Plectic Hodge theory I

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Introduction

This is the first of a series of preprints providing detailed statements and proofs of the results announced in sections 16–17 of [9]. Here we state and prove the structure theorem for real plectic mixed Hodge structures, describe the extension groups, and show that the singular cohomology of a Hilbert modular variety carries a canonical plectic mixed Hodge structure.

Subsequent papers will deal with plectic Deligne cohomology of Shimura varieties, the plectic polylogarithm sheaf and arithmetic applications.

1 I-filtrations

1.1 I-prefiltrations

(1.1.1) Let I be a locally finite lattice; i.e., I is a poset, and for every i, $j \in I$:

(i) the inf and sup $i \wedge j$ and $i \vee j$ exist;

(ii) $\{k \in I \mid i < k < j\}$ is finite.

We view I as a category, with one morphism from i to j whenever $i \leq j$.

Let $\mathcal{C}$ be an abelian category, $X \in \operatorname{ob} \mathcal{C}$. Denote by $\mathbf{sub}\, X$ the category of subobjects of X. We assume $\mathbf{sub}\, X$ has arbitrary colimits.

(1.1.2) Definition. An (increasing) *I-prefiltration* on X is a functor

$$F\colon I \to \mathbf{sub}\, X.$$

We usually write $F_i X$ instead of $F(i)$. A decreasing I-prefiltration is an I^{op}-prefiltration (usually written $F^i X$).

The I-prefiltered objects (X, F) of $\mathcal{C}$ form a category $\mathbf{preFil}_I(\mathcal{C})$. Say that $f\colon (X, F) \to (X', F')$ is *strict* if

$$\forall i \in I, \quad f(F_i X) = f(X) \cap F'_i X'.$$

Say that the sequence in $\mathbf{preFil}_I(\mathcal{C})$

$$0 \to (X', F') \xrightarrow{\ f\ } (X, F) \xrightarrow{\ g\ } (X'', F'') \to 0$$

is *exact* if

(i) the underlying sequence $0 \to X' \xrightarrow{f} X \xrightarrow{g} X'' \to 0$ is an exact sequence in $\mathcal{C}$, and

(ii) f and g are strict.

Condition (ii) is equivalent to the exactness of the sequence

$$0 \to F_i X' \to F_i X \to F_i X'' \to 0.$$

for every $i \in I$. This definition makes $\mathbf{preFil}_I(\mathcal{C})$ into an exact category.

(1.1.3) Let $I_1, \dots, I_r$ be locally finite lattices, and $I = \prod_{\alpha=1}^r I_\alpha$ with the product ordering: $\underline{i} \le \underline{j} \iff (\forall \alpha)\ i_\alpha \le j_\alpha$. Suppose that $({}_\alpha F)_\alpha$ are I_α-prefiltrations on X. We define an I-prefiltration $F = \bigcap_\alpha {}_\alpha F$ on X by

$$F_{\underline{i}} X = \bigcap_\alpha {}_\alpha F_{i_\alpha} X.$$

(1.1.4) Pedantic note: we are considering filtrations as functors whose values are subobjects, rather than isomorphism classes of subobjects. Therefore certain constructions which follow, involving sums or intersections of subobjects, will only be well-defined up to unique isomorphism.

1.2 I-filtrations

(1.2.1) Definition. (i) A *weak I-filtration* on X is an I-prefiltration F which commutes with finite limits (i.e. $\forall i, j \in I$, $F_{i \wedge j} X = F_i X \cap F_j X$).

(ii) An *I-filtration* on X is an I-prefiltration F such that:

for every finite nonempty subset $J = \{j_0, \dots, j_n\} \subset I$, the sequence

$$\cdots \to \bigoplus_{\alpha<\beta<\gamma} F_{j_\alpha \wedge j_\beta \wedge j_\gamma} X \to \bigoplus_{\alpha<\beta} F_{j_\alpha \wedge j_\beta} X \to \bigoplus_\alpha F_{j_\alpha} X \to \sum_\alpha F_{j_\alpha} X \to 0 \tag{1.2.1.1}$$

is exact.

The maps in (1.2.1.1) are the alternating sums of the inclusions. (Observe that condition (ii) implies that F commutes with finite limits.)

(1.2.2) For $J = (j_0, \dots, j_n)$ a finite family in I, let $K_\bullet(J) = K_\bullet(X, F, J)$ denote the (homological) complex (1.2.1.1), with $\sum_\alpha F_{j_\alpha} X$ in degree -1. Suppose that $j_0 = j_1$ and $J^* = (j_1, \dots, j_n)$. Then $K_\bullet(J^*) \subset K_\bullet(J)$ and the quotient $K_\bullet(J)/K_\bullet(J^*)$ is easily seen to be acyclic. Hence, if (1.2.1.1) is exact for every finite subset of I, it is exact for every finite family in I. Standard simplicial arguments show also that the condition is independent of the ordering of J.

Every filtration in the usual sense is automatically a $\mathbb{Z}$-filtration:

(1.2.3) Proposition. *Suppose I is totally ordered. Then any I-prefiltration is an I-filtration.*

Proof. It suffices to check that for any finite filtration (in the usual sense)

$$0 = F_{-1}X \subset F_0 X \subset \cdots \subset F_n X = X$$

the sequence

$$\cdots \to \bigoplus_{0 \le i < j < k \le n} F_i X \to \bigoplus_{0 \le i < j \le n} F_i X \to \bigoplus_{0 \le i \le n} F_i X \to X \to 0$$

is exact. Filtering this complex termwise by F, and applying the spectral sequence of a filtered complex, it is enough to prove it when $0 = F_{m-1} X \subset F_m X = X$ for some m. In this case the complex becomes

$$\cdots \to \bigoplus_{m \le i < j < k \le n} X \to \bigoplus_{m \le i < j \le n} X \to \bigoplus_{m \le i \le n} X \to X \to 0$$

which is acyclic (it is the chain complex with coefficients in X for the standard simplex). □

(1.2.4) Let $I = I_1 \times \cdots \times I_r$ as in (1.1.3). Suppose that $({}_\alpha F)_\alpha$ are weak I_α-filtrations on X. Then by definition $F = \bigcap_\alpha {}_\alpha F$ commutes with finite limits, hence is a weak I-filtration. Conversely:

(1.2.5) Proposition. *Let F be a weak I-filtration on X, where $I = \prod_{\alpha=1}^r I_\alpha$. Then there exist unique weak I_α-filtrations ${}_\alpha F$ such that $F = \bigcap_\alpha {}_\alpha F$.*

Proof. Simply define

$${}_\alpha F_j = \sum_{\substack{\underline{i} = (i_1, \dots, i_r) \in I \\ i_\alpha = j}} F_{\underline{i}}. \qquad \square$$

(1.2.6) In particular, every weak $\mathbb{Z}^r$-filtration is the intersection of r $\mathbb{Z}$-filtrations.

Weak $\mathbb{Z}^2$-filtrations were considered in [2, X,§2], where they were called *bifiltrations*. (What is called there an M-filtration coincides with our notion of weak I-filtration.) Moreover one has:

(1.2.7) Proposition. *Any weak $\mathbb{Z}^2$-filtration is a $\mathbb{Z}^2$-filtration.*

Proof. By the above, $F = G \cap H$ for $\mathbb{Z}$-filtrations G, H. Replacing X by the subobject $\varinjlim_i F_i X$ we may assume that $X = \varinjlim_{m \in \mathbb{Z}} G_m X$. Let $J = \{j_\alpha = (m_\alpha, n_\alpha) \mid 0 \le \alpha \le r\} \subset \mathbb{Z}^2$ be a finite subset, with $m_0 \le m_1 \cdots \le m_r$. Consider the modification of the complex (1.2.1.1):

$$\tilde{K}(X, F, J) = \Big[\cdots \to \bigoplus_{\alpha<\beta<\gamma} F_{j_\alpha \wedge j_\beta \wedge j_\gamma} X \to \bigoplus_{\alpha<\beta} F_{j_\alpha \wedge j_\beta} X \to \bigoplus_{\alpha} F_{j_\alpha} X \to X \Big]$$

with X in homological degree -1. For F to be an I-filtration is equivalent to this complex being acyclic in degree ≥ 0. Let $\tilde{G}$ be the truncated filtration on X:

$$\tilde{G}_p X = \begin{cases} X & (p > m_r) \\ G_p X & (m_0 \le p \le m_r) \\ 0 & (p < m_0) \end{cases}$$

which induces a (finite) filtration on $F_j X$ and therefore on the complex $\tilde{K}(X, F, J)$. By the spectral sequence for a filtered complex, it is enough to show that for each p, the associated graded $\mathrm{gr}_p^{\tilde{G}} \tilde{K}(X, F, J)$ is acyclic in degree ≥ 0. For this, compute:

$$\mathrm{gr}_p^G F_{m,n} X = \begin{cases} 0 & (m < p) \\ \mathrm{gr}_p^G H_n X & (m \ge p) \end{cases}$$

and if $\alpha < \beta$ then $F_{j_\alpha \wedge j_b} X = G_{m_\alpha} X \cap H_{n_\alpha \wedge n_\beta} X$ and so $\mathrm{gr}_p^{\tilde{G}} \tilde{K}(X, F, J)$ equals, for $m_0 < p \le m_r$,

$$\cdots \to \bigoplus_{\omega \le \alpha<\beta<\gamma} \mathrm{gr}_p^G H_{n_\alpha \wedge n_\beta \wedge n_\gamma} X \to \bigoplus_{\omega \le \alpha<\beta} \mathrm{gr}_p^G H_{n_\alpha \wedge n_\beta} X \to \bigoplus_{\omega \le \alpha} \mathrm{gr}_p^G H_{n_\alpha} X \to X \tag{1.2.7.1}$$

where ω is the least α such that $m_\alpha \ge p$. For $p = m_0$ it equals

$$\cdots \to \bigoplus_{\alpha<\beta<\gamma} G_{m_0} H_{n_\alpha \wedge n_\beta \wedge n_\gamma} X \to \bigoplus_{\alpha<\beta} G_{m_0} H_{n_\alpha \wedge n_\beta} X \to \bigoplus_{\omega \le \alpha} G_{m_0} H_{n_\alpha} X \to G_{m_0} X \tag{1.2.7.2}$$

and for $p \notin [m_0, m_r]$ it is zero.

Let J' be the family $(n_\alpha)_{1 \le \alpha \le r}$. Then (1.2.7.2) is just the complex $\tilde{K}(G_{m_0} X, H, J')$ which is acyclic in degree ≥ 0 by (1.2.3). Moreover $\mathrm{gr}_p^G H_n X = \bar{H}_n \mathrm{gr}_p^G X$ where $\bar{H}$ is the induced filtration on $\mathrm{gr}_p^G X$. (We are using [4, (1.1.9)], which states that the two different ways of inducing a filtration on subquotients give the same answer.) So (1.2.7.1) is the complex $\tilde{K}(\mathrm{gr}_p^G X, \bar{H}, J'_{\ge \omega})$ where $J'_{\ge \omega} = (n_\alpha)_{\omega \le \alpha \le r}$, hence by (1.2.3) it is acyclic in degree ≥ 0. $\square$

(1.2.8) If $r > 2$, it is not the case that every weak $\mathbb{Z}^r$-filtration is a $\mathbb{Z}^r$-filtration. (The proof above, taking now H to be a weak $\mathbb{Z}^{r-1}$-filtration, breaks down at the last step.) The simplest counterexample is: take for X a 2-dimensional vector space $Kx \oplus Ky$, and put on it the weak $\mathbb{Z}^3$-filtration which is the intersection of the $\mathbb{Z}$-filtrations F, G, H with $F_0X = G_0X = H_0X = 0$, $F_2X = G_2X = H_2X = X$ and $F_1X = Kx$, $G_1X = Ky$, $H_1X = K(x + y)$.

(1.2.9) Proposition/Definition. *The exact category* $\mathbf{Fil}_I(\mathcal{C})$ *of* I*-filtered objects in* $\mathcal{C}$ *is closed under finite sums; it is closed under arbitrary sums if they exist in* $\mathcal{C}$*.*

(1.2.10) Proposition. *Let*

$$0 \to (X', F') \to (X, F) \to (X'', F'') \to 0 \tag{1.2.10.1}$$

be an exact sequence in $\mathbf{preFil}_I(\mathcal{C})$*, and assume that* F'' *is an* I*-filtration. If one of* F*,* F' *is an* I*-filtration, then so is the other.*

Proof. F is an I-filtration on X if and only if $\tilde{K}(X, F, J))$ is acyclic in degree ≥ 0 for all J. Applying the functor $\tilde{K}(-, -, J)$ to the exact sequence (1.2.10.1) we obtain an exact sequence of (homological) complexes

$$0 \to \tilde{K}(X', F', J) \to \tilde{K}(X, F, J) \to \tilde{K}(X'', F'', J) \to 0$$

By assumption, $H_i(\tilde{K}(X'', F'', J)) = 0$ from which the result follows. □

(1.2.11) Let F be an I-filtration on X, $i \in I$ and $I_{\leq i} = \{j \in I \mid j \leq i\}$. Then F induces both an I-filtration and an $I_{\leq i}$-filtration on F_iX.

On an arbitrary subobject $X' \subset X$, F induces a weak I-filtration, but in general this need not be an I-filtration.

1.3 Associated graded objects

(1.3.1) Definition. Let X be an object of $\mathcal{C}$, F an I-prefiltration on X. Define for $i \in I$

$$F_{<i}X = \sum_{j<i} F_jX, \quad \mathrm{gr}_i^F X := \frac{F_iX}{F_{<i}X}$$

(assuming that the sum exists).

(1.3.2) Definition. An I-prefiltration F on X is *bounded below* if there exists $j \in I$ such that $F_iX \neq 0$ implies $i > j$. It is *exhaustive* if $\varinjlim_{i \in I} F_iX = X$. It is *finite* if it is bounded below and

(i) there exists $k \in I$ such that $F_kX = X$; and

(ii) $\mathrm{wts}(X,F)\colon = \{i \in I \mid \mathrm{gr}_i^F X \neq 0\}$ is finite.

The elements of $\mathrm{wts}(X,F)$ will be called the *weights* of F.

(1.3.3) Proposition/Definition. *Let $X = \bigoplus_{i\in I} X(i)$ be an I-graded object of $\mathcal{C}$, and set $F_iX := \bigoplus_{j\leq i} X(j)$. Then F is an I-filtration, satisfying $\mathrm{gr}_i^F X = X(i)$. It is finite if and only if $J = \{i \mid X(i) \neq 0\}$ is finite and there exists $j \in I$ with $j < \inf J$. An I-filtration of this type is said to be* splittable .

Proof. For $k \in I$ define $F^{(k)}\colon I \to \mathbf{sub}\, X(k)$ by

$$F_i^{(k)} X(k) = \begin{cases} X(k) & \text{if } i \geq k \\ 0 & \text{otherwise.} \end{cases}$$

Let $J = \{j_0, \dots, j_n\} \subset I$, ordered so that $j_\alpha \geq k \iff 0 \leq \alpha \leq m$. Then $K(X(k), F^{(k)}, J)$ is

$$\cdots \to \bigoplus_{0\leq\alpha<\beta<\gamma\leq m} X(k) \to \bigoplus_{0\leq\alpha<\beta\leq m} X(k) \to \bigoplus_{0\leq\alpha\leq m} X(k) \to X(k)$$

which is exact, hence $F^{(k)}$ is an I-filtration on $X(k)$. As $(X,F) = \bigoplus (X(k), F^{(k)})$, F is an I-filtration on X. The rest follows immediately from the definitions. □

(1.3.4) Remark. If $f = \sum f_i \colon \bigoplus X(i) \to \bigoplus Y(i)$ is a graded morphism of I-graded objects, then it is easy to see that the morphism of associated I-filtered objects is strict.

(1.3.5) Proposition. *Let*

$$0 \to (X', F') \to (X,F) \to (X'', F'') \to 0$$

be an exact sequence of I-filtered objects of $\mathcal{C}$. Then for every i the sequence

$$0 \to \mathrm{gr}_i^{F'} X' \to \mathrm{gr}_i^F X \to \mathrm{gr}_i^{F''} X'' \to 0$$

is exact.

Proof. Let $i \in I$, and let $\{j_\alpha\} \subset I$ be any finite subset such that $j_\alpha < i$ for all α. We then have a long exact sequence

$$\cdots \to \bigoplus_{\alpha<\beta} F_{j_\alpha \wedge j_\beta} X \to \bigoplus_{\alpha} F_{j_\alpha} X \to F_i X \to F_i X / \sum_{\alpha} F_{j_\alpha} X \to 0$$

and similarly for X' and X''. Since $0 \to F_j' X'' \to F_j X \to F_j' X' \to 0$ is exact for every j, we have the corresponding exactness for $F_i / \sum_\alpha F_{j_\alpha}$. Passing to the direct limit over all such finite subsets $\{j_\alpha\}$ gives the exactness for gr_i^F. □

For an object Y of $\mathcal{C}$, let $[Y] \in K_0\mathcal{C}$ denote the class of Y in the Grothendieck group of $\mathcal{C}$.

(1.3.6) Theorem. *Let F be an I-filtration on X, bounded below and exhaustive.*

(i) If F is finite, $[X] = \sum_i [\mathrm{gr}_i^F X]$.

(ii) If $\mathrm{gr}_i^F X = 0$ for every $i \in I$, then $X = 0$.

(iii) Assume that $\mathcal{C}$ is semisimple. Then (X, F) is splittable.

Proof. Since F is bounded below, replacing I by a suitable sublattice we may assume that the lattice I has a minimal element.

(i) Let us show by induction that for every i, $[F_i X] = \sum_{j \le i} [\mathrm{gr}_j^F X]$. Let $\{j_\alpha\} = \{j \in I \mid j < i\}$, which is finite since I is locally finite and bounded below. By definition of gr^F and since F is an I-filtration, we have a (finite) long exact sequence

$$\cdots \to \bigoplus_{\alpha<\beta} F_{j_\alpha \wedge j_\beta} X \to \bigoplus_\alpha F_{j_\alpha} X \to F_i X \to \mathrm{gr}_i^F X \to 0 \tag{1.3.6.1}$$

giving an equality in $K_0 \mathcal{C}$

$$[F_i X] = [\mathrm{gr}_i^F X] + \sum_\alpha [F_{j_\alpha} X] - \sum_{\alpha<\beta} [F_{j_\alpha \wedge j_\beta} X] + \cdots .$$

By induction we may assume that for every $j < i$, $[F_j X] = \sum_{j' \le j} [\mathrm{gr}_{j'}^F X]$. Then the equality becomes

$$\begin{aligned}[F_i X] &= [\mathrm{gr}_i^F X] + \sum_\alpha \sum_{j \le j_\alpha} [\mathrm{gr}_j^F X] - \sum_{\alpha<\beta} \sum_{j \le j_\alpha \wedge j_\beta} [\mathrm{gr}_j^F X] + \cdots . \\ &= [\mathrm{gr}_i^F X] + \sum_{j<i} \big(\#\{\alpha \mid j_\alpha \ge j\} - \#\{(\alpha < \beta) \mid j_\alpha,\ j_\beta \ge j\} + \dots \big) [\mathrm{gr}_j^F X]\end{aligned}$$

and the parenthesised sum equals 1.

(ii) The exact sequence (1.3.6.1) (which does not require F to be finite) and induction shows that $F_i X = 0$ for all i, and so $X = 0$ as F is exhaustive.

(iii) Choose for each i a splitting $F_i X = X(i) \oplus \sum_{j<i} F_j X$, so that by induction $F_i X = \sum_{j \le i} X(j)$. Write $\widetilde{X} = \bigoplus X(i)$, $\widetilde{F}_i \widetilde{X} = \bigoplus_{j \le i} X(i)$. Then the obvious map $f \colon (\widetilde{X}, \widetilde{F}) \to (X, F)$ is surjective (as F is exhaustive) and is strict. Therefore $\ker f = (Y, G)$ is I-filtered, by (1.2.10). Since f is an isomorphism on associated gradeds, by (1.3.5) $\mathrm{gr}_i^G Y = 0$ for every i and $Y = 0$, and therefore $X = \bigoplus X(i)$. □

(1.3.7) Proposition. *Let F be a finite weak I-filtration on X. Let $Y = F_j X$ for some $j \in I$, with the filtration induced from F. Then*

$$\mathrm{wts}(Y, F) = \{i \in \mathrm{wts}(X, F) \mid i \le j\}.$$

Proof. if $i \le j$ then obviously $\mathrm{gr}_i^F Y = \mathrm{gr}_i^F X$. If $i \not\le j$, then $i \wedge j < i$, and $F_i Y = F_{i \wedge j} Y$ so $\mathrm{gr}_i^F Y = 0$. □

(1.3.8) From now on, we assume all (pre-)filtrations to be finite.

1.4 Opposed filtrations

(1.4.1) Let $F^\bullet$, $G_\bullet$ be decreasing (resp. increasing) I-prefiltrations on X. Consider the $I^{opp} \times I$-prefiltration (well-defined up to unique isomorphism)

$$F \cap G \colon (i,j) \mapsto F^i X \cap G_j X.$$

and $(\mathrm{gr}^i_{j,F\cap G} X)_{i,j}$ the associated graded.

(1.4.2) Definition. F and G are *opposed I-filtrations* if

(i) $F \cap G$ is an $I^{opp} \times I$-filtration; and

(ii) $i \neq j \implies \mathrm{gr}^i_{j,F\cap G} X = 0$.

The terminology is justified by the following theorem. We assume that either (a) $\mathcal{C}$ is semisimple, or (b) for every nonzero X in $\mathcal{C}$, the class $[X] \in K_0\mathcal{C}$ is nonzero.

(1.4.3) Proposition. *(i) Let (F, G) be opposed I-filtrations on X, and set $X(i) = F^i X \cap G_i X$. Then $X = \bigoplus_i X(i)$ and*

$$F^i X = \sum_{k \geq i} X(k), \quad G_i X = \sum_{k \leq j} X(k).$$

In particular, F and G are splittable I-filtrations.

(ii) The category of triples (X, F, G), where (F, G) are opposed I-filtrations on objects X of $\mathcal{C}$, is equivalent to the category of finitely I-graded objects of $\mathcal{C}$.

Proof. (i) Since $F \cap G$ is an $I^{opp} \times I$-filtration, by (1.3.6)

$$[F^i \cap G_j X] = \sum_{\substack{i' \geq i \\ j' \leq j}} [\mathrm{gr}^{i'}_{j',F\cap G} X] = \sum_{i \leq k \leq j} [\mathrm{gr}^k_{k,F\cap G} X]. \tag{1.4.3.1}$$

Therefore

$$[F^i \cap G_j X] = 0 \qquad \text{if } i \not\leq j \tag{1.4.3.2}$$

and

$$[F_i \cap G_i X] = [\mathrm{gr}^i_{i,F\cap G} X].$$

So in case (b), we deduce $F^i \cap G_i X \xrightarrow{\sim} \mathrm{gr}^i_{i,F\cap G} X$. In case (a) we obtain the same isomorphism by splitting the filtration $F \cap G$.

Now let $J = \{j_1, \ldots, j_n\} \subset I$. The sequence

$$\bigoplus_{\alpha<\beta} (F^{j_\alpha} \cap G_{j_\alpha})X \cap (F^{j_\beta} \cap G_{j_\beta})X \to \bigoplus_\alpha X(j_\alpha) \to X$$

is exact by condition (ii), and if $\alpha \neq \beta$

$$(F^{j_\alpha} \cap G_{j_\alpha})X \cap (F^{j_\beta} \cap G_{j_\beta}) = F^{j_\alpha \vee j_\beta} X \cap G_{j_\alpha \wedge j_\beta} X = 0$$

by (1.4.3.2). Therefore

$$\bigoplus_{k \in J} X(k) \xrightarrow{\sim} \sum_{k \in J} X(k)$$

and so

$$\bigoplus_{i \le k \le j} X(k) \hookrightarrow (F^i \cap G_j)X$$

and by (1.4.3.1) this is an isomorphism.

(ii) follows from (i). $\square$

2 Plectic Hodge structures

2.1 Pure plectic Hodge structures over $\mathbb{C}$

(2.1.1) All vector spaces in this section will be assumed to be finite-dimensional.

We denote the natural generators of $\mathbb{Z}^r$ as $\underline{1}_j$ $(1 \le j \le r)$.

If $\underline{n} = (n_j) \in \mathbb{Z}^r$, write $|\underline{n}| = \sum_{j=1}^r n_j$.

If W is a $\mathbb{Z}^r$-filtration on X, we define the *simple filtration sW associated to W* by

$$sW_w X = \sum_{|\underline{n}| \le w} W_{\underline{n}} X.$$

The inclusions $W_{\underline{n}} X \subset W_{|\underline{n}|} X$ induce an isomorphism

$$\bigoplus_{|\underline{n}|=w} \mathrm{gr}^W_{\underline{n}} X \xrightarrow{\sim} \mathrm{gr}^{sW}_w X. \tag{2.1.1.1}$$

(2.1.2) If F, $\bar{F}$ are decreasing $\mathbb{Z}^r$-prefiltrations, and $\underline{n} \in \mathbb{Z}^r$, say that $(F, \bar{F})$ are *$\underline{n}$-opposed* if $(F^\bullet, \bar{F}^{\underline{n}-\bullet})$ are $\mathbb{Z}^r$-opposed filtrations.

(2.1.3) Definition. (i) An *unmixed plectic $\mathbb{C}$-Hodge structure* is a $\mathbb{Z}^{2r}$-graded $\mathbb{C}$-vector space $V = \bigoplus_{\underline{p},\underline{q} \in \mathbb{Z}^r} V^{\underline{p}\,\underline{q}}$.

(ii) V is *pure of plectic weight* $\underline{n} \in \mathbb{Z}^r$ if $V^{\underline{p}\,\underline{q}} = 0$ unless $\underline{p} + \underline{q} = \underline{n}$.

(iii) V is *pure of simple weight* $w \in \mathbb{Z}$ if $V^{\underline{p}\,\underline{q}} = 0$ unless $\left|\underline{p} + \underline{q}\right| = w$.

(2.1.4) For each $\underline{p}$, $\underline{q} \in \mathbb{Z}^r$ we write $\mathbb{C}(\underline{p}, \underline{q})$ for the unmixed plectic Hodge structure over $\mathbb{C}$ of dimension 1 and degree $(\underline{p}, \underline{q})$.

(2.1.5) Every unmixed plectic $\mathbb{C}$-Hodge structure V has:

- a $\mathbb{Z}$-grading $V = \bigoplus_{w\in\mathbb{Z}} V_{(w)}$, where each $V_{(w)}$ is pure of simple weight $w \in \mathbb{Z}$, and
- a $\mathbb{Z}^r$-grading $V = \bigoplus_{\underline{n}\in\mathbb{Z}^r} V_{(\underline{n})}$, where each $V_{(\underline{n})}$ is pure of plectic weight $\underline{n}$

and $V_{(w)} = \sum_{|\underline{n}|=w} V_{(\underline{n})}$. We shall refer to these as the gradings by plectic and simple weight, respectively.

(2.1.6) Let V be an unmixed plectic $\mathbb{C}$-Hodge structure. Define decreasing $\mathbb{Z}$-filtrations F_j, $\bar{F}_j$ $(1 \le j \le r)$ by

$$F_j^p V = \bigoplus_{p_j \ge p} V^{\underline{p}\,\underline{q}}, \quad \bar{F}_j^q V = \bigoplus_{q_j \ge q} V^{\underline{p}\,\underline{q}}$$

and $\mathbb{Z}^r$-filtrations F_J, $\bar{F}_J$ (for $J \subset \{1, \dots, r\}$ and J^c the complementary subset)

$$F_J^{\underline{a}} V = \bigcap_{j\notin J} F_j^{a_j} V \cap \bigcap_{j\in J} \bar{F}_j^{a_j} V = \bigoplus_{\substack{p'_j \ge a_j \ (j\notin J) \\ q'_j \ge a_j \ (j\in J)}} V_J^{\underline{p}'\underline{q}'}, \quad \bar{F}_J = F_{J^c}.$$

Set $F = F_\emptyset$, $\bar{F} = \bar{F}_\emptyset$.

(2.1.7) If V is pure of plectic weight $\underline{n}$, then for every J, $(F_J, \bar{F}_J)$ are $\underline{n}$-opposed, since

$$F_J^{\underline{a}} V \cap \bar{F}_J^{\underline{n}-\underline{a}} V = V^{\underline{p}'\underline{q}'} \quad \text{with} \quad (p'_j, q'_j) = \begin{cases} (a_j, n_j - a_j) & (j \notin J) \\ (n_j - a_j, a_j) & (j \in J) \end{cases}$$

(2.1.8) Conversely, if $(F, \bar{F})$ are $\underline{n}$-opposed filtrations on V, then by (1.4.3)

$$V = \bigoplus_{\underline{p}+\underline{q}=\underline{n}} F^{\underline{p}} V \cap \bar{F}^{\underline{q}} V$$

is a plectic Hodge structure of plectic weight $\underline{n}$.

(2.1.9) If V is a pure plectic $\mathbb{C}$-Hodge structure of simple weight $w \in \mathbb{Z}$, then it is still the case that $V^{\underline{p}\,\underline{q}} = F^{\underline{p}} V \cap \bar{F}^{\underline{q}} V$ whenever $\left|\underline{p} + \underline{q}\right| = w$. Conversely, suppose that $(F, \bar{F})$ are decreasing $\mathbb{Z}^r$-prefiltrations on a vector space V and that for some $w \in \mathbb{Z}$,

$$V = \bigoplus_{\left|\underline{p}+\underline{q}\right|=w} F^{\underline{p}} V \cap \bar{F}^{\underline{q}} V.$$

Then V is a plectic Hodge structure which is pure of simple weight w, and F, $\bar{F}$ are the associated $\mathbb{Z}^r$-filtrations.

(2.1.10) The unmixed plectic $\mathbb{C}$-Hodge structures form a $\mathbb{C}$-linear semisimple Tannakian category, which is equivalent to $\mathbf{Rep}_{\mathbb{C}}(\mathbb{G}_m^{2r})$. The plectic weight is given by the character of $\mathbb{G}_m^r \subset \mathbb{G}_m^{2r}$, and the simple weight by the character of $\mathbb{G}_m$ diagonally embedded. Its simple objects are the $\mathbb{C}(\underline{p}, \underline{q})$.

2.2 Pure plectic $\mathbb{R}$-Hodge structures

(2.2.1) Let V be an $\mathbb{R}$-vector space, and $V_{\mathbb{C}} = \bigoplus V^{\underline{p}\,\underline{q}}$ an unmixed plectic Hodge structure on $V_{\mathbb{C}} = V \otimes_{\mathbb{R}} \mathbb{C}$. The following conditions are equivalent:

(i) $\forall (\underline{p}, \underline{q}),\ \overline{V^{\underline{p}\,\underline{q}}} = V^{\underline{q}\underline{p}}$.

(ii) $\forall j,\ \bar{F}_j =$ complex conjugate of F_j.

(iii) $\forall J,\ \bar{F}_J =$ complex conjugate of F_J.

We say that V is a real unmixed plectic Hodge structure (or unmixed plectic $\mathbb{R}$-Hodge structure) if these hold.

(2.2.2) The unmixed plectic $\mathbb{R}$-Hodge structures form an $\mathbb{R}$-linear Tannakian category, equivalent to $\mathbf{Rep}_{\mathbb{R}}(\mathbb{S}^r)$, where $\mathbb{S} = R_{\mathbb{C}/\mathbb{R}}\mathbb{G}_m$. The gradings by plectic and simple weight are given in the same way as in (2.1.10).

(2.2.3) We define an unmixed plectic $\mathbb{R}$-Hodge structure over $\mathbb{R}$ to be an unmixed plectic Hodge structure over $\mathbb{R}$, together with r commuting involutions $\tau_j \in \mathrm{Aut}_{\mathbb{R}}(V)$ satisfying

$$\tau_j(F_i^p) = \begin{cases} \bar{F}_i^p & \text{if } i = j \\ F_i^p & \text{otherwise.} \end{cases}$$

These form a Tannakian category, equivalent to the category of representations of the group $(\mathbb{S} \rtimes \mathbb{Z}/2\mathbb{Z})^r$, where the nonzero element of $\mathbb{Z}/2\mathbb{Z}$ acts on $\mathbb{S}$ by complex conjugation.

2.3 Mixed structures

(2.3.1) In this and the following section, all vector spaces will be over a field k of characteristic zero. In preparation for the theory of mixed plectic $\mathbb{R}$-Hodge structures, we first review some of Deligne's theory for usual mixed Hodge structures [4, 5], and discuss its plectic variant.

(2.3.2) Recall from [5] that if V is a finite dimensional vector space over k, the following structures are equivalent:

(a) Three filtrations $(F^\bullet, \bar{F}^\bullet, W_\bullet)$ which are *opposed*; i.e.,

$$\begin{aligned} &\mathrm{gr}_n^W F^p V \cap \mathrm{gr}_n^W \bar{F}^q V = 0 \quad \text{if } p+q=n+1 \\ \text{and} \quad &\mathrm{gr}_n^W V = \bigoplus_{p+q=n} \mathrm{gr}_n^W F^p V \cap \mathrm{gr}_n^W \bar{F}^q V \end{aligned}$$

(b) A bigrading $V = \bigoplus V_F^{pq}$, plus a nilpotent endomorphism ∂ of V satisfying

$$\partial(V_F^{pq}) \subset \bigoplus_{p'<p,\, q'<q} V_F^{p'q'};$$

(c) A representation $\rho\colon \mathcal{G} \to GL(V)$, where $\mathcal{G} = \mathbb{G}_m^2 \ltimes U$, and U is the pro-unipotent group whose Lie algebra is freely generated by elements ∂^{ab} of bidegree (a,b), $a<0$ and $b<0$.

(2.3.3) The dictionary (a) $\Longrightarrow$ (b) is given by

$$F^p V = \bigoplus_{p' \geq p} V_F^{p'q}, \qquad W_n V = \bigoplus_{p+q \leq n} V_F^{pq} \tag{2.3.3.1}$$

$$\bar{F}^q = \bigoplus_{q' \geq q} V_{\bar{F}}^{pq'}, \qquad V_{\bar{F}}^{pq} = (\exp \partial) V_F^{pq} \tag{2.3.3.2}$$

and in the other direction by setting

$$V_F^{pq} = (F^p V \cap W_{p+q} V) \cap (\bar{F}^q V \cap W_{p+q} V + \sum_{i \geq 0} \bar{F}^{q-i} \cap W_{p+q-i-1}) \tag{2.3.3.3}$$

$$V_{\bar{F}}^{pq} = (\bar{F}^q V \cap W_{p+q} V) \cap (F^p V \cap W_{p+q} V + \sum_{i \geq 0} F^{p-i} \cap W_{p+q-i-1}) \tag{2.3.3.4}$$

from which it follows that

$$V_{\bar{F}}^{pq} \subset V_F^{pq} + \sum_{p'<p, q'<q} V_F^{p'q'}$$

and that there is a unique endomorphism ∂ as in (b) satisfying $(\exp \partial) V_F^{pq} = V_{\bar{F}}^{pq}$.

These constructions give rise to equivalences of abelian categories [4, 5], and in particular it follow that a morphism of trifiltered structures (a) is automatically strictly compatible with the filtrations.

(2.3.4) Let $G^\bullet$ be the convolution of the filtrations $\bar{F}^\bullet$ and $W_\bullet$:

$$G^q V = \sum_n \bar{F}^{n+q} V \cap W_n V.$$

A conceptual way to view the formula (2.3.3.3) is to observe that the filtrations $(F^\bullet, G^\bullet)$ on the filtered space $(V, W_\bullet)$ are 0-opposed: for every p, $n \in \mathbb{Z}$ one has

$$W_n V = (F^p V \cap W_n V) \oplus (G^{-p-1} V \cap W_n V)$$

giving a splitting of the filtration $F^\bullet$:

$$F^pV \cap G^{-p}V \xrightarrow{\sim} \mathrm{gr}^p_F V. \tag{2.3.4.1}$$

Since on $\mathrm{gr}^p_F V$ the filtrations induced by $\bar{F}^\bullet$ and $W_\bullet$ are $(-p)$-opposed, they induce a grading on $\mathrm{gr}^p_F V$, hence via the splitting (2.3.4.1) they induce a bigrading on V, which is the bigrading by the V_F^{pq}.

(2.3.5) The equivalence of (b) and (c) is formal, and admits the following simple generalisation:

(2.3.6) Proposition. *Let r be a positive integer. The following categories are equivalent:*

(i) $\mathbf{Rep}_k\, \mathcal{G}^r$.

(ii) The category of triples $(V, (V^{\underline{p}\underline{q}}), (\partial_j))$, where

$$V = \bigoplus_{\underline{p},\underline{q}\in\mathbb{Z}^r} V^{\underline{p}\,\underline{q}}$$

is a $\mathbb{Z}^{2r}$-graded k-vector space of finite dimension, and $(\partial_j)_{1\le j\le r}$ are commuting endomorphisms of V such that for each j,

$$\partial_j(V^{\underline{p}\,\underline{q}}) \subset \sum_{a,b>0} V^{\underline{p}-a\underline{1}_j,\underline{q}-b\underline{1}_j} \tag{2.3.6.1}$$

□

(2.3.7) If V is a representation of $\mathcal{G}^r$, define the *plectic Hodge numbers* of V to be $d^{\underline{p}\,\underline{q}}(V) = \dim V^{\underline{p}\,\underline{q}}$ ($\underline{p}$, $\underline{q} \in \mathbb{Z}^r$).

(2.3.8) The simple objects of $\mathbf{Rep}_k\, \mathcal{G}^r$ are the 1-dimensional representations $V = k(\underline{p},\underline{q})$ ($\underline{p}$, $\underline{q} \in \mathbb{Z}^r$) with $V = V^{\underline{p}\,\underline{q}} = k$, $\partial_j = 0$.

(2.3.9) Write $S = \{1, \dots, r\}$. Let V be a representation of $\mathcal{G}^r$. For each $J \subset S$ let $\partial_J = \sum_{j\in J} \partial_j$, and define

$$V_J^{\underline{p}\,\underline{q}} := (\exp \partial_J) V^{\underline{p}\,\underline{q}} \subset V^{\underline{p}\,\underline{q}} + \sum_{\substack{\underline{p}'<\underline{p}\\ \underline{q}'<\underline{q}}} V^{\underline{p}'\underline{q}'} \tag{2.3.9.1}$$

giving a family $(V_J^{**})_J$ of $\mathbb{Z}^{2r}$-gradings of V, indexed by subsets $J \subset S$.

Conversely, given gradings $V = \bigoplus_{\underline{p},\underline{q}} V_J^{\underline{p}\,\underline{q}}$ for all $J \subset S$ there exists at most one family of nilpotent operators $(\partial_J)_J$ of negative degree for which $V_J^{\underline{p}\,\underline{q}} = (\exp \partial_J) V^{\underline{p}\,\underline{q}}$ for every J and $(\underline{p}, \underline{q})$.

(2.3.10) The representation V of $\mathcal{G}^r$ now carries several filtrations:

- The *plectic weight filtration* $W_\bullet$, a $\mathbb{Z}^r$-filtration given by

$$W_{\underline{n}}V = \sum_{\substack{\underline{p},\underline{q}\in\mathbb{Z}^r \\ \underline{p}+\underline{q}\le\underline{n}}} V_?^{\underline{p}\,\underline{q}} \qquad (\underline{n}\in\mathbb{Z}^r) \tag{2.3.10.1}$$

  (where, for each $(\underline{p},\underline{q})$, $V_?^{\underline{p}\,\underline{q}}$ can be any of the $V_J^{\underline{p}\,\underline{q}}$).

- The associated *simple weight filtration* $sW_\bullet$:

$$sW_nV = \sum_{\substack{\underline{p},\underline{q}\in\mathbb{Z}^r \\ |\underline{p}+\underline{q}|\le n}} V_?^{\underline{p}\,\underline{q}} \qquad (\underline{n}\in\mathbb{Z}^r) \tag{2.3.10.2}$$

- For each $J\subset S$, a $\mathbb{Z}^r$-filtration $F_J^\bullet$ given by

$$F_J^{\underline{a}}V = \sum_{\substack{\underline{p},\underline{q}\in\mathbb{Z}^r \\ p_j\ge a_j\ (j\notin J) \\ q_j\ge a_j\ (j\in J)}} V_J^{\underline{p}\,\underline{q}}. \tag{2.3.10.3}$$

  Write $\bar{F}_J^\bullet = F_{J^c}^\bullet$, $F^\bullet = F_\emptyset^\bullet$.

- For each $j\in S$, $\mathbb{Z}$-filtrations $F_j^\bullet$, $\bar{F}_j^\bullet$ given by

$$F_j^pV = \sum_{\substack{\underline{p}',\underline{q}'\in\mathbb{Z}^r \\ p'_j\ge p}} V_J^{\underline{p}'\underline{q}'} \qquad \text{for any } J \text{ with } j\notin J$$

$$\bar{F}_j^qV = \sum_{\substack{\underline{p}',\underline{q}'\in\mathbb{Z}^r \\ q'_j\ge q}} V_J^{\underline{p}'\underline{q}'} \qquad \text{for any } J \text{ with } j\in J. \tag{2.3.10.4}$$

(The condition (2.3.6.1) ensures that the first two filtrations are indeed independent of J.) By definition, for every J, $W_\bullet\cap F_J^\bullet$ is a $\mathbb{Z}^{2r}$-filtration on V.

(2.3.11) The filtrations (2.3.10.3) and (2.3.10.4) determine one another by (1.2.5), since

$$F_J = \bigcap_{j\notin J} F_j \cap \bigcap_{j\in J} \bar{F}_j.$$

If $r=1$ we simply have $F_1 = F = F_\emptyset = \bar{F}_{\{1\}}$, $\bar{F}_1 = \bar{F} = F_{\{1\}}$.

(2.3.12) If $\pi\colon U\to V$ is any morphism of representations of $\mathcal{G}^r$, then as π commutes with the ∂_j, we have $\pi(U_J^{\underline{p}\,\underline{q}})\subset V_J^{\underline{p}\,\underline{q}}$, and therefore π is strictly compatible with all the filtrations. If $V=\sum V_\alpha$ is a sum of subrepresentations then $V^{\underline{p}\,\underline{q}} = \sum V_\alpha^{\underline{p}\,\underline{q}}$, $F_j^\bullet V = \sum F_j^\bullet V_\alpha$.

(2.3.13) The associated graded objects $\mathrm{gr}^W_* V$ and $\mathrm{gr}^{sW}_* V$ are then what we have called (for $k = \mathbb{C}$) unmixed plectic Hodge structures, and they are canonically isomorphic by (2.1.1.1).

(2.3.14) From (2.3.10.3) one sees that for each $J \subset S$, there is an increasing $\mathbb{Z}^r$-prefiltration G^J, complementary to F_J, given by

$$G^J_{\underline{p}} = \bigoplus_{\underline{p}' \not\geq \underline{p}} V_J^{\underline{p}'\underline{q}'}.$$

Write $G_{\underline{p}} = G^{\emptyset}_{\underline{p}}$. (Note that in general G^J is not even a weak $\mathbb{Z}^r$-filtration.)

(2.3.15) For $J \subset S$ and $\underline{p}$, $\underline{q} \in \mathbb{Z}^r$, let $\sigma_J(\underline{p}, \underline{q}) = (\underline{a}, \underline{b})$ where

$$(a_j, b_j) = \begin{cases} (p_j, q_j) & \text{if } j \notin J \\ (q_j, p_j) & \text{if } j \in J \end{cases}$$

(2.3.16) Proposition. *(i) Let $\underline{p} + \underline{q} = \underline{n}$. Then*

$$V^{\underline{p}\,\underline{q}} = (F^{\underline{p}} V \cap W_{\underline{n}} V) \cap \big((\bar{F}^{\underline{q}} V \cap W_{\underline{n}} V) + (G_{\underline{p}} V \cap W_{<\underline{n}} V) \big).$$

(ii) Let $J \subset S$, and $(\underline{a}, \underline{b}) = \sigma_J(\underline{p}, \underline{q})$. Then

$$V_J^{\underline{p}\,\underline{q}} = (F_J^{\underline{a}} V \cap W_{\underline{n}} V) \cap \big((F_{J^c}^{\underline{b}} V \cap W_{\underline{n}} V) + \sum_{\substack{\underline{a}' + \underline{b}' < \underline{n} \\ \underline{a}' \not\geq \underline{a}}} V^{\sigma_J(\underline{a}'\underline{b}')} \big). \tag{2.3.16.1}$$

Proof. (i) We have

$$W_{<\underline{n}} V = (F^{\underline{p}} V \cap W_{<\underline{n}} V) \oplus \sum_{\substack{\underline{p}' + \underline{q}' < \underline{n} \\ \underline{p}' \not\geq \underline{p}}} V^{\underline{p}'\underline{q}'}$$

and by (2.3.9.1)

$$\bar{F}^{\underline{q}} V \cap W_{<\underline{n}} V = \sum_{\substack{\underline{p}' + \underline{q}' < \underline{n} \\ \underline{q}' \geq \underline{q}}} V_S^{\underline{p}'\underline{q}'} \subset \sum_{\substack{\underline{p}' + \underline{q}' < \underline{n} \\ \underline{p}' \not\geq \underline{p}}} V^{\underline{p}'\underline{q}'} = G_{\underline{p}} V \cap W_{<\underline{n}} V$$

and

$$V^{\underline{p}\,\underline{q}} \subset V_S^{\underline{p}\,\underline{q}} + \sum_{\substack{\underline{p}' + \underline{q}' < \underline{n} \\ \underline{p}' \not\geq \underline{p}}} V^{\underline{p}'\underline{q}'} \subset (\bar{F}^{\underline{q}} V \cap W_{\underline{n}} V) + \sum_{\substack{\underline{p}' + \underline{q}' < \underline{n} \\ \underline{p}' \not\geq \underline{p}}} V^{\underline{p}'\underline{q}'}$$

Recall (cf. [4, (1.2.9)]) a simple lemma from linear algebra:

> If A, B, U are subspaces of a vector space V such that $U = (A \cap U) \oplus (B \cap U)$ then $(A + U) \cap (B + U) = (A \cap B) \oplus U$.

Apply this with $U = W_{<\underline{n}}V$, $A = F^{\underline{p}}V \cap W_{\underline{n}}V$, $B = (\bar{F}^{\underline{q}}V \cap W_{\underline{n}}V) + (G_{\underline{p}}V \cap W_{<\underline{n}}V)$. Then $A \cap U = F^{\underline{p}}V \cap W_{<\underline{n}}$ and $B \cap U = G_{\underline{p}} \cap W_{<\underline{n}}$, so by (2.3.14), the conditions are satisfied. Therefore, letting $X = A \cap B$ be the right-hand side of (i), this gives $X \cap W_{<\underline{n}}V = 0$ and $(X + W_{<\underline{n}}V)/W_{<\underline{n}}V = (gr^W_{\underline{n}}V)^{\underline{pq}}$. As $V^{\underline{p}\,\underline{q}} \subset X$ this implies $X = V^{\underline{p}\,\underline{q}}$.

(ii) follows by applying the symmetry σ_J to (i). $\square$

(2.3.17) It follows that from the filtrations $((F_j)_j, W)$ one may recover the subspaces $V_J^{\underline{p}\,\underline{q}}$, and hence the action of $\mathcal{G}^S$, inductively by the formula (2.3.16.1). More precisely (cf. [5, (1.1.1)]), the map d_J is the automorphism of V given as the composite $a_J^{-1}a$:

$$V = \bigoplus V^{\underline{p}\,\underline{q}} \xrightarrow[a]{\sim} \mathrm{gr}^W_* V = \bigoplus (\mathrm{gr}^W_{\underline{p}+\underline{q}} V)^{\underline{p}\,\underline{q}} \xleftarrow[a_J]{\sim} \bigoplus V_J^{\underline{p}\,\underline{q}} = V \tag{2.3.17.1}$$

They may equally be recovered from the filtrations $((F_j)_j, sW)$ by the following variant of the previous proposition (whose proof is the same):

(2.3.18) Proposition. *(i) Let* $\left|\underline{p} + \underline{q}\right| = w$. *Then*

$$V^{\underline{p}\,\underline{q}} = (F^{\underline{p}}V \cap sW_wV) \cap \big((\bar{F}^{\underline{q}}V \cap sW_wV) + (G_{\underline{p}}V \cap sW_{w-1}V)\big).$$

(ii) Let $J \subset S$, *and* $(\underline{a}, \underline{b}) = \sigma_J(\underline{p}, \underline{q})$. *Then*

$$V_J^{\underline{p}\,\underline{q}} = (F_J^{\underline{a}}V \cap sW_wV) \cap \Big((F_{J^c}^{\underline{b}}V \cap sW_wV) + \sum_{\substack{|\underline{a}'+\underline{b}'|<w \\ \underline{a}' \not\geq \underline{a}}} V^{\sigma_J(\underline{a}'\underline{b}')}\Big). \quad \square$$

(2.3.19) Let $V \in \mathrm{ob}\,\mathbf{Rep}_k\,\mathcal{G}^r$. Define subspaces

$$R_J(V) = \bigoplus_{\substack{p_i=q_i=0 \ (i\notin J) \\ p_i,\ q_i<0 \ (i\in J)}} V^{\underline{p}\,\underline{q}}$$

$$R(V) = \sum_{J\subset S} R_J(V).$$

Write $R_j(V) = R_{\{j\}}(V) = \sum_{a,b>0} V^{-a\underline{1}_j, -b\underline{1}_j}$. The definitions imply:

(2.3.20) Proposition. *$R(V)$ is a subobject of V, and $V \mapsto R(V)$ is an exact endofunctor of* $\mathbf{Rep}_k\,\mathcal{G}^r$. $\square$

(2.3.21) Lemma. *Assume $r > 1$. If $V = W_{<\underline{0}}V$ and $R(V) = 0$, then*

$$Q(V) := \bigcap_i \left(\bar{F}_i^0 V + \bigcap_{j\neq i} F_j^0 V \right) = F^{\underline{0}}V \oplus \bar{F}^{\underline{0}}V.$$

Proof. Observe first that $Q(V) \supset F^{\underline{0}}V + \bar{F}^{\underline{0}}V$, and that $F^{\underline{0}}V \cap \bar{F}^{\underline{0}} = 0$ since $V = W_{<\underline{0}}V$. Suppose V is unmixed (i.e., the action of $\mathcal{G}^r$ factors through $\mathbb{G}_m^{2r}$). Then all the F_j^p, $\bar{F}_j^q$ are sums of some of the $V^{\underline{p}\,\underline{q}}$, so in the expression for $Q(V)$, intersection is distributive over sum, and therefore

$$Q(V) = \sum_{J \subset S} \left(\bigcap_{j \notin J} \bar{F}_j^0 V \cap \bigcap_{i \in J} \bigcap_{j \neq i} F_j^0 V \right)$$

The term $J = \emptyset$ equals $\bar{F}^{\underline{0}}$. The term $J = \{i\}$ equals

$$\bigcap_{j \neq i} F_j^0 V \cap \bar{F}_j^0 V = \sum_{p_j,\, q_j \geq 0 \ (j \neq i)} V^{\underline{p}\,\underline{q}} = \sum_{p_j = q_j = 0 \ (j \neq i)} V^{\underline{p}\,\underline{q}}$$

since $V = W_{<\underline{0}}V$. This is contained in $F^{\underline{0}}V + \bar{F}^{\underline{0}}V$, since $R_{\{i\}}(U) = 0$. The term $J = S$ equals $F^{\underline{0}}V$. If $1 < \#J < r$ then $\bigcap_{i \in J} \bigcap_{j \neq i} F_i^0 V = F^{\underline{0}}V$. So the conclusion holds for V unmixed.

For the general case, consider any nontrivial subobject $0 \neq U \subsetneq V$. By the previous Proposition, $R(U) = 0 = R(V/U) = 0$, so by induction we may assume that the conclusion of the Lemma holds for $Q(U)$ and $Q(V/U)$. Let $\pi \colon V \to V/U$ be the canonical map. Then as π compatible with all the filtrations (2.3.12), we have $\pi(Q(V)) \subset Q(V/U)$. Moreover

$$\bar{F}_i^0 V + \sum_{j \neq i} F_j^0 V = \sum_{q_i \geq 0} V_{\{i\}}^{\underline{p}\,\underline{q}} + \sum_{p_j \geq 0 \ (j \neq i)} V_{\{i\}}^{\underline{p}\,\underline{q}}$$

and also for U. Since $U_J^{\underline{pq}} = U \cap V_J^{\underline{p}\,\underline{q}}$, we get

$$U \cap (\bar{F}_i^0 V + \sum_{j \neq i} F_j^0 V) = \bar{F}_i^0 U + \sum_{j \neq i} F_j^0 U.$$

Therefore $Q(U) = U \cap Q(V)$. We thus have a commutative diagram with exact rows:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & Q(U) & \longrightarrow & Q(V) & \xrightarrow{\pi} & Q(V/U) & & \\
 & & \| & & \bigcup & & \| & & \\
0 & \longrightarrow & F^{\underline{0}}U \oplus \bar{F}^{\underline{0}}U & \longrightarrow & F^{\underline{0}}V \oplus \bar{F}^{\underline{0}}V & \xrightarrow{\pi} & F^{\underline{0}}(V/U) \oplus \bar{F}^{\underline{0}}(V/U) & \longrightarrow & 0
\end{array}$$

and so by the 5-lemma, equality holds in the middle. □

(2.3.22) Lemma. *Let I, $J \subset S$ be disjoint, and define*

$$P_{I,J}(V) = \sum_{i \in I, j \in J} \sum_{p_i, q_i, p_j, q_j < 0} V^{\underline{p}\,\underline{q}}.$$

Then the sum of the inclusions

$$R_I(V) \oplus R_J(V) \oplus P_{I,J}(V) \to V$$

is injective.

Proof. This follows immediately from the definitions. □

2.4 The structure theorem

(2.4.1) In this section we define a category which encapsulates the linear algebra data (2.3.10) arising from representations of $\mathcal{G}^r$. The main theorem (2.4.7) states that this category (which is the plectic analogue of the category of vector spaces with three opposed filtrations) is equivalent to $\mathbf{Rep}_k\,\mathcal{G}^r$.

(2.4.2) Define a k-linear category $\mathcal{C}_k^r$ as follows: its objects are tuples $(V, (F_j^\bullet)_j, (\bar{F}_j^\bullet)_j, W_\bullet)$, where:

- V is a finite-dimension k-vector space.
- $F_j^\bullet$, $\bar{F}_j^\bullet$ $(1 \le j \le r)$ are decreasing filtrations on V, and $W_\bullet$ is an increasing $\mathbb{Z}^r$-filtration on V.

subject to the condition:

- For each $\underline{n} \in \mathbb{Z}^r$, there is a $\mathbb{Z}^{2r}$-grading

  $$\mathrm{gr}_{\underline{n}}^W V = \bigoplus_{\underline{p}+\underline{q}=\underline{n}} (\mathrm{gr}_{\underline{n}}^W V)^{\underline{pq}}$$

  such that for every $J \subset S$ and $\underline{a} \in \mathbb{Z}^r$,

  $$\mathrm{gr}_{\underline{n}}^W F_J^{\underline{a}} V = \sum_{\substack{\underline{p},\underline{q} \in \mathbb{Z}^r \\ p_j \ge a_j \ (j \notin J) \\ q_j \ge a_j \ (j \in J)}} (\mathrm{gr}_{\underline{n}}^W V)^{\underline{pq}}. \tag{2.4.2.1}$$

  where $F_J^\bullet$ is defined as in (2.3.11).

The morphisms of $\mathcal{C}_k^r$ are k-linear maps compatible with all of the filtrations.

It is convenient to define a futher category $\mathcal{C}_k^{r,wk}$: the definition is the same as for $\mathcal{C}_k^r$ except that W is only required to be a weak $\mathbb{Z}^r$-filtration. It obviously contains $\mathcal{C}_k^r$ as a full subcategory.

If V is an object of $\mathcal{C}_k^r$ we define

$$d^{\underline{p}\,\underline{q}}(V) = \dim(\mathrm{gr}_{\underline{p}+\underline{q}}^W V)^{\underline{p}\,\underline{q}}$$

(2.4.3) Remark. It is not enough in the definition of $\mathcal{C}_k^r$ to require merely that $\mathrm{gr}_\bullet^W F_j$ and $\mathrm{gr}_\bullet^W \bar{F}_j$ are given by the grading of $\mathrm{gr}^W V$ (since the formation of F_J from $(F_j, \bar{F}_j)$ does not *a priori* commute with passage to $\mathrm{gr}_\bullet^W$).

(2.4.4) From the definition and (1.2.11), if V is an object of $\mathcal{C}_k^r$ then so is every $W_{\underline{n}}V$ with the induced filtrations. By (1.3.7), if $V \ne W_{\underline{n}}$ then the plectic weights of $W_{\underline{n}}V$ form a proper subset of those of V.

(2.4.5) Let V be an object of $\mathcal{C}_k^r$ and $\underline{a}$, $\underline{b} \in \mathbb{Z}^r$. Define an object $V' = V(\underline{a}, \underline{b})$ of the same category, given by the following data: the underlying vector space is V, and the filtrations are given by

$$F_j^p V' = F_j^{p+a_j} V,\ \bar{F}_j^q V' = \bar{F}_j^{q+b_j} V,\ W_{\underline{n}} V' = W_{\underline{n}-\underline{a}-\underline{b}} V$$

Then $V \mapsto V(\underline{a}, \underline{b})$ is an autoequivalence of $\mathcal{C}_k^r$.

If $J \subset S$ we may define another object $\mu_J V$ of $\mathcal{C}_k^r$ as follows: its underlying space is V with the same weight filtrations, and the Hodge filtrations are given by

$$F_j^p(\mu_J V) = \begin{cases} F_j^p V & (j \notin J) \\ \bar{F}_j^p V & (j \in J) \end{cases}, \qquad \bar{F}_j^p(\mu_J V) = \begin{cases} \bar{F}_j^p V & (j \notin J) \\ F_j^p V & (j \in J) \end{cases}.$$

(2.4.6) The construction (2.3.10) defines a functor $\Phi\colon \mathbf{Rep}_k\, \mathcal{G}^r \to \mathcal{C}_k^r$, which is obviously faithful. We denote the image under Φ of $k(\underline{p}, \underline{q})$ (2.3.8) by the same symbol. From the definitions, $\Phi(V \otimes k(\underline{p}, \underline{q})) = \Phi(V)(\underline{p}, \underline{q})$, and the dimensions $d^{\underline{p}\,\underline{q}}$ for V and $\Phi(V)$ are the same.

(2.4.7) Theorem. *The functor Φ is an equivalence of categories.*

(2.4.8) Corollary. *(i) Let V be an object of $\mathcal{C}_k^r$. Then the weak $\mathbb{Z}^r$-filtrations $F_J^\bullet$ are $\mathbb{Z}^r$-filtrations, the weak $\mathbb{Z}^{2r}$-filtrations $F_J^\bullet \cap W_\bullet$ are $\mathbb{Z}^{2r}$-filtrations.*

(ii) The category $\mathcal{C}_k^r$ is abelian, and has tensor product and internal Hom functors, induced by the usual ones on $\mathbf{Vec}_k$. The morphisms in $\mathcal{C}_k^r$ are strictly compatible with all the filtrations in (i).

(2.4.9) First we show that Φ is fully faithful. Let U, V be representations of $\mathbf{Rep}_k\, \mathcal{G}^r$ and $f\colon \Phi(U) \to \Phi(V)$ a morphism in $\mathcal{C}_k^r$. Consider the decompositions $U = \bigoplus U_J^{\underline{p}\,\underline{q}}$, $V = \bigoplus V_J^{\underline{p}\,\underline{q}}$ from (2.3.17). These depend only on the filtrations, and since f is compatible with the filtrations, the definitions imply that $f(U_J^{\underline{p}\,\underline{q}}) \subset V_J^{\underline{p}\,\underline{q}}$. Then the maps (2.3.17.1) fit into a commutative diagram:

$$\begin{array}{ccccc}
U = \bigoplus U^{\underline{p}\,\underline{q}} & \xrightarrow[a]{\sim} & \mathrm{gr}_*^W U = \bigoplus (\mathrm{gr}_{\underline{p}+\underline{q}}^W U)^{\underline{p}\,\underline{q}} & \xleftarrow[a_J]{\sim} & \bigoplus U^{\underline{p}\,\underline{q}} = U \\
{\scriptstyle f}\downarrow & & {\scriptstyle \mathrm{gr}(f)}\downarrow & & {\scriptstyle f}\downarrow \\
V = \bigoplus V^{\underline{p}\,\underline{q}} & \xrightarrow[a]{\sim} & \mathrm{gr}_*^W V = \bigoplus (\mathrm{gr}_{\underline{p}+\underline{q}}^W V)^{\underline{p}\,\underline{q}} & \xleftarrow[a_J]{\sim} & \bigoplus V^{\underline{p}\,\underline{q}} = V
\end{array}$$

So f commutes with every $d_J = a_J^{-1} a$, and therefore by (2.3.6) f is a morphism in $\mathbf{Rep}_k\, \mathcal{G}^r$.

(2.4.10) We now show that Φ is essentially surjective. The proof is in several steps. If V has a single plectic weight — in other words, $V = \mathrm{gr}_{\underline{n}}^W V$ for some $\underline{n}$ — then V has the bigrading $\bigoplus_{\underline{p}+\underline{q}=\underline{n}} (\mathrm{gr}_{\underline{n}}^W V)^{\underline{p},\underline{q}}$, hence comes from a representation of $\mathbb{G}_m^r$. The proof will proceed by induction on the number of plectic weights of V.

(2.4.11) The main inductive step is the following: let $\underline{n} = \underline{a} + \underline{b}$, and $V = W_{\underline{n}} V$ be an object of $\mathcal{C}_k^r$ such that $\mathrm{gr}_{\underline{n}}^W V$ is 1-dimensional of degree $(\underline{a}, \underline{b})$. Assume that $U = W_{<\underline{n}} V$ is in the essential image of Φ. In paragraphs (2.4.12)–(2.4.18) we will show that V is as well.

(2.4.12) Replacing V by the twist $V(-\underline{a}, -\underline{b})$ from (2.4.5), we will assume that $\underline{a} = \underline{b} = \underline{n} = \underline{0}$. For technical reasons we will merely assume that $V \in \mathcal{C}_k^{r,wk}$.

(2.4.13) According to the equivalence (2.3.6) we have a decomposition $U = \bigoplus U^{\underline{pq}}$ together with commuting endomorphisms (∂_j^U) of U, satisfying (2.3.6.1). To extend the action of $\mathcal{G}^r$ from U to V is equivalent to giving a subspace $V^{\underline{00}} \subset V$ complementary to U, together with commuting extensions ∂_j^V of ∂_j^U to V, satisfying (2.3.6.1).

(2.4.14) Lemma. *Suppose $R(U) = 0$. Then V is isomorphic in $\mathcal{C}_k^{r,wk}$ to $k(\underline{0}, \underline{0}) \oplus U$, so in particular lies in $\mathcal{C}_k^r$ and is contained the essential image of Φ.*

Proof. Choose elements $x_J \in F_J^{\underline{0}} V \setminus F_J^{\underline{0}} U$ which have the same image in $\mathrm{gr}_{\underline{0}}^W V = V/U$. Then

$$x_J - x_I \in U \cap (F_I^{\underline{0}} V + F_J^{\underline{0}} V) \subset U \cap \bigcap_{j \notin I \cup J} F_j^0 V \cap \bigcap_{j \in I \cap J} \bar{F}_j^0 V = \bigcap_{j \notin I \cup J} F_j^0 U \cap \bigcap_{j \in I \cap J} \bar{F}_j^0 U.$$

In particular, for any i,

$$x_S - x_\emptyset = (x_S - x_{\{i\}}) + (x_{\{i\}} - x_\emptyset) \in \bar{F}_i^0 U + \bigcap_{j \neq i} F_j^0 U$$

and so by Lemma (2.3.21) $x_S - x_\emptyset \in Q(U) = F^{\underline{0}} U \oplus \bar{F}^{\underline{0}} U$. Applying this instead to $\mu_J V$ as in (2.4.5) we obtain the relation, for any $J \subset S$:

$$x_J - x_{J^c} \in F_J^{\underline{0}} U \oplus F_{J^c}^{\underline{0}} U.$$

Write $x_J - x_{J^c} = y_J - y_{J^c}$ where $y_J \in F_J^{\underline{0}} U$. and set $x'_J = x_J - y_J = x'_{J^c}$. We claim $x = x'_J$ is independent of J. Assuming this, then for any J, $x \in F_J^{\underline{a}} V$ if and only if all a_j are ≤ 0, and so x defines an isomorphism $k(\underline{0}, \underline{0}) \oplus U \simeq V$. As $\mathrm{Hom}_{\mathcal{C}_r^k}(k(\underline{0}, \underline{0}), U) = 0$ (since any morphism is strictly compatible with W) x is unique up to a scalar multiple.

To prove the claim, we have $x'_J = x'_{J^c} \in F_J^{\underline{0}} V \cap F_{J^c}^{\underline{0}} V$, and

$$\bigcap_{j \notin J} F_j^0 U \ni x'_J - x'_\emptyset = x'_{J^c} - x'_\emptyset \in \bigcap_{j \in J} F_j^0 U$$

and therefore $x'_J - x'_\emptyset \in F^{\underline{0}} U$. Likewise,

$$x'_J - x'_S = x'_{J^c} - x'_S \in \bigcap_{j \notin J} \bar{F}_j^0 U \cap \bigcup_{j \in J} \bar{F}_j^0 U = \bar{F}^{\underline{0}} U$$

and so since $x'_S = x'_\emptyset$,

$$x'_J - x'_\emptyset = x'_J - x'_S \in F^{\underline{0}}U \cap \bar{F}^{\underline{0}}U = \{0\}.$$

$\square$

(2.4.15) In the general case, since $R(U)$ is a subrepresentation of U, we may consider the space $V/R(U)$ together with the filtrations induced from (F_j), W. Assuming the truth of the following lemma, the proof of previous lemma applied to $V/R(U)$ then shows that there exists, up to scalar multiplication, a unique family $(x_J) \in \prod_J F_J^{\underline{0}}V$, such that for every I and J, $x_J - x_I \in R(U)$.

(2.4.16) Lemma. *$V/R(U)$ together with the induced filtrations is an object of $\mathcal{C}_k^{r,wk}$.*

Proof. First we check that the $\mathbb{Z}^r$-prefiltration on $V/R(U)$ induced from W commutes with finite limits; in other words, that for every $\underline{m}$, $\underline{n}$ we have $(W_{\underline{m}}V + R(U)) \cap (W_{\underline{n}}V + R(U)) = W_{\underline{m}\wedge\underline{n}}V \cap R(U)$. If $\underline{m} \geq 0$ or $\underline{n} \geq 0$ then as $V = W_{\underline{0}}V$ this is trivial. Otherwise, since W commutes with finite limits and $W_{\underline{0}}V = V$, we have $W_{\underline{m}}V = W_{\underline{m}\wedge\underline{0}}V \subset W_{<\underline{0}}V = U$ and likewise for $\underline{n}$, and so we are reduced to the equality $W_{\underline{m}}(U/R(U)) \cap W_{\underline{n}}(U/R(U)) = W_{\underline{m}\wedge\underline{n}}(U/R(U))$, which holds by (2.3.10.1) since $U/R(U) \in \mathbf{Rep}_k\, \mathcal{G}^r$.

We then need to verify the condition (2.4.2.1). For $\underline{n} < \underline{0}$ it is equivalent to the same condition for $U/R(U)$, which holds since $U/R(U) \in \mathbf{Rep}_k\, \mathcal{G}^r$. For $\underline{n} \not\leq \underline{0}$ we have $W_{\underline{n}}(V/R(U)) = W_{\underline{n}\wedge\underline{0}}(V/R(U))$ and $\underline{n} \wedge \underline{0} < \underline{n}$, hence $\mathrm{gr}^W_{\underline{n}}(V/R(U)) = 0$. It remains to consider $\underline{n} = \underline{0}$. As $\mathrm{gr}^W_{\underline{0}}(V/R(U))$ is 1-dimensional of degree $(\underline{0}, \underline{0})$, (2.4.2.1) can be written as

$$F_J^{\underline{a}}(V/R(U)) \begin{cases} \subset U/R(U) & \text{if some } a_j > 0 \\ \not\subset U/R(U) & \text{otherwise} \end{cases} \tag{2.4.16.1}$$

Now by the same condition for V, if all $a_j \leq 0$ then $F_J^{\underline{a}}V \not\subset U$, so certainly $F_J^{\underline{a}}(V/R(U)) \not\subset U/R(U)$. Otherwise suppose say that $a_1 > 0$. It is enough to show that $F_1^1V + \bar{F}_1^1V \subset U$. But $F_1^1V = F^{(1,-N,\ldots,-N)}V$ for N sufficiently large, and if this were not in U we would have $F^{(1,-N,\ldots,-N)}\mathrm{gr}^W_{\underline{0}}V \neq 0$, which is impossible as $\mathrm{gr}^W_{\underline{0}}V$ is purely of degree $(\underline{0}, \underline{0})$. $\square$

(2.4.17) Let $d_J^U = \exp \partial_J^U = \exp \sum_{j\in J} \partial_j^U$. Extend d_J^U to an automorphism d_J^V of V by requiring that $x_J = d_J^V(x_\emptyset)$. Let

$$y_J = x_J - x_\emptyset \in R(U) \cap \bigcap_{j\notin J} F_j^0U = R_J(U).$$

In terms of the decomposition $V = kx_\emptyset \oplus U$, we have matrix representations:

$$d_J^V = \begin{pmatrix} 1 & 0 \\ y_J & d_J^U \end{pmatrix}, \qquad d_I^V \circ d_J^V = \begin{pmatrix} 1 & 0 \\ y_I + d_I^U(y_J) & d_I^U d_J^U \end{pmatrix}.$$

(2.4.18) Suppose $I \cap J = \emptyset$. Then

$$x_{I\cup J} - x_J \in R(U) \cap \bigcap_{j\in J} \bar{F}_j^0 U \cap \bigcap_{j\notin I\cup J} F_j^0 U = d_J^U(R_I(U))$$

so for some $\xi \in R_I(U)$, $x_{I\cup J} - x_J = d_J^U \xi$. Likewise, for some $\eta \in R_J(U)$, $x_{I\cup J} - x_I = d_I^U \eta$. Now

$$(d_J^U - 1)\xi = \sum_{n\geq 1} \frac{1}{n!}(\partial_J^N)^n \xi \in P_{I,J}(U)$$

and similarly $(d_I^U - 1)\eta \in P_{I,J}(U)$. Write

$$\begin{aligned} y_{I\cup J} &= (x_{I\cup J} - x_J) + y_J = \xi + y_J + (d_J^U - 1)\xi \\ &= (x_{I\cup J} - x_I) + y_I = y_I + \eta + (d_I^U - 1)\eta. \end{aligned}$$

Then by Lemma (2.3.22) both right hand expressions are decompositions in $R_I(U) \oplus R_J(U) \oplus P_{I,J}(U)$, and so $\xi = y_I$, $\eta = y_J$, and $(d_J^U - 1)\xi = (d_I^U - 1)\eta$. Therefore

$$y_{I\cup J} = y_I + d_I^U(y_J) = y_J + d_J^U(y_I)$$

or equivalently, $d_{I\cup J}^V = d_I^V \circ d_J^V = d_J^V \circ d_I^V$. If we write $d_{\{j\}}^V = \exp \partial_j^V$ then $[\partial_{\{i\}}^V, \partial_{\{j\}}^V] = 0$. Now $y_{\{j\}} \in R_j(U)$ and $\partial_j^U(R_j(U)) \subset R_j(U)$, so since $y_{\{j\}} = (\exp \partial_j^V - 1)(x_\emptyset)$, one has

$$\partial_j^V(x_\emptyset) = \frac{\partial_j^U}{\exp \partial_j^U - 1}(y_{\{j\}}) \in R_J(U).$$

Therefore the commuting family of maps (∂_j^V) satisfies (2.3.6.1), and thus V is in the image of Φ. This completes the proof of the step (2.4.11).

(2.4.19) We now consider a general $V \in \operatorname{ob} \mathcal{C}_k^r$, and proceed by induction on the size of the set $\operatorname{wts}(V, W)$ of plectic weights of V. If V has only one plectic weight then V is in the image of Φ. We may therefore assume by induction and (2.4.4) that for every $\underline{n} \in \mathbb{Z}^r$ for which $W_{\underline{n}} V \neq V$, $W_{\underline{n}} V$ is in the image of Φ. There are two cases:

(i) Suppose that $T = \operatorname{wts}(V, W)$ contains $s > 1$ maximal elements $\underline{n}(\alpha)$ $(1 \leq \alpha \leq s)$. Then for every $\underline{n} \in T$, $W_{\underline{n}} V$ is in the essential image of Φ. The complex $\tilde{K}(V, W, \{\underline{n}(\alpha)\})$ of vector spaces (1.2.1.1) is exact since W is a $\mathbb{Z}^r$-filtration. Let V' be the cokernel in $\mathbf{Rep}_k\, \mathcal{G}^r$ of the map

$$\tilde{K}_2 = \bigoplus_{\alpha<\beta} W_{\underline{n}(\alpha)\wedge \underline{n}(\beta)} V \to \bigoplus_{\alpha} W_{\underline{n}(\alpha)} V = \tilde{K}_1.$$

Then V' has the same underlying vector space as V. For every $\underline{n}$ with $\underline{n} \leq \underline{n}(\alpha)$, some α, one has $W_{\underline{n}} V' = W_{\underline{n}} V$ by construction, so $\operatorname{gr}_{\underline{n}}^W V' = \operatorname{gr}_{\underline{n}}^W V$ for all $\underline{n} \in T$. By (2.3.12) the Hodge filtrations on V' are given by

$$F_j^\bullet V' = \sum_\alpha F_j^\bullet W_{\underline{n}(\alpha)} V \subset F_j V.$$

Therefore $(\mathrm{gr}^W_{\underline{n}} V')^{\underline{p}\,\underline{q}} \subset (\mathrm{gr}^W_{\underline{n}} V)^{\underline{p}\,\underline{q}}$ for every $(\underline{p}, \underline{q})$, and so equality holds. Therefore $F^\bullet_j V = F^\bullet_j V'$ and hence $V = V'$ is in the essential image of Φ.

(ii) Otherwise, V has a unique maximal plectic weight $\underline{n}$, so $V = W_{\underline{n}} V$ and $\mathrm{gr}^W_{\underline{n}} V \neq 0$, and by induction $U = W_{<\underline{n}} V$ is in the essential image of Φ. Let $\{x_\alpha \mid 1 \leq \alpha \leq N\}$ be a basis for $\mathrm{gr}^W_{\underline{n}} V$ adapted to the grading (2.4.2.1). Let $\pi \colon V \to \mathrm{gr}^W_{\underline{n}} V$ be the quotient, and $V_\alpha = \pi^{-1}(kx_\alpha)$. Then with the induced filtrations, V_α is in $\mathcal{C}^r_k$ and satisfies the hypotheses of (2.4.11), so is in the essential image of Φ. The kernel of the vector space surjection $\bigoplus V_\alpha \to V$ is the sum-zero subspace $(U^N)^{\Sigma=0}$. Then taking V' to be the cokernel in $\mathbf{Rep}_k\, \mathcal{G}^r$ of $(U^N)^{\Sigma=0} \to \bigoplus V_\alpha$ the same argument as in (i) shows that $V = \sum V_\alpha$ is also in the essential image of Φ.

2.5 Real mixed plectic Hodge structures

(2.5.1) We define a mixed plectic $\mathbb{C}$-Hodge structure to be an object of any of the equivalent categories in Theorem (2.4.7), with $k = \mathbb{C}$.

(2.5.2) We may then define the categories of mixed plectic $\mathbb{R}$-Hodge structures and mixed plectic $\mathbb{R}$-Hodge structures over $\mathbb{R}$ in exactly the same way as (2.2). Specifically, a mixed plectic $\mathbb{R}$-Hodge structure is a finite-dimensional real vector space V, together with

(i) decreasing $\mathbb{Z}$-filtrations $F^\bullet_j$ $(1 \leq j \leq r)$ on $V_\mathbb{C} = V \otimes_\mathbb{R} \mathbb{C}$; and

(ii) a $\mathbb{Z}^r$-filtration W on V

such that $(V_\mathbb{C}, (F_j), (\bar{F}_j), W)$ is a mixed plectic $\mathbb{C}$-Hodge structure. A mixed plectic $\mathbb{R}$-Hodge structure over $\mathbb{R}$ is a mixed plectic $\mathbb{R}$-Hodge structure $(V, (F_j), W)$ together with r commuting involutions $\tau_j \in \mathrm{Aut}_\mathbb{R}(V)$ satisfying

$$\tau_j(F^p_i) = \begin{cases} \bar{F}^p_i & \text{if } i = j \\ F^p_i & \text{otherwise.} \end{cases}$$

(2.5.3) These objects form $\mathbb{R}$-linear Tannakian categories $\mathbb{R}$–HS_r, $\mathbb{R}$–HS^+_r respectively (with the obvious morphisms). They are equivalent to the categories of real representations of the groups $\mathcal{G}^r_\mathbb{R}$ and $(\mathcal{G}_\mathbb{R} \rtimes \mathbb{Z}/2\mathbb{Z})^r$ respectively, where $\mathcal{G}_\mathbb{R}$ is the real form of $\mathcal{G}$ which is denoted $\mathfrak{M}$ in [5, §2].

(2.5.4) There are plectic Tate Hodge structures $\mathbb{R}(\underline{n})$ $(\underline{n} \in \mathbb{Z}^r)$ in both categories. The underlying vector space is $(2\pi i)^{|\underline{n}|}\mathbb{R}$, and the plectic Hodge structure is the unique one for which

$$\mathbb{R}(\underline{n})_\mathbb{C} = (\mathrm{gr}^W_{-2\underline{n}} \mathbb{R}(\underline{n}))^{-\underline{n}, -\underline{n}}.$$

In $\mathbb{R}$–HS^+_r, the real Frobenius τ_j acts as $(-1)^{n_j}$.

In the next section we compute the Ext-groups in these categories.

3 Extensions

3.1 Extensions in the category $\mathcal{C}_k^r$

(3.1.1) Let $(V, (F_j)_j, (\bar{F}_j)_j, W)$ be an object in $\mathcal{C}_k^r$. Let $\mathbf{1}$ be the unit object of $\mathcal{C}_k^r$. We will give an explicit complex computing $R\operatorname{Hom}_{\mathcal{C}_k^r}(\mathbf{1}, V)$.

(3.1.2) We will need the results and methods of [1]. In particular, we start from the more-or-less well-known quasi-isomorphism (in the derived category of k-vector spaces) for the case $r = 1$:

$$R\operatorname{Hom}_{\mathcal{C}_k^1}(\mathbf{1}, V) \xrightarrow{\sim} \left[F^0W_0V \oplus \bar{F}^0W_0V \xrightarrow{(x,y)\mapsto x-y} W_0V\right]. \tag{3.1.2.1}$$

(3.1.3) In [1] this is proved for $\mathbb{R}$-Hodge structures, and the formulation is slightly different. As we will use the same argument below for general r, we spell it out here.

(3.1.4) Let $\tilde{\Gamma}(V) = [\tilde{\Gamma}^0(V) \to \tilde{\Gamma}^1(V)]$ be the complex on the right-hand side of (3.1.2.1). The functors $\tilde{\Gamma}^i\colon \mathcal{C}_k^1 \to \mathbf{Vec}_k$ are exact, since the morphisms in $\mathcal{C}_k^1$ are strictly compatible with the filtrations. There is a canonical isomorphism of functors

$$\operatorname{Hom}_{\mathcal{C}_k^1}(\mathbf{1}, -) \xrightarrow{\sim} \ker(d\colon \tilde{\Gamma}^0 \to \tilde{\Gamma}^1) \tag{3.1.4.1}$$

which takes $f\colon \mathbf{1} = k(0,0) \to V$ to $(f(1), f(1)) \in \ker(d)$.

(3.1.5) Extend $\tilde{\Gamma}$ to a functor $C(\mathcal{C}_k^1) \to C(\mathbf{Vec}_k)$, where $C(-)$ is the category of chain complexes. It is easily seen to have the following properties:

(i) $\tilde{\Gamma}$ takes quasi-isomorphisms to quasi-isomorphisms.

(ii) If $K^\bullet \in C(\mathcal{C}_k^1)$ is bounded below (resp. above) then so is $\tilde{\Gamma}(K^\bullet)$.

Therefore it descends to a functor $\bar{\Gamma}\colon D(\mathcal{C}_k^1) \to D(\mathbf{Vec}_k)$, which preserves boundedness conditions. The isomorphism (3.1.4.1) gives by derivation a map

$$R\operatorname{Hom}_{\mathcal{C}_k^1}(\mathbf{1}, -) \to \bar{\Gamma}(-).$$

This is an isomorphism of functors. To see this, it is enough to know that for an object $V \in \mathcal{C}_k^1$, it induces an isomorphism on H^i. We have already seen this for H^0, and the usual argument for Hodge structures (cf. [1, Lemma 1.8]) shows that it is an isomorphism on H^1. Since $\mathcal{C}_k^1 = \mathbf{Rep}_k\,\mathcal{G}$ and $\mathcal{G}$ is the semidirect product of $\mathbb{G}_m$ and a free prounipotent group, $H^i = 0$ for $i > 1$.

(3.1.6) Define for any $I \subset S = \{1, \dots, r\}$ and any $J \subset I$ the space

$$\tilde{\Gamma}_{I,J}(V) = \bigcap_{j \in I \setminus J} F_j^0 W_{\underline{0}} V \cap \bigcap_{j \in J} \bar{F}_j^0 W_{\underline{0}} V$$

and set $\tilde{\Gamma}_I(V) = \bigoplus_{J \subset I} \Gamma_{I,J}(V)$. These are functors $\tilde{\mathcal{C}}_k^r \to \mathbf{Vec}_k$, and since the morphisms in $\mathcal{C}_k^r$ are strictly compatible with the filtrations, they are exact.

(3.1.7) We may then define a multidimensional complex indexed by $\{0,1\}^r$, with $\tilde{\Gamma}_I(V)$ placed in degree $(\varepsilon_1, \dots, \varepsilon_r)$ where $I = \{j \mid \varepsilon_j = 0\}$. The differentials in the i-direction are

$$d_i \colon \tilde{\Gamma}_I(V) \to \tilde{\Gamma}_{I \setminus \{i\}}(V), \quad (x_J)_{J \subset I} \mapsto (y_J)_{J \subset I \setminus \{i\}}$$

where $y_J = x_J - x_{J \cup \{i\}}$.

(3.1.8) Let $\tilde{\Gamma}_\bullet(V)$ be the associated simple complex. It is a functor from $\mathcal{C}_k^r$ to $C(\mathbf{Vec}_k)$, the category of chain complexes of k-vector spaces, and so extends to a functor $\tilde{\Gamma} \colon C(\mathcal{C}_k^r) \to C(\mathbf{Vec}_k)$. If $r = 1$, this agrees with the functor considered in (3.1.4).

(3.1.9) Lemma. *(i) There is a functorial isomorphism* $\mathrm{Hom}_{\mathcal{C}_k^r}(\mathbf{1}, V) \to H^0(\tilde{\Gamma}_\bullet(V))$.

(ii) $\tilde{\Gamma}$ *takes quasi-isomorphisms to quasi-isomorphisms.*

Proof. (i) In degrees 0 and 1, $\tilde{\Gamma}_\bullet(V)$ is

$$\bigoplus_J F_J^0 W_{\underline{0}} V \to \bigoplus_{i \in S} \bigoplus_{J \subset S \setminus \{i\}} \left[\bigcap_{j \notin J \cup \{i\}} F_j^0 W_{\underline{0}} V \cap \bigcap_{j \in J} \bar{F}_j^p W_{\underline{0}} V \right]$$

which shows that $H^0(\tilde{\Gamma}_\bullet(V)) = \bigcap_j F_j^0 V \cap \bar{F}_j^0 V \cap W_{\underline{0}} V = \mathrm{Hom}(\mathbf{1}, V)$.

(ii) Taking cones, it is enough to check that if $K^\bullet \in C(\mathcal{C}_k^r)$ is acyclic, then so is $\tilde{\Gamma}(K^\bullet)$. This holds since every $F_J \cap W$ is a $\mathbb{Z}^{2r}$-filtration with respect to which morphisms in $\mathcal{C}_k^r$ are strict (2.4.8). $\square$

(3.1.10) By (ii) and the exactness of $\tilde{\Gamma}_I$, just as in the case $r = 1$, the functor $\tilde{\Gamma}$ descends to a functor $\bar{\Gamma}$ on derived categories, preserving boundedness conditions.

(3.1.11) Theorem. *The map of functors* $R\,\mathrm{Hom}_{\mathcal{C}_k^r}(\mathbf{1}, -) \to \bar{\Gamma}$ *induced by (3.1.9)(i) is an isomorphism.*

Proof. It is enough to show that for any simple object of $\mathcal{C}_k^r$, the map induces isomorphisms on H^*. Consider the external tensor product $\boxtimes \colon \mathcal{C}_k^1 \times \cdots \times \mathcal{C}_k^1 \to \mathcal{C}_k^r$. The simple object $k(\underline{p}, \underline{q})$ is isomorphic to $\boxtimes k(p_i, q_i)$. Let us prove the isomorphism of H^* for $V = \boxtimes V_i$, for any $V_i \in \mathcal{C}_k^1$ $(1 \le i \le r)$. But $\tilde{\Gamma}(V)$ is then simply the tensor product of the complexes $\tilde{\Gamma}(V_i)$, and by the Künneth formula for cohomology of $\mathcal{G}^r$,

$$R\,\mathrm{Hom}_{\mathcal{C}_k^r}(\mathbf{1}, \boxtimes V_i) = \bigotimes R\,\mathrm{Hom}_{\mathcal{C}_k^1}(\mathbf{1}, V_i).$$

So the result follows from (3.1.2.1). $\square$

3.2 Extensions of mixed plectic Hodge structure

(3.2.1) Let V be a mixed plectic $\mathbb{R}$-Hodge structure. Then the filtrations F_j, $\bar{F}_j$ are complex conjugate, and W is real. Therefore we may define a complex conjugation c on the complex $\tilde{\Gamma}(V_\mathbb{C})$ by:

$$c\colon \tilde{\Gamma}_{I,J}(V_\mathbb{C}) \to \tilde{\Gamma}_{I,I\setminus J}(V_\mathbb{C}), \qquad x \mapsto (-1)^{r-\#I}\bar{x}$$

and set

$$\tilde{\Gamma}_\mathbb{R}(V) = \tilde{\Gamma}(V_\mathbb{C})^{c=1}.$$

If $r = 1$ then $\tilde{\Gamma}_\mathbb{R}(V) = [F^0 W_0 V_\mathbb{C} \xrightarrow{d} W_0 V(1)]$, with d given by (twice) the projection $V_\mathbb{C} = V \oplus V(1) \to V(1)$ onto the imaginary part.

As passing to c-invariants is exact, we obtain a functor $\bar{\Gamma}_\mathbb{R}$ on the derived category, and one obtains:

(3.2.2) Corollary. *There is a canonical isomorphism of functors*

$$R\operatorname{Hom}_{\mathbb{R}\text{–HS}_r}(\mathbb{R}(\underline{0}), -) \xrightarrow{\sim} \bar{\Gamma}_\mathbb{R}(-)$$

(3.2.3) Likewise, let $V \in \mathbb{R}\text{–HS}_r^+$. Then the real Frobenii τ_j act on $\tilde{\Gamma}_\mathbb{R}(V)$. Let $\tilde{\Gamma}^+_\mathbb{R}(V)$ be the invariants under the groups they generate. In the same way we obtain a functor $\bar{\Gamma}^+_\mathbb{R}$ on the derived category of $\mathbb{R}\text{–HS}_r^+$ and an isomorphism

$$R\operatorname{Hom}_{\mathbb{R}\text{–HS}_r^+}(\mathbb{R}(\underline{0}), -) \xrightarrow{\sim} \bar{\Gamma}^+_\mathbb{R}(-).$$

(3.2.4) Let us compute the Ext-groups for plectic Tate mixed Hodge structures. Now $\mathbb{R}(\underline{n}) = \boxtimes_j \mathbb{R}(n_j)$ is the external tensor product of the Tate objects $\mathbb{R}(n_j)$ in the category $\mathbb{R}\text{–HS}$ of (usual) $\mathbb{R}$-Hodge structures. Therefore

$$R\operatorname{Hom}_{\mathbb{R}\text{–HS}_r}(\mathbb{R}(\underline{0}), \mathbb{R}(\underline{n})) \simeq \bigotimes R\operatorname{Hom}_{\mathbb{R}\text{–HS}}(\mathbb{R}(0), \mathbb{R}(n_j))$$

Using the well-known formulae for the Ext-groups in $\mathbb{R}\text{–HS}$ and $\mathbb{R}\text{–HS}^+$, we see at once that if some n_j is < 0, then $R\operatorname{Hom}_{\mathbb{R}\text{–HS}_r}(\mathbb{R}(\underline{0}), \mathbb{R}(\underline{n})) = 0$. Otherwise it has exactly one non-vanishing cohomology group $\operatorname{Ext}^d_{\mathbb{R}\text{–HS}_r}(\mathbb{R}(\underline{0}), \mathbb{R}(\underline{n})) \simeq \mathbb{R}$ in degree $d = \#\{j \mid n_j > 0\}$.

So passing to invariants under the τ_j, one sees that $\operatorname{Ext}^d_{\mathbb{R}\text{–HS}_r^+}(\mathbb{R}(\underline{0}), \mathbb{R}(\underline{n})) \simeq \mathbb{R}$ if each n_j is either zero or is odd and positive (with d as before), and vanishes otherwise.

3.3 Hilbert modular varieties

(3.3.1) Let F be a totally real field of degree $r > 1$, with ring of integers $\mathfrak{o}$, and $\Sigma = \operatorname{Hom}(F, \mathbb{R})$ the set of real embeddings. Let X be a (complex, smooth, connected) Hilbert modular variety. Thus $X = \Gamma\backslash\mathfrak{H}^\Sigma$ for a (sufficiently small) subgroup $\Gamma \subset SL_2(\mathfrak{o})$ acting on r copies of $\mathfrak{H}$, the complex upper half-place. In this section we will construct a mixed plectic $\mathbb{R}$-structure on $H^*(X,\mathbb{R})$, compatible with the usual mixed Hodge structure, which is in some sense canonical (for example, one easily can show that it is preserved by Hecke operators).

(3.3.2) We first review the cohomology of X. For further details, see [7] for the topological and [6, Ch. III] for the Hodge theory. Let $X \xrightarrow{j} \overline{X}$ be the minimal compactification of X. Its boundary is a finite set Y of points of X; let $i\colon Y \longrightarrow X$ be the inclusion, and $\nu = \#Y$. One has the usual long exact sequence of boundary cohomology (with coefficients in a field E of characteristic zero, say):

$$\cdots \to H^n_c(X,E) \xrightarrow{\rho^n} H^n(X,E) \xrightarrow{\sigma^n} H^n(Y, i^*Rj_*E) \to \dots \qquad (3.3.2.1)$$

Write $H^n_!(X,E) = \operatorname{im}(H^n_c(X,E) \to H^n(X,E))$ for the interior cohomology.

The boundary cohomology was computed by Harder [7, 8], together with the maps ρ^n, σ^n, using the Borel-Serre compactification. For each $y \in Y$, the Borel-Serre boundary component is a bundle over a real torus isogenous to $\mathfrak{o}^* \otimes_{\mathbb{Z}} (\mathbb{R}/\mathbb{Z})$, with fibre isogenous to $\mathfrak{o} \otimes_{\mathbb{Z}} (\mathbb{R}/\mathbb{Z})$. Its rational cohomology is

$$(R^\bullet j_*\mathbb{Q})_y \simeq \textstyle\bigwedge^\bullet_{\mathbb{Z}} \mathfrak{o}^* \otimes_{\mathbb{Z}} (\mathbb{Q}e_0 \oplus \mathbb{Q}e_r)$$

with e_n in degree n. Moreover the maps ρ^n, σ^n have the property:

- For $r < n \le 2r-1$, ρ^n is injective.
- For $r \le n < 2r-1$, σ^n is surjective.
- The image of σ^{2r-1} has codimension 1.

In particular, by Poincaré duality one has that $H^n_!(X,\mathbb{Q}) = H^n(X,\mathbb{Q})$ for $0 < n < r$, and for $r \le n \le 2r-2$ there is an exact sequence

$$0 \to H^n_!(X,E) \to H^n(X,E) \to H^n(Y, i^*Rj_*E) \to 0.$$

(3.3.3) Suppose from now on that $E \subset \mathbb{R}$. Then the exact sequence (3.3.2.1) is an exact sequence of mixed E-Hodge structure. The Hodge structure on the boundary cohomology is [6, §III.7]

$$(R^n j_*E)_y = \begin{cases} \bigwedge^n_{\mathbb{Z}} \mathfrak{o}^* \otimes_{\mathbb{Z}} E(0) & (0 \le n \le r-1) \\ \bigwedge^{n-r}_{\mathbb{Z}} \mathfrak{o}^* \otimes_{\mathbb{Z}} E(-r) & (r \le n \le 2r-1) \end{cases}$$

(3.3.4) The interior cohomology is pure, and is a direct sum of two parts. The first of these (denote H_A^* in [7]) comes from algebraic classes associated to the r standard line bundles $\mathcal{L}_\iota$ ($\iota \in \Sigma$) on X. Their cohomology classes ξ_ι belong to $H_!^2(X, \mathbb{Q}(1))$. The E-subalgebra generated by $\{(2\pi i)^{-1}\xi_\iota\}$ is the sub-E-Hodge structure

$$H_A^*(X, E) \simeq (E \oplus E(-1))^{\otimes\Sigma}_{\deg<2r}.$$

of $H_!^*(X, E)$ (where $E(-1)$ is in degree 2).

(3.3.5) The second part is the cuspidal cohomology $H^r_{\mathrm{cusp}}(X, E)$. It has dimension 2^r times the dimension of the space of holomorphic cusp forms of weight $(2, \dots, 2)$ for Γ, and for $E = \mathbb{R}$ can be interpreted as relative Lie algebra cohomology for $\mathfrak{gl}_2(\mathbb{R})^\Sigma$. As explained in [9, §2], there is a tensor product decomposition of the Lie algebra cohomology which puts on $H^r_{\mathrm{cusp}}(X, \mathbb{R})$ a canonical pure plectic $\mathbb{R}$-structure, of plectic weight $(1, \dots, 1)$.

(3.3.6) The real boundary cohomology can be given a canonical plectic Hodge structure, as a sum of copies of $\mathbb{R}(0, \dots, 0)$ and $\mathbb{R}(-1, \dots, -1)$.

(3.3.7) The algebraic part H_A^* also carries a canonical plectic Hodge structure, for which ξ_ι (which is represented by a $(1,1)$-form on the ι-th copy of $\mathfrak{H}$) is of Hodge type $(\underline{1}_\iota, \underline{1}_\iota)$. So $H_A^{2m}(X, \mathbb{R})$ is pure of simple weight $2m$ but has several plectic weights.

(3.3.8) The "Manin-Drinfeld principle" shows that $H^r_{\mathrm{cusp}}(X, \mathbb{Q})$ is (uniquely) a direct factor of $H^r(X, \mathbb{Q})$ (not just of the interior cohomology), and thus the mixed Hodge structure on X is completely determined by the extension of the boundary cohomology by H_A^*.

(3.3.9) In many degrees this is enough to put a canonical plectic Hodge structure on $H^n(X, \mathbb{R})$. The cases $n = 0$, $2r$ are trivial, and in degree $1 \le n < r$, $H^n(X, \mathbb{R}) = H_!^n(X, \mathbb{R})$ so has a canonical plectic pure $\mathbb{R}$-Hodge structure. In odd degrees n with $r < n < 2r-1$, $H^n(X, \mathbb{R})$ equals the boundary cohomology, so therefore has a canonical plectic Hodge structure. The same holds for $n = 2r-1$ because of the exact sequence

$$0 = H_!^{2r-1}(X, \mathbb{R}) \to H^{2r-1}(X, \mathbb{R}) \to \bigoplus_{y\in Y} \mathbb{R}(-r) \xrightarrow{\Sigma} \mathbb{R}(-r) = H_c^{2r}(X, \mathbb{R}) \to 0.$$

(3.3.10) To describe the mixed Hodge structure completely, it therefore suffices to consider the following possible extensions: if r is even

$$0 \to H_A^r(X, \mathbb{R}) \to H^r(X, \mathbb{R})/H^r_{\mathrm{cusp}}(X, \mathbb{R}) \to \mathbb{R}(-r)^\nu \to 0 \tag{3.3.10.1}$$

and for $\frac{r}{2} < n < r$

$$0 \to H_!^{2n}(X, \mathbb{R}) = H_A^{2n}(X, \mathbb{R}) \to H^{2n}(X, \mathbb{R}) \to {\textstyle\bigwedge}_{\mathbb{Z}}^{2n-r} \mathfrak{o}^* \otimes_{\mathbb{Z}} \mathbb{R}(-r)^\nu \to 0. \tag{3.3.10.2}$$

(3.3.11) Theorem. *The extensions* (3.3.10.1), (3.3.10.2) *are (uniquely) split, with the exception of degree* $2r-2$*, when the class of the extension is described in (3.3.13) below.*

For $r = 2$ this was proved by Caspar [3], using Eisenstein cohomology. The general case (which is proved the same way) will be found in the forthcoming PhD thesis of C. Davidescu. It follows that in these degrees, apart from $2r-2$, $H^n(X, \mathbb{R})$ carries a unique (unmixed) plectic $\mathbb{R}$-Hodge structure for which these sequences are exact sequences in $\mathbb{R}\text{–HS}_r$.

(3.3.12) In degree $2r-2$ the extension may be written (after fixing an orientation $\wedge^{r-1}\mathfrak{o}^*/(\text{torsion}) \simeq \mathbb{Z}$) as

$$0 \to \bigoplus_{\iota\colon F\to\mathbb{R}} \mathbb{R}(1) \to H^{2r-2}_?(X,\mathbb{R})(r) \to \mathrm{Hom}(\mathfrak{o}^*,\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{R}(0) \to 0$$

where $H^{2r-2}_? = H^{2r-2}$ if $r \neq 2$, and $H^2_? = H^2/H^2_{\text{cusp}}$. The ι-component of the direct sum is generated by $\wedge_{\alpha\neq\iota}\xi_\alpha$. The extension is determined by the pushouts to these components, for each $\iota\colon F \to \mathbb{R}$:

$$0 \to \mathbb{R}(1) \to E_\iota \to \mathrm{Hom}(\mathfrak{o}^*,\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{R}(0) \to 0.$$

The extension class of E_ι is an element of

$$\mathrm{Hom}(\mathfrak{o}^*,\mathbb{Z}) \otimes_{\mathbb{Z}} \mathrm{Ext}^1_{\mathbb{R}\text{–HS}}(\mathbb{R}(0),\mathbb{R}(1)) = \mathrm{Hom}(\mathfrak{o}^*,\mathbb{R}).$$

(3.3.13) Theorem. *The image of the class of* E_ι *in* $\mathrm{Hom}(\mathfrak{o}^*,\mathbb{R})$ *is a non-zero rational multiple of the regulator homomorphism* $u \mapsto \log|\iota(u)|$.

(3.3.14) The subspace of cohomology generated by $\wedge_{\alpha\neq\iota}\xi$ is isomorphic, as plectic Hodge structure, to the Tate object $\mathbb{R}(-\underline{n})$ with $n_\alpha = 1$ $(\alpha \neq \iota)$, $n_\iota = 0$. The boundary cohomology in degree $(2r-2)$ is a sum of copies of $\mathbb{R}(-1,\ldots,-1)$. To give a canonical plectic mixed $\mathbb{R}$-Hodge structure on $H^{2r-2}(X,\mathbb{R})$, it therefore is enough to show that each extension E_ι comes from a unique extension of plectic Hodge structures

$$0 \to \mathbb{R}(\underline{1}_\iota) \to E_\iota \to \mathrm{Hom}(\mathfrak{o}^*,\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{R}(\underline{0}) \to 0.$$

But the Künneth formula gives

$$\mathrm{Ext}^1_{\mathbb{R}\text{–HS}_r}(\mathbb{R}(\underline{0}),\mathbb{R}(\underline{1}_\iota)) = \mathrm{Ext}^1_{\mathbb{R}\text{–HS}}(\mathbb{R}(0),\mathbb{R}(1)) \bigotimes_{\alpha\neq\iota} \mathrm{Hom}_{\mathbb{R}\text{–HS}}(\mathbb{R}(0),\mathbb{R}(0)) \simeq \mathbb{R}$$

and so the forgetful map $\mathrm{Ext}^1_{\mathbb{R}\text{–HS}_r}(\mathbb{R}(\underline{0}),\mathbb{R}(\underline{1}_\iota)) \to \mathrm{Ext}^1_{\mathbb{R}\text{–HS}}(\mathbb{R}(0),\mathbb{R}(1))$ is an isomorphism; so E_ι lifts to a unique element of $\mathbb{R}\text{–HS}_r$.

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