), \mathbf{Q}(1)) = 0$, there is a unique mixed motive E over $\mathbf{Z}$ with

$$\mathrm{Gr}^W_0 E = G' \otimes \mathbf{Q}(0), \quad \mathrm{Gr}^W_{-1} E = M, \quad \mathrm{Gr}^W_{-2} E = G^* \otimes \mathbf{Q}(1)$$

and $\mathrm{Gr}^W_j E = 0$ for $j > 0$ or $j < -2$, such that

$$W_{-1}E = \widetilde{M}, \quad E/W_{-2}E = M^\dagger.$$

Setting $\rho = \dim_{\mathbf{Q}} G$, $\rho' = \dim_{\mathbf{Q}} G'$, we find that E is highly critical and that

$$L(E, s) = L(M, s) \cdot \zeta(s+1)^\rho \cdot \zeta(s)^{\rho'}.$$

Now M is itself critical. Examining the period mapping for E shows that conjecture C holds for E if and only if a certain connecting homomorphism

$$\Omega_M : G' \otimes \mathbf{R} \longrightarrow G^*$$

is an isomorphism, and $c^+(E)$ is then $c^+(M) \cdot \det \Omega_M$.

Now consider the particular case $M = h^{2n-1}(X)(n)$, $n \geq 1$. In this case we should have $G' = CH^n(X)^0 \otimes \mathbf{Q}$, and $G = CH_{n-1}(X)^0 \otimes \mathbf{Q}$. As a first attempt to construct E, choose $Y \subset X$ of codimension n and $Z \subset X$ of dimension $n-1$ such that $Y \cap Z = \emptyset$ and every cycle of codimension n (respectively dimension $n-1$) is rationally equivalent to a cycle supported in Y (resp. Z). Then in any of the various cohomology theories, $E' = H^{2n-1}(X - Y \mathrm{rel} Z)(n)$ has three nonzero steps in its weight filtration:

$$\begin{aligned} &\mathrm{Gr}^W_0 E' = H^{2n}_Y(X)^0(n) \stackrel{\mathrm{def}}{=} \ker\{H^{2n}_Y(X)(n) \longrightarrow H^{2n}(X)(n)\}; \\ &\mathrm{Gr}^W_{-1} E' = H^{2n-1}(X)(n); \\ &\mathrm{Gr}^W_{-2} E' = H^{2n-2}(Z)_0(n) \stackrel{\mathrm{def}}{=} \mathrm{coker}\{H^{2n-2}(X)(n) \longrightarrow H^{2n-2}(Z)(n)\}. \end{aligned}$$

By choosing suitable maps $G' \to H^{2n}_Y(X)^0(n)$ and $G \otimes H^{2n-2}(Z)_0(n) \to \mathbf{Q}(1)$, and taking the associated pullback and pushout, we obtain an object

E'' with the correct graded pieces. The homorphism $\Omega_{E''}$ is essentially the infinite component of the height pairing between cycles supported in Y and in Z.

However E'' need not be a mixed motive over $\mathbf{Z}$. The obstruction is a certain element of $\mathrm{Ext}^1_{\mathbf{Q}}(\mathbf{Q}(0)^{\rho'}, \mathbf{Q}(1)^{\rho})$. If X satisfies some reasonable hypotheses (essentially those required to define tha global height pairing) this obstruction can be explicitly constructed in terms of local heights, and can be realised as a suitable sum of 1-motives $[\mathbf{Z} \overset{1 \mapsto p}{\longrightarrow} \mathbf{G}_m]$, for various primes p (as described at the end of §II). Therefore in order to obtain E itself we must twist by the inverse of this obstruction. This will change the period mapping by appropriate multiples of $\log p$, and it turns out that Ω_E is the homomorphism attached to the complete height pairing (including the contributions from the finite primes). Consequently:-

- Firstly, E is critical if and only if the global height pairing of Beilinson-Bloch-Gillet-Soulé ([**Be3**] [**Bl2**] [**GS**])

$$< \cdot , \cdot > : G \otimes G' \longrightarrow \mathbf{R}$$

  is non-singular;

- Secondly, if E is indeed critical, then

$$c^+(E) = c^+(h^{2n-1}(X)(n)) \cdot \det < \ \ > .$$

In other words, conjectures A, B and C imply the Beilinson-Bloch generalisation of the Birch-Swinnerton-Dyer conjectures.

It is possible to rewrite the construction of the extensions of this and the previous section in a unified way in a (hypothetical) derived category of motives, but we shall not attempt to describe this here.

Finally we remark here that one can consider the p-adic periods of mixed motives in the same way as above. The essential point (which was shown to me by U. Jannsen) is that if the l-adic realisations of a motive E (pure or mixed) are unramified at p, then the p-adic representation E_p of $\mathrm{Gal}(\overline{\mathbf{Q}}_p/\mathbf{Q}_p)$ should be crystalline. In this case there is then a p-adic period map

$$I_p^+ : E_B^+ \otimes B_{\mathrm{cris}} \longrightarrow \frac{E_{DR}}{F^0} \otimes B_{\mathrm{cris}}$$

which should be an isomorphism if and only if E is critical. Now if we start with a pure motive M whose l-adic realisations are unramified at p, the same will be true of the various universal extensions constructed here. By considering the associated period mappings, one then obtains B_{cris}-valued regulators and height pairings. (These have been directly constructed by P. Schneider.)

VI. Some compatibilities.

In this section we show that the only consequences of conjecture (A) are those described above, and that the conjecture is compatible with the functional equation. In order to give the statements of the theorems below some actual meaning, we will take for $\mathcal{MM}_{\mathbf{Q}}$ the category of mixed motives defined by absolute Hodge cycles, as considered by Deligne and Jannsen [**J**]. We will restrict our attention to motives whose l-adic realisations are independent of l, so as to be able to discuss L-functions, and we shall assume the analytic continuation and functional equation for any L-functions that may arise. Of the remaining desirable properties this category might enjoy, we assume (only) the following:

Hypotheses.

(a) $\mathrm{Ext}^1_{\mathbf{Q}}(\mathbf{Q}(0),\mathbf{Q}(1))$ is generated by the classes of 1-motives of type (iv) below.

(b) If M is pure of weight -1, then $\mathrm{ord}_{s=0} L(M,s) \geq \dim \mathrm{Ext}^1_{\mathbf{Z}}(M,\mathbf{Q}(1)) = \dim \mathrm{Ext}^1_{\mathbf{Q}}(M,\mathbf{Q}(1))$.

(c) If M is pure of weight ≤ -2 then the realisation map $\mathrm{Ext}^1_{\mathbf{Z}}(\mathbf{Q}(0),M)\otimes \mathbf{R} \to \mathrm{Ext}^1_{\mathbf{R}-\mathrm{Hdg}}(\mathbf{R}(0),M_{\mathbf{R}})$ is injective.

The first hypothesis implies that $\mathrm{Ext}^1_{\mathbf{Q}}(\mathbf{Q}(0),\mathbf{Q}(1)) = \mathbf{Q}^{(\mathcal{S})}$ where $\mathcal{S}$ denotes the set of rational primes. Hypothesis (b) includes a weak form of conjecture (B) for M. (See also III above.) Finally, (c) is equivalent to the statement that $I^+_\infty(M^\dagger)$ is injective, hence is a weak form of conjecture (C). (The reader who is sceptical of the general finiteness of groups such as Ш will be reassured that we do not require equality in (b).)

Theorem 1. Assume the truth of hypotheses (a)–(c) above. Let E be a critical mixed motive over $\mathbf{Q}$, with $L(E,0)\neq 0$. Then there is a filtration $K_\bullet$ of E with the following properties:

- The graded pieces $E_i = \mathrm{Gr}^K_i E$ are critical motives with $L(E_i,0)\in\mathbf{R}^*$;
- $\prod L(E_i,0)\cdot L(E,0)^{-1}\in\mathbf{Q}^*$;
- Each E_i is one of the following:
  - (i) An extension of a pure motive M of weight ≥ 0 by a sum of copies of $\mathbf{Q}(1)$, with $\mathrm{Hom}(\mathbf{Q}(0),M)=0$;
  - (ii) An extension of a sum of copies of $\mathbf{Q}(0)$ by a pure motive M of weight ≤ -2 with $\mathrm{Hom}(\mathbf{Q}(1),M)=0$;
  - (iii) A motive whose nonzero graded pieces in the weight filtration are a sum of copies of $\mathbf{Q}(1)$, a pure motive of weight -1, and a sum of copies of $\mathbf{Q}(0)$;

(iv) The 1-motive $[\mathbf{Z}\xrightarrow{\phi}\mathbf{G}_m]$, $\phi(1)=p$.

- In cases (i)–(iii) E_i is a motive over $\mathbf{Z}$.

If x, $y\in\mathbf{R}$, we write $x\sim y$ if $x=ay$ for some $a\in\mathbf{Q}^*$.

Theorem 2. The hypotheses being as in Theorem 1, assume the truth of conjecture 6.6 of [**Del2**] (that every motive of rank 1 and weight 0 is an Artin motive). Suppose that $L(E^\vee(1),0)$ is also nonzero. Then $L(E,0)\cdot c^+(E)^{-1}\sim L(E^\vee(1),0)\cdot c^+(E^\vee(1))^{-1}$.

Before giving the proofs, we make some general observations. We say that a motive is supercritical (subcritical) if the period mapping is surjective (injective). Suppose $A\to B\to C$ is a short exact sequence of motives. Then by applying the snake lemma to the ladder

$$\begin{array}{ccccc} A_B^+\otimes\mathbf{R} & \longrightarrow & B_B^+\otimes\mathbf{R} & \longrightarrow & C_B^+\otimes\mathbf{R} \\ \downarrow & & \downarrow & & \downarrow \\ F^0\backslash A_{DR}\otimes\mathbf{R} & \longrightarrow & F^0\backslash B_{DR}\otimes\mathbf{R} & \longrightarrow & F^0\backslash B_{DR}\otimes\mathbf{R} \end{array}$$

we see that if B is supercritical, then C is supercritical and $\dim\ker I_\infty^+(C)\geq\dim\operatorname{coker} I_\infty^+(A)$; there is a dual statement if B is subcritical.

We recall the conjectural analytic continuation and functional equation of a (pure) motive. If A is pure of weight w and contains no direct factors of $\mathbf{Q}(0)$ or $\mathbf{Q}(1)$ then the conjectures imply:

- If $w\geq 0$ then A is supercritical and $\operatorname{ord}_{s=0}L(A,s)=\dim\ker I_\infty^+(A)$;
- If $w\leq -2$ then A is subcritical and $\operatorname{ord}_{s=0}L(A,s)=0$.

Now suppose that A is mixed of weight ≥ 0, and $\operatorname{Hom}(\mathbf{Q}(0),A)=0$. Then as $\prod L(\operatorname{Gr}_q^W A,s)$ is the product of $L(A,s)$ with a finite number of Euler factors, we have $\operatorname{ord}_{s=0}L(A,s)\geq\dim\ker I_\infty^+(A)$; likewise if A is mixed of weight ≤ -2 and $\operatorname{Hom}(A,\mathbf{Q}(-1))-0$ then $\operatorname{ord}_{s=0}L(A,s)\geq 0$. In each case equality holds if and only if the multiplicities of the eigenvalue 1 of Frob_p on $\operatorname{Gr}_\bullet^W(A_l)^{\mathcal{I}_p}$ and on $A_l^{\mathcal{I}_p}$ are equal.

Construction of the filtration. We first introduce a convenient notation. By the symbol

$$\left\{\begin{array}{l}\underline{A_k}\\ \vdots\\ \overline{A_1}\end{array}\right.$$

we mean any mixed motive with an increasing filtration (not necessarily the weight filtration!) whose i^{th} graded constituent is isomorphic to A_i.

By virtue of the weight filtration, and the fact that pure motives are semisimple, we can write:

$$E = \begin{cases} \overline{\quad A \quad} \\ N = \begin{cases} \mathbf{Q}(0)^d \\ \hline M \\ \hline \mathbf{Q}(1)^e \end{cases} \\ \hline \quad B \end{cases}$$

where M is pure of weight -1 (and therefore critical), A is mixed of weight ≥ 0 with $\mathrm{Hom}(\mathbf{Q}(0), A) = 0$, and B is mixed of weight ≤ -2 with $\mathrm{Hom}(B, \mathbf{Q}(1)) = 0$. Since E is critical, A and B are respectively supercritical and subcritical; let $f = \dim\ker I^+_\infty(A)$, $g = \dim\operatorname{coker} I^+_\infty(B)$. We have $f + d = e + g$.

By removing direct factors we can also write

$$N = \mathbf{Q}(0)^{d'} \oplus \mathbf{Q}(1)^{e'} \oplus M' \oplus C$$

with

$$M' = \begin{cases} X = \begin{cases} \mathbf{Q}(0)^{q'} \\ \hline M \end{cases} \\ \hline \mathbf{Q}(1)^{q''} \end{cases} \Bigg\} = Y \quad , \qquad C = \begin{cases} \mathbf{Q}(0)^m \\ \hline \mathbf{Q}(1)^n \end{cases}$$

and $\mathrm{Hom}(\mathbf{Q}(0), X) = \mathrm{Hom}(Y, \mathbf{Q}(1)) = \mathrm{Hom}(\mathbf{Q}(0), C) = \mathrm{Hom}(C, \mathbf{Q}(1)) = 0$. By hypothesis *(a)*, the extension C is classified by some

$$\phi = (\phi_p) \in \mathrm{Hom}(\mathbf{Q}^m, \mathbf{Q}^n)^{(\mathcal{S})} = \mathrm{Ext}^1_{\mathbf{Q}}(\mathbf{Q}(0)^m, \mathbf{Q}(1)^n).$$

Kummer theory gives an isomorphism

$$\mathrm{Ext}^1_{\mathcal{I}_p}(\mathbf{Q}_l(0)^m, \mathbf{Q}_l(1)^n) \xrightarrow{\sim} \mathrm{Hom}(\mathbf{Q}_l^m, \mathbf{Q}_l^n);$$

under this isomorphism, the representation C_l of the inertia group at p is classified by the homomorphism $\phi_p : \mathbf{Q}_l^m \to \mathbf{Q}_l^n$.

Proposition. Let R_p denote the rank of ϕ_p. Then:

(i) $\mathrm{ord}_{s=0} L(C, s) = \sum_{p \in \mathcal{S}} R_p - n$;

(ii) $\sum_{p \in \mathcal{S}} R_p \geq \max(m, n)$;

(iii) If C is critical and $L(C, 0) \neq 0$ then $C = \bigoplus_p [\mathbf{Z} \xrightarrow{1 \mapsto p} \mathbf{G}_m]^{R_p}$.

Proof. (i) Consider the short exact sequence

$$0 \longrightarrow \mathbf{Q}_l(1)^n \longrightarrow C_l \longrightarrow \mathbf{Q}_l(0)^m \longrightarrow 0$$

of $\mathcal{I}_p$-modules. R_p is the dimension of the image of the boundary homomorphism

$$\mathbf{Q}_l(0) \longrightarrow \mathrm{Ext}^1_{\mathcal{I}_p}(\mathbf{Q}_l(0)^m, \mathbf{Q}_l(1)^m)$$

and therefore equals the codimension of the image of $(C_l)^{\mathcal{I}_p}$ in $\mathbf{Q}_l(0)^m$. Therefore the Euler factor of $L(C,s)$ at p is

$$\left(1-p^{-1-s}\right)^{-n}\left(1-p^{-s}\right)^{R_p-n}$$

whence

$$L(C,s) = \zeta(s)^m \zeta(s+1)^n \prod_p \left(1-p^{-s}\right)^{R_p}$$

and (i) follows.

(ii) Write $U = \mathbf{Q}^m$, $V = \mathbf{Q}^n$. Since $\mathrm{Hom}(\mathbf{Q}(0), C) = \mathrm{Hom}(C, \mathbf{Q}(1)) = 0$, the homomorphisms

$$\phi' : U \xrightarrow{x \mapsto (\phi_p(x))} V^{(\mathcal{S})} \quad \text{and} \quad \phi'' : V^* \xrightarrow{x \mapsto ({}^t\phi_p(x))} (U^*)^{(\mathcal{S})}$$

attached to the classifying map ϕ are injective. Since the images of these are contained in the subspaces

$$\bigoplus_p \phi_p(U) \subset V^{(\mathcal{S})}, \quad \bigoplus_p {}^t\phi_p(V^*) \subset (U^*)^{(\mathcal{S})}$$

we have the inequality (ii).

(iii) The period mapping for C is

$$\sum_{p \in \mathcal{S}} \log p \cdot \phi_p : U \otimes \mathbf{R} \longrightarrow V \otimes \mathbf{R}.$$

If C is critical and $L(C,0)$ is nonzero, $m = n = \sum R_p$. In this case the image of ϕ' is precisely $\bigoplus_p \phi_p(U)$, hence we can write $U = \bigoplus_p U_p$ in such a way that ϕ_p factors through the projection:

$$\phi_p : U = \bigoplus U_p \longrightarrow\!\!\!\!\to U_p \hookrightarrow V.$$

For $\sum \log p \cdot \phi_p$ to be an isomorphism we must therefore have $V = \bigoplus_p \phi_p(U_p)$, which implies (iii). ■

Now let $q = \mathrm{ord}_{s=0} L(M,s)$. By hypothesis (b) for M and the properties of M' we have q', $q'' \leq q$. Also the conjectural analytic continuation and functional equation give

$$\mathrm{ord}_{s=0} L(A,s) \geq f \quad \text{and} \quad \mathrm{ord}_{s=0} L(B,s) \geq 0.$$

Because of the decomposition of N we can rearrange the filtration on E as:

$$E = \begin{cases} A' = \left\{ \frac{A}{\mathbf{Q}(1)^{e'}} \right. \\ \hline M' \oplus C \\ \hline B' = \left\{ \frac{\mathbf{Q}(0)^{d'}}{B} \right. \end{cases}.$$

Then A' is supercritical and B' is subcritical. Therefore $f \geq e'$ and $g \geq d'$, with equality if and only if A' and B' are critical, respectively. This gives

$$\begin{aligned} 0 &\geq \mathrm{ord}_{s=0} L(E,s) \\ &\geq \mathrm{ord}_{s=0} L(A',s) + \mathrm{ord}_{s=0} L(M',s) + \mathrm{ord}_{s=0} L(C,s) + \mathrm{ord}_{s=0} L(B',s) \\ &\geq (f-e') + (q-q'') + (\textstyle\sum R_p - n) + 0 \\ &\geq 0. \end{aligned}$$

Therefore $L(E,0) \neq \infty$, and $L(E,0)$ is nonzero if and only if $f = e'$, $q = q''$ and $\sum R_p = n$. Then

$$\sum_{p \in \mathcal{S}} R_p = n = e - f - g = d - g - q \leq d - d' - q' = m.$$

By part (ii) of the proposition, this implies also $g = d'$, $q = q'$, $m = n$. Therefore A' and B' are critical. Hence so are M' and C. By part (iii) of the proposition, C is a sum of motives of type (iv). Moreover we have $\mathrm{ord}_{s=0} L(M',s) = 0$. Now by hypothesis (b)

$$\mathrm{ord}_{s=0} L(M',s) \geq \mathrm{ord}_{s=0} L(Y,s) \geq \mathrm{ord}_{s=0} L(M,s) + q \geq 0$$

and the first inequality is an equality only if the extension of $\mathcal{I}_p$-modules $Y_l \to M_l \to \mathbf{Q}_l(0)^q$ splits. Also the hypothetical equality of $\mathrm{Ext}^1_{\mathbf{Z}}(M, \mathbf{Q}(1))$ and $\mathrm{Ext}^1_{\mathbf{Q}}(M, \mathbf{Q}(1))$ shows that the extension $\mathbf{Q}_l(1)^q \to Y_l \to M_l$ splits over $\mathcal{I}_p$, hence M' is a motive over $\mathbf{Z}$ of the type (iii).

It finally remains to analyze the motives A' and B'. Consider first B'. By the above we have $\mathrm{ord}_{s=0} L(B',s) = 0$. Moreover as $\mathrm{Hom}(B', \mathbf{Q}(1)) = 0$ we have $\mathrm{ord}_{s=0} L(B,s) \geq 0$. Consider the extension $B_l \to B'_l \to \mathbf{Q}_l(0)^{d'_1}$ of modules under the inertia group at $p \neq l$. Since the local Euler factors for $\mathbf{Q}(0)$ each have a pole at $s = 0$, this extension must split (otherwise $L(B',0)$ would vanish). Now let w be the highest weight of B, and write $B_1 = \mathrm{Gr}^W_w(B)$, $Z = W_{w-1}(B)$. Then

$$B = \left\{ \frac{B_1}{Z} \right.$$

and we can then rewrite B' as:

$$B' = \left\{ \frac{\mathbf{Q}(0)^{d'}}{\dfrac{B_1}{Z}} \right. = \left\{ \frac{\mathbf{Q}(0)^{d'-d'_1} \oplus \left\{ \dfrac{\mathbf{Q}(0)^{d'_1}}{B_1} \right.}{Z} \right. = \left\{ \frac{B'_1 = \left\{ \dfrac{\mathbf{Q}(0)^{d'_1}}{B_1} \right.}{Z' = \left\{ \dfrac{\mathbf{Q}(0)^{d'-d'_1}}{Z} \right.} \right.$$

Continuing in this way we obtain a filtration:

$$B' = \left\{ \begin{array}{c} B'_1 \\ \hline \vdots \\ \hline B'_r \end{array} \right. , \quad B'_i = \left\{ \frac{\mathbf{Q}(0)^{d'_i}}{B_i} \right.$$

where each B_i is pure, $\mathrm{Hom}(\mathbf{Q}(0), B'_i) = 0$, and $\sum d'_i = d'$. By the above, each B'_i is a motive over $\mathbf{Z}$, so is a submotive of the universal extension $B^\dagger_i$ of section IV. The hypothesis (c) implies that $B^\dagger_i$ is subcritical. Therefore each B'_i is subcritical, hence critical (since B' itself is critical).

Now consider A'. In the same way we can write

$$A' = \left\{ \begin{array}{c} A'_1 \\ \hline \vdots \\ \hline A'_t \end{array} \right. , \quad A'_i = \left\{ \frac{A_i}{\mathbf{Q}(0)^{e'_i}} \right.$$

with A_i pure, $\mathrm{Hom}(A'_i, \mathbf{Q}(1)) = 0$, and $\sum e'_i = e'$. Now

$$\mathrm{ord}_{s=0} L(A_i, s) = f_i = \dim \ker I^+_\infty(A'_i)$$

Therefore $\mathrm{ord}_{s=0} L(A'_i, s) \geq f_i - e'_i$, with equality if and only if the $(\mathrm{Frob}_p = 1)$-eigenspace of $(A'_{i,l})^{\mathcal{I}_p}$ maps onto that of $(A_{i,l})^{\mathcal{I}_p}$. In view of the exact sequence

$$(A_{i,l})^{\mathcal{I}_p} \longrightarrow (A'_{i,l})^{\mathcal{I}_p} \longrightarrow H^1(\mathcal{I}_p, \mathbf{Q}_l(1)^{e'_i}) = \mathbf{Q}_l(0)^{e'_i}$$

this holds if and only if A'_i is a motive over $\mathbf{Z}$.

Now

$$0 = \mathrm{ord}_{s=0} L(A', s) \geq \sum \mathrm{ord}_{s=0} L(A'_i, s) \geq \sum (f_i - e'_i) = f - e' = 0,$$

so we must have equality at each stage. Therefore each A'_i is a motive over $\mathbf{Z}$, and therefore is a quotient of the universal extension $\widetilde{A}_i$. So by hypothesis (c) each A'_i is supercritical, and therefore also critical. Also $e'_i = f_i$, whence $L(A'_i, 0) \in \mathbf{R}^*$. This concludes the proof of Theorem 1. ■

Proof of Theorem 2. Recall the definition of the constant $\delta(M)$ ([**Del2**], 1.7.3); we extend this definition to mixed motives as well. The proof of Proposition 5.1 of [**Del2**] then gives:

Proposition. Let E be a critical (mixed) motive. Then $E^\vee(1)$ is also critical, and

$$c^+(E)\cdot c^+(E^\vee(1))^{-1} \sim (2\pi i)^{-d^-(E)}\cdot\delta(E)$$

(where $d^-(E)=\dim E_B^-$).

Now suppose that E satisfies the hypotheses of the theorem. Write $L^*(-)$ for the leading coefficient in the Laurent expansion of $L(-,s)$. By the proof of Theorem 1, we have

$$L(E,0)=L^*(E)=\prod_q L^*(\mathrm{Gr}_q^W E)\cdot\prod_p(\log p)^{R_p}$$

and likewise

$$L(E^\vee(1),0)=L^*(E^\vee(1))=\prod_q L^*(\mathrm{Gr}_q^W E^\vee(1))\cdot\prod_p(\log p)^{R_p}.$$

It therefore suffices to prove that for any pure motive M of weight w

$$L^*(M)\cdot L^*(M^\vee(1))^{-1}\sim(2\pi i)^{-d^-(M)}\cdot\delta(M)$$

which by under our assumptions (as in 5.6 of [**Del2**]) is equivalent to

$$L_\infty^*(M)\cdot L_\infty^*(M^\vee(1))^{-1}\sim(2\pi)^{d^-(M)+\frac{1}{2}wd(M)}.$$

This in turn is a trivial extension of Proposition 5.4 of [**Del2**]. ∎

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