Motives for modular forms

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Introduction

In [**DeFM**], Deligne constructs l-adic parabolic cohomology groups attached to holomorphic cusp forms of weight ≥ 2 on congruence subgroups of $SL_2(\mathbf{Z})$. These groups occur in the l-adic cohomology of certain smooth projective varieties over $\mathbf{Q}$—the Kuga-Sato varieties— which are suitably compactified families of products of elliptic curves. In view of Grothendieck's conjectural theory of motives it is natural to hope that the parabolic cohomology groups can be directly constructed as the kernel of some projectors (in a suitable ring of algebraic correspondences) acting on the cohomology of these varieties. In this note we show that this can be done; in fact the projector we use belongs to the group algebra of a finite group of automorphisms of the Kuga-Sato variety.

The existence of such motives has been speculated for some time (see for instance the introduction to [**La**]). In [**Ja2**] Jannsen has shown how to construct motives for modular forms in the category of motives defined by absolute Hodge cycles; his construction is very general and should apply to automorphic forms on other groups.

In §1 we state our results. As an application, we exhibit a relation between the the p-adic representation of $\mathrm{Gal}(\overline{\mathbf{Q}}_p/\mathbf{Q}_p)$ attached to a modular form, and the action of the Hecke operator T_p. This was suggested by Fontaine, and I am very grateful to him for discussions on this topic. I am also indebted to Messing for drawing my attention to the results of [**GM**], and to Jannsen, Rapoport and Schappacher for useful discussions.

1. Definitions and results.

1.0.0. Consider integers $n \geq 3$, $k \geq 1$. (We do not treat here the case $k = 0$, which corresponds to cusp forms of weight 2; the associated motives are then given by the Jacobians of modular curves, and are well understood.) The properties of modular curves and universal families of elliptic curves used below can be found in [**DeFM**], [**DR**] and [**KM**].

1.0.1. Denote by M_n the modular curve over $\mathbf{Q}$ parametrising elliptic curves with level n structure, and let $j : M_n \hookrightarrow \overline{M}_n$ be its smooth compactification (classifying generalised elliptic curves). Then M_n is the complement in $\overline{M}_n$ of the cuspidal subscheme M_n^∞, a finite sum of copies of $\operatorname{Spec}\mathbf{Q}(\zeta_n)$.

1.0.2. Write $\pi : X_n \to M_n$ for the universal elliptic curve with level n-structure (which exists as $n \geq 3$), and $\bar\pi : \overline{X}_n \to \overline{M}_n$ for the universal generalised elliptic curve. Thus $\overline{X}_n$ is a smooth and proper $\mathbf{Q}$-scheme. The open subscheme $\overline{X}_n^*$ on which $\bar\pi$ is smooth is the Néron model of X_n over $\overline{M}_n$, and the fibres of $\bar\pi$ over M_n^∞ are isomorphic to the standard Néron n-gon, which we denote C_n.

1.0.3. Let $\bar\pi_k : \overline{X}_n^k \to \overline{M}_n$ be the k-fold fibre product of $\overline{X}_n$ with itself over $\overline{M}_n$, and $X_n^k = \bar\pi_k^{-1}(M_n)$. If $k \geq 2$ then $\overline{X}_n^k$ is singular; denote by $\overline{\overline{X}}_n^k$ the canonical desingularisation of $\overline{X}_n^k$ constructed by Deligne in [**DeFM**], Lemmes 5.4, 5.5. (In §3 below we shall give an alternative description of $\overline{\overline{X}}_n^k$.) Let

$$\begin{aligned}\overline{X}_n^{k,\infty} &= \bar\pi_k^{-1}(M_n^\infty) \\ &= \overline{X}_n^k - X_n^k\end{aligned}$$

and write $\overline{\overline{X}}_n^{k,\infty} = \overline{\overline{X}}_n^k - X_n^k$.

1.1.0. The level n structure on $\overline{X}_n$ gives a homomorphism of group schemes over $\overline{M}_n$

$$(\mathbf{Z}/n)^2 \times \overline{M}_n \hookrightarrow \overline{X}_n^*.$$

Therefore $(\mathbf{Z}/n)^2$ acts by translations on $\overline{X}_n$. Inversion in the fibres defines an involution of $\overline{X}_n$, and we obtain an action of the semi-direct product $(\mathbf{Z}/n)^2 \rtimes \boldsymbol{\mu}_2$ on $\overline{X}_n$.

1.1.1. Let Σ_k be the symmetric group on k letters, acting on $\overline{X}_n^k$ by permuting the factors of the fibre product. Then the wreath product

$$\begin{aligned}\Gamma_k &\stackrel{\text{def}}{=} ((\mathbf{Z}/n)^2 \rtimes \boldsymbol{\mu}_2) \wr \Sigma_k \\ &= ((\mathbf{Z}/n)^2 \rtimes \boldsymbol{\mu}_2)^k \rtimes \Sigma_k\end{aligned}$$

acts on $\overline{X}_n^k$ by automophisms in the fibres of $\bar{\pi}_k$. By the canonical nature of the desingularisation, this extends to an action of Γ_k on $\overline{\overline{X}}_n^k$.

1.1.2. Let $\epsilon : \Gamma_k \to \{\pm 1\}$ be the homomorphism which is trivial on $(\mathbf{Z}/n)^{2k}$, is the product map on $\boldsymbol{\mu}_2^k$ and is the sign character on Σ_k. Let $\Pi_\epsilon \in \mathbf{Z}[1/2n.k!][\Gamma_k]$ be the projector attached to ϵ, and for any $\mathbf{Z}[1/2n.k!]$-module V on which Γ_k acts, write $V(\epsilon)$ for $\Pi_\epsilon(V)$.

1.2.0. Recall the parabolic cohomology groups (in Betti and l-adic theories) attached to the space of cusp forms of weight $k+2$ and level n:

$$\begin{aligned}{}_n^kW_B &= H^1(\overline{M}_n(\mathbf{C}), j_* \operatorname{Sym}^k R^1\pi_*\mathbf{Q}) \\ {}_n^kW_l &= H^1_{\text{ét}}(\overline{M}_n \otimes \overline{\mathbf{Q}}, j_* \operatorname{Sym}^k R^1\pi_*\mathbf{Q}_l) \\ &\simeq {}_n^kW_B \otimes_{\mathbf{Q}} \mathbf{Q}_l.\end{aligned}$$

They are subquotients of $H_B^{k+1}(\overline{\overline{X}}_n^k)$, $H_l^{k+1}(\overline{\overline{X}}_n^k)$ respectively. In [**DeFM**] these groups are defined instead as the image of $H^1_c(M_n, -)$ in $H^1(M_n, -)$, but it is well known that these definitions are equivalent. There is also a long exact sequence (in, say, l-adic cohomology)

$$0 \longrightarrow {}_n^kW_l \longrightarrow H^1_{\text{ét}}(M_n \otimes \overline{\mathbf{Q}}, \operatorname{Sym}^k R^1\pi_*\mathbf{Q}_l) \longrightarrow H^0_{\text{ét}}(M_n^\infty \otimes \overline{\mathbf{Q}}, \mathbf{Q}_l(-k-1)) \longrightarrow 0.$$

1.2.1. Theorem. *For $? = B$ or l,*

$${}_n^kW_? = H_?^*(\overline{\overline{X}}_n^k)(\epsilon).$$

1.2.2. We call a *Chow motive* an object of the category of motives [**Ma**] over $\mathbf{Q}$ defined by taking as morphisms

$$\operatorname{Hom}(h(Y), h(X)) = CH^{\dim Y}(X \times Y) \otimes \mathbf{Q},$$

the Chow group of cycles of codimension $\dim Y$ modulo rational equivalence. The pair $(\overline{\overline{X}}_n^k, \Pi_\epsilon)$ then defines a Chow motive, which we denote ${}_n^k\mathcal{W}$. The theorem shows that the realisations of ${}_n^k\mathcal{W}$ are then the parabolic cohomology groups.

1.2.3. We call a *Grothendieck motive* (with coefficients in a number field L) an object of the category of motives over $\mathbf{Q}$ in which $\operatorname{Hom}(h(Y), h(X))$ is the group of algebraic cycles on $X \times Y$ of codimension $\dim Y$, tensored with L, modulo homological equivalence. The image of ${}_n^k\mathcal{W}$ in this category can be decomposed under action of the Hecke algebra for suitable L, and we shall obtain the following result.

1.2.4. Theorem. *Let $f = \sum_{m=1}^{\infty} a_m q^m$ be a normalised newform of weight w, level n and character χ. Let L be the field generated by the coefficients a_m, $m \geq 1$. Then there is a Grothendieck motive $M(f)$ over $\mathbf{Q}$ with coefficients in E, with the following properties:*

(i) If $p \nmid nl$, and λ is a prime of L dividing l, then the λ-adic realisation $H_\lambda(M(f))$ of $M(f)$ is unramified at p, and the characteristic polynomial of a geometric Frobenius at p is the Hecke polynomial

$$\mathbf{T}_p(X) = X^2 - a_pX + \chi(p)p^{w-1}.$$

(ii) If $p \nmid n$ (and $p \geq w$) and π is a prime of L dividing p, then the π-adic realisation $H_\pi(M(f))$ is a crystalline representation of $\operatorname{Gal}(\overline{\mathbf{Q}}_p/\mathbf{Q}_p)$, and the characteristic polynomial of ϕ on the associated ϕ-filtered module is equal to $\mathbf{T}_p(X)$.

1.2.5. *Remark.* The hypothesis $p \geq w$ is required as the results of Fontaine and Messing [**FM**] are only proved for primes p which are greater than the dimensions of the varieties in question (which is $w-1$ in our case). This restriction will be removed by forthcoming work of Faltings.

1.2.6. *Remark.* One should be able to decompose ${}^k_n\mathcal{W}$ already in the category of Chow motives (as is the case for $k=0$); but this seems very hard without assuming the standard conjectures.

1.3.0. Theorem 1.2.1 will be a consequence of a slightly more general result. Consider a twisted Poincaré duality theory H^*, H_* (in the sense of [**BlO**]) for varieties over $\mathbf{Q}$ with coefficients in a field of characteristic zero. Assume that the projective bundle axiom holds. Then for any smooth S there is a canonical decomposition

$$H_*(\mathbf{G}_m \times S, \bullet) = H_{*-2}(S, \bullet - 1) \oplus H_{*-1}(S, \bullet)$$

in which inversion $x \mapsto x^{-1}$ on $\mathbf{G}_m$ acts by $+1$ on the first summand, and by -1 on the second. As a simple consequence, we have:

1.3.1. Lemma. *i) Let ϵ_k be the restriction of the character ϵ to the subgroup $\boldsymbol{\mu}_2^k \rtimes \Sigma_k = \boldsymbol{\mu}_2 \wr \Sigma_k$ of Γ_k. Then for any k,*

$$H_*(\mathbf{G}_m^k \times S, \bullet)(\epsilon_k) = H_{*-k}(S, \bullet).$$

ii) Let $D \simeq \mathbf{G}_m^{k-1}$ be the kernel of the product map

$$\mathbf{G}_m^k \to \mathbf{G}_m, \qquad (x_1, \ldots, x_k) \mapsto x_1 \cdots x_k$$

which is stable under the action of Σ_k. Then

$$H_*(D \times S, \bullet)(\mathrm{sgn}) = H_{*-k+1}(S, \bullet). \qquad \blacksquare$$

1.3.2. Since $\overline{X}_n^{1,\infty}$ is a union of Néron n-gons there is a (non-canonical) isomorphism

$$(\overline{X}_n^{k,\infty})^{\mathrm{reg}} \xrightarrow{\sim} M_n^\infty \times (\mathbf{G}_m \times \mathbf{Z}/n)^k.$$

Define a map ρ as the composite

$$\begin{array}{ccccc} H_*(X_n^k, \bullet) & \xrightarrow{\partial} & H_{*-1}(\overline{X}_n^{k,\infty}, \bullet)(\epsilon) & \longrightarrow & H_{*-1}(M_n^\infty \times (\mathbf{G}_m \times \mathbf{Z}/n)^k, \bullet)(\epsilon) \\ & & \rho & & \downarrow \simeq \\ & & & & H_{*-k-1}(M_n^\infty, \bullet) \end{array}$$

(the vertical arrow being an isomorphism by 1.3.1). Let $j\colon X_n^k \hookrightarrow \overline{\overline{X}}_n^k$ be the inclusion morphism.

1.3.3. Theorem. *There is a long exact sequence*

$$\ldots \longrightarrow H_*(\overline{\overline{X}}_n^k, \bullet)(\epsilon) \xrightarrow{j^*} H_*(X_n^k, \bullet)(\epsilon) \xrightarrow{\rho} H_{*-k-1}(M_n^\infty, \bullet) \longrightarrow H_{*-1}(\overline{\overline{X}}_n^k, \bullet)(\epsilon) \longrightarrow \ldots.$$

1.3.4. To deduce 1.2.1, recall that translation by sections of finite order acts trivially on $R^1\pi_*\mathbf{Q}_l$, and by the Künneth formula and the Leray spectral sequence (cf. the proof of 5.3 of [**DeFM**]) one has

$$H^*(X_n^k \otimes \overline{\mathbf{Q}}, \mathbf{Q}_l)(\epsilon) \xrightarrow{\sim} H^1(M_n \otimes \overline{\mathbf{Q}}, \mathrm{Sym}^k R^1\pi_*\mathbf{Q}_l)$$

and similarly in Betti cohomology. From 1.3.3 we deduce

$$H_l^i({}^k_n\mathcal{W}) = 0 \qquad \text{for} \quad i \neq k+1,\, k+2$$

and an exact sequence

$$\begin{aligned} 0 \longrightarrow H_l^{k+1}({}^k_n\mathcal{W}) \longrightarrow H^1(M_n \otimes \overline{\mathbf{Q}}, \mathrm{Sym}^k R^1\pi_*\mathbf{Q}_l) \xrightarrow{\rho} \\ H^0(M_n^\infty \otimes \overline{\mathbf{Q}}, \mathbf{Q}_l(-k-1)) \xrightarrow{\sigma} H_l^{k+2}({}^k_n\mathcal{W}) \longrightarrow 0. \end{aligned}$$

Since $k \neq 0$ the source and target of σ have different weights, so $\sigma = 0$ and the result follows by 1.2.0.

1.4.0. Other cohomology theories can be used in 1.3.3. For example, consider Beilinson's motivic cohomology $H_\mathcal{M}$ (denoted $H_\mathcal{A}$ in [**Be1**])

$$H^i_\mathcal{M}(X, \mathbf{Q}(j)) \stackrel{\mathrm{def}}{=} K_{2j-i}^{(j)}(X)$$

the q^j-eigenspace for the Adams operators ψ^q on $K_{2j-i}(X) \otimes \mathbf{Q}$. By Borel's theorem $H^0_\mathcal{M}(M_n^\infty, \mathbf{Q}(j)) = 0$ if $j > 0$, and 1.3.1 then gives:

1.4.1. Corollary. *There is, for every $l \geq 0$, an exact sequence*

$$0 \longrightarrow H^{k+2}_{\mathcal{M}}({}^k_n\mathcal{W}, \mathbf{Q}(k+l+2)) \longrightarrow H^{k+2}_{\mathcal{M}}(X^k_n, \mathbf{Q}(k+l+2))(\epsilon) \xrightarrow{\rho} H^1_{\mathcal{M}}(M^\infty_n, \mathbf{Q}(l+1)).$$

1.4.2. An analogous sequence exists in Deligne-Beilinson ("absolute Hodge") cohomology $H^*_{\mathcal{D}}(-, \mathbf{R}(\bullet))$. Using ideas of Beilinson (notably the Eisenstein symbol map [**Be2**]) one may then obtain results concerning the leading coefficients at $s = -l$ of the L-series $L({}^k_n\mathcal{W}, s)$ (which are products of Hecke L-series for cusp forms of weight $k+2$). The details will appear elsewhere [**Sc3**].

2. Products of double points.

2.0.0. For this section, S denotes the spectrum of a discrete valuation ring R with uniformiser π and residue field κ.

2.0.1. For an integer $k \geq 1$, write

$$E^k = S[x_1, y_1, \ldots, x_k, y_k]/(x_i y_i - \pi)_{1 \leq i \leq k}.$$

Let F^k be the special fibre $\{\pi = 0\}$ of X; it has a filtration

$$F^k = F^k_k \supset F^k_{k-1} \supset \ldots \supset F^k_0 \supset F^k_{-1} = \emptyset$$

where F^k_p is the closed subscheme of F^k on which some $(k-p)$ pairs of coordinates (x_i, y_i) simultaneously vanish. The singular locus of E^k is F^k_{k-2}.

2.0.2. Proposition. *Let $\phi : \hat{E}^k \to E^k$ be the blowing-up of the reduced point F^k_0, and write $\hat{F}^k_p$ for the proper transform of F^k_p under ϕ, for $1 \leq p \leq k-1$.*

i) $\hat{E}^k$ has an open covering by $2k$ copies U_i of $\mathbf{A}^1 \times E^{k-1}$.

ii) $U_i \cap \hat{F}^k_p = \mathbf{A}^1 \times F^{k-1}_{p-1}$ for $1 \leq p \leq k-1$. ■

2.1.0. Now fix k, and define $E\langle 0\rangle = E^k$, $F\langle 0\rangle = F^k_0$. For $1 \leq p \leq k-1$, define inductively $\phi_p : E\langle p\rangle \to E\langle p-1\rangle$ to be the blowing-up along the closed subscheme $F\langle p-1\rangle \subset E\langle p-1\rangle$, and $F\langle p\rangle$ to be the proper transform of F^k_p in $E\langle p\rangle$. Induction on p and 2.0.2 show:

2.1.1. Proposition. *For $0 \leq p \leq k-1$, there is an open covering of $E\langle p\rangle$ by copies V_i of $\mathbf{A}^p \times E^{k-p}$ such that $F\langle p\rangle \cap V_i = \mathbf{A}^p \times F^{k-p}_0$. In particular, $F\langle p\rangle \subseteq E\langle p\rangle^{\text{sing}}$ for $0 \leq p \leq k-2$, and $\widetilde{E} \overset{\text{def}}{=} E\langle k-1\rangle$ is regular.* ■

2.1.2. Remark. The desingularisation $\widetilde{E}$ is the same as that of Lemme 5.5 of [**DeFM**]; we leave the verification of this as an exercise.

2.1.3. Define $G\langle p\rangle \subset E\langle p\rangle$ to be the locally closed subscheme

$$G\langle p\rangle = E\langle p\rangle^{\text{reg}} \cap \phi_p^{-1}\left(F\langle p-1\rangle\right).$$

By 2.1.1, the composite

$$\psi_p \overset{\text{def}}{=} \phi_{k-1} \circ \cdots \circ \phi_{p+1} : \widetilde{E} \longrightarrow E\langle p\rangle$$

is an isomorphism over $E\langle p\rangle^{\text{reg}} \supset G\langle p\rangle$. If we define $W_p \overset{\text{def}}{=} \psi_p^{-1}\left(E\langle p\rangle^{\text{sing}}\right)$ for $0 \leq p \leq k-2$, then the special fibre $\widetilde{F}$ of $\widetilde{E}$ over S has a filtration

$$\widetilde{F} \supset W_0 \supset W_1 \supset \ldots \supset W_{k-2} \supset W_{k-1} = \emptyset$$

by closed subschemes such that

$$W_p - W_{p+1} \xrightarrow{\sim} G\langle p\rangle \quad \text{for } 0 \leq p \leq k-2$$

and

$$\widetilde{F} - W_0 \xrightarrow{\sim} F_k^k - F_{k-2}^k.$$

2.2.0. Consider the projective space $\mathbf{P}_\kappa^{2r-1}$ over κ with homogeneous coordinates x_1, y_1,..., x_l, y_l, and let P_r be the closed subscheme defined by the equations

$$x_1 y_1 = \ldots = x_r y_r;$$

thus P_k is the projectivised tangent cone to E^k at the point F_0^k. Define open subschemes:

$$\begin{aligned} P_r' &= \{\text{no two pairs } (x_i, y_i) \text{ of coordinates vanish simultaneously}\} \\ P_r'' &= \{\text{no pair } (x_i, y_i) \text{ of coordinates vanishes}\} \\ P_r^* &= \{\text{no single coordinate } x_i \text{ or } y_i \text{ vanishes}\}. \end{aligned}$$

Then by 2.1.1 and 2.1.3, the fibration $G\langle p\rangle \to F\langle p\rangle$ is locally on $F\langle p\rangle$ isomorphic to the product $P_{k-p}' \times \mathbf{A}^p \to \mathbf{A}^p$.

2.3.0. The wreath product $\boldsymbol{\mu}_2 \wr \Sigma_k$ acts on E^k and on P_k by permutations of the coordinates which leave the set of pairs (x_i, y_i) unchanged. It permutes the $2^p \cdot \binom{k}{p}$ irreducible components of F_p^k transitively, and the stabiliser of a component is isomorphic to $(\boldsymbol{\mu}_2 \wr \Sigma_{k-p}) \times \Sigma_p$, the subgroup $\boldsymbol{\mu}_2 \wr \Sigma_{k-p}$ acting trivially on the component.

The construction of the resolution $\widetilde{E} \to E^k$ is invariant under $\boldsymbol{\mu}_2 \wr \Sigma_k$. In particular the group acts on the fibration $G\langle p\rangle \to F\langle p\rangle$.

2.3.1. Proposition. *Let C be an irreducible component of $F\langle p\rangle$, and consider the subgroup $\boldsymbol{\mu}_2 \wr \Sigma_{k-p}$ of its stabiliser. The fibration*

$$G\langle p\rangle \times_{F\langle p\rangle} C \longrightarrow C$$

together with the action of $\boldsymbol{\mu}_2 \wr \Sigma_{k-p}$, is locally isomorphic on C to the product $P_{k-p}' \times C \longrightarrow C$. ∎

2.4.0. Let ϵ_r be the character of $\boldsymbol{\mu}_2 \wr \Sigma_r$ defined in 1.3.1, and let H^*, H_* be a Poincaré duality theory for varieties over κ satisfying the conditions of 1.3.0.

2.4.1. Proposition. *For any smooth T over κ,*

$$H_*(P_r' \times T)(\epsilon_r) = 0.$$

This is a consequence of the following three lemmas.

2.4.2. Lemma. $H_*((P_r' - P_r'') \times T)(\epsilon_r) = 0$.

Proof. $P_r' - P_r''$ is a disjoint union of $r \cdot 2^{r-1}$ copies of $\mathbf{G}_m^{r-2}$, permuted transitively by $\boldsymbol{\mu}_2 \wr \Sigma_r$. The component

$$\{x_1 = y_1 = y_2 = \cdots = y_r = 0,\ x_2, \ldots, x_r \neq 0\}$$

is acted on trivially by the transposition $x_1 \leftrightarrow y_1$, and the lemma follows. ∎

2.4.3. Lemma. *$H^*(P_r^* \times T)(\epsilon_r)$ is a free module over $H^*(T)$ generated by the image of $\frac{x_1}{y_1} \cup \ldots \cup \frac{x_r}{y_r}$ in $H^*(P_r^*, r)$.*

Proof. The morphism

$$\begin{gathered} \mathbf{G}_{m/\kappa}^r \to P_r^* \\ (z_1 \ldots, z_r) \mapsto (z_1, z_1^{-1}; \ldots; z_r, z_r^{-1}) \end{gathered}$$

is an isogeny with kernel $\boldsymbol{\mu}_2$, hence

$$H^*(P_r^* \times T)(\epsilon_r) \xrightarrow{\sim} H^*(\mathbf{G}_m^r \times T)(\epsilon_r)$$

and we can apply 1.3.1. ∎

2.4.4. Lemma. *$H^*((P''_r - P^*_r) \times T)(\epsilon_r)$ is free over $H^*(T)$ of rank one, and the boundary map*

$$H^*(P^*_r \times T)(\epsilon_r) \to H^{*-1}((P''_r - P^*_r) \times T)(\epsilon_r)$$

is an isomorphism.

Proof. $P''_r - P^*_r$ is the disjoint union of 2^r copies of $\mathbf{G}_m^{r-1}$, permuted transitively by $\boldsymbol{\mu}_2 \wr \Sigma_r$. Therefore

$$H^*((P''_r - P^*_r) \times T)(\epsilon_r) = H^*(\mathbf{G}_m^{r-1} \times T)(\mathrm{sgn})$$

and the latter is free of rank one over $H^*(T)$ by 1.3.1. Consider the component

$$Q = \{x_1 = x_2 = \ldots = x_r = 0,\ y_1, \ldots, y_r \neq 0\}$$

of $P''_r - P^*_r$. It belongs to the open set $\operatorname{Spec} R \subset P''_r$, where

$$R = F\left[\frac{x_1}{y_1}, \frac{y_2}{y_1}, \left(\frac{y_2}{y_1}\right)^{-1}, \ldots, \frac{y_r}{y_1}, \left(\frac{y_r}{y_1}\right)^{-1}\right]$$

and a local equation for Q on $\operatorname{Spec} R$ is $\dfrac{x_1}{y_1} = 0$. The generator of $H^*(P^*_r \times T)(\epsilon_r)$ may be written

$$\begin{aligned} \psi &= \frac{x_1}{y_1} \cup \ldots \cup \frac{x_r}{y_r} \\ &= \frac{x_1}{y_1} \cup \frac{x_1}{y_1}\left(\frac{y_1}{y_2}\right)^2 \cup \ldots \cup \frac{x_1}{y_1}\left(\frac{y_1}{y_r}\right)^2 \\ &= 2^{r-1} \cdot \frac{x_1}{y_1} \cup \frac{y_1}{y_2} \cup \ldots \cup \frac{y_1}{y_r} \end{aligned}$$

and the component along Q of the boundary of ψ is

$$2^{r-1} \cdot \frac{y_1}{y_2} \cup \ldots \cup \frac{y_1}{y_r}$$

which generates $H^*(Q \times T)(\mathrm{sgn})$ over $H^*(T)$ as required. ■

3. Homology at infinity.

3.0. Resume the notations of §1. Write for convenience $X = \overline{X}^k_n$, $Y = \overline{X}^{k,\infty}_n$, and filter Y by closed subschemes

$$Y = Y_k \supset Y_{k-1} \supset \ldots \supset Y_0 \supset Y_{-1} = \emptyset$$

where Y_p is the set of $(x_1 \ldots, x_k) \in Y$ such that at least $(k-p)$ of the components x_i are singular points of the corresponding Néron polygon. Define inductively $X\langle p\rangle$, $Y\langle p\rangle$, $Z\langle p\rangle$ in a manner analogous to 2.1.0 above; more precisely, $Y\langle 0\rangle = Y_0 \subset X\langle 0\rangle = X$; the morphism $\phi_p : X\langle p\rangle \to X\langle p-1\rangle$ is the blowing-up of $Y\langle p-1\rangle$, and $Y\langle p\rangle$ is the proper transform of Y_p in $X\langle p\rangle$; and

$$Z\langle p\rangle = \phi_p^{-1}(Y\langle p-1\rangle) \cap X\langle p\rangle^{\text{reg}} \subset X\langle p\rangle.$$

Write

$$\widetilde{X} = X\langle k-1\rangle \xrightarrow{\psi_p} X\langle p\rangle;$$
$$W\langle p\rangle = \psi_p^{-1}\left(X\langle p\rangle^{\text{sing}}\right).$$

3.1.0. Theorem. *i) $\widetilde{X}$ is smooth over $\mathbf{Q}$, and the action of $((\mathbf{Z}/n)^2 \rtimes \boldsymbol{\mu}_2) \wr \Sigma_k = \Gamma_k$ on X extends to an action on $\widetilde{X}$.*

ii) $\psi_0 : \widetilde{X} \to X$ is an isomorphism over X^{reg}, inducing an isomorphism

$$H^*(\widetilde{X})(\epsilon) \xrightarrow{\sim} H^*(X^{\text{reg}})(\epsilon).$$

iii) The inclusion of the connected component X^* of the Néron model of X^k_n over $\overline{M}_n$ in X^{reg} induces an isomorphism

$$H^*(X^{\text{reg}})(\epsilon) \xrightarrow{\sim} H^*(X^*)(\epsilon_k).$$

Proof. i) Recall ([**DR**], Ch. VII) that the formal completion of $\overline{X}_n$ along any singular fibre is isomorphic to the n-sided Tate curve $\overline{\mathcal{G}}_m^{q^{1/n}}/q^{\mathbf{Z}}$ over $\mathbf{Q}(\zeta_n)[[q^{1/n}]]$. Therefore the formal completion of X along Y is locally isomorphic to the formal completion of E^k along F^k (§2.0.1), and the result of 2.1.1 is applicable.

ii) We have a filtration

$$Y \supset W\langle 0\rangle \supset W\langle 1\rangle \supset \ldots \supset W\langle k-2\rangle \supset W\langle k-1\rangle = \emptyset$$

such that $W\langle p\rangle - W\langle p+1\rangle = Z\langle 0\rangle$ is a fibration over $Y\langle p\rangle$ for the Zariski topology with fibre P'_{k-p}. The components of $Y\langle p\rangle$ over a fixed cusp are permuted transitively by Γ_k, and the stabiliser of a component contains a subgroup $\boldsymbol{\mu}_2 \wr \Sigma_{k-p}$ acting trivially on the base. By Proposition 2.4.1, $H_*(W\langle p\rangle - W\langle p+1\rangle)(\epsilon)$ is trivial for $0 \le p \le k-2$, and the result follows from the long exact sequence of cohomology.

iii) Lying over a fixed cusp there are kn^k components of $Y_{k-1} - Y_{k-2}$, each isomorphic to $\mathbf{G}_m^{k-1}$, permuted transitively by Γ_k. The stabiliser of a component is $\boldsymbol{\mu}_2 \times (\boldsymbol{\mu}_2 \wr \Sigma_{k-1})$, the first factor acting trivially. Therefore $H^*(Y_{k-1} - Y_{k-2})(\epsilon) = 0$, and by the exact cohomology sequence,

$$H^*(X^{\text{reg}})(\epsilon) = H^*(X - Y_{k-2})(\epsilon) \xrightarrow{\sim} H^*(X - Y_{k-1})(\epsilon).$$

Now the translations $(\mathbf{Z}/n)^{2k}$ act trivially on $H^*(X^k_n) = H^*(X - Y)$. We have $X^* \times_{\overline{M}_n} M_n^\infty = \mathbf{G}_m^k \times M_n^\infty$, $(X - Y_{k-1}) \times_{\overline{M}_n} M_n^\infty = \mathbf{G}_m^k \times (\mathbf{Z}/n)^k \times M_n^\infty$, and the inclusion $X^* \hookrightarrow X - Y_{k-1}$ induces an isomorphism

$$H^*(X^* \times_{\overline{M}_n} M_n^\infty) \xrightarrow{\sim} H^*((X - Y_{k-1}) \times_{\overline{M}_n} M_n^\infty)^{(\mathbf{Z}/n)^{2k}};$$

the result follows from the 5-lemma applied to the long exact homology sequences for the inclusions $X - Y \hookrightarrow X^*$, $X - Y \hookrightarrow X^{\text{reg}}$. ■

3.1.1. Now applying the long exact cohomology sequence for the inclusion of X^* in $\tilde{X}$ and Poincaré duality gives Theorem 1.3.3. ■

4. Hecke operators.

4.0.0. For this section, p will be a prime not dividing n. Let $M_{n,p}$ be the modular curve over $\mathbf{Q}$ classifying elliptic curves E with level n structure and a subgroup $C \subset E$ of order p. The fibre product $X_{n,p} = X_n \times_{M_n} M_{n,p}$ is canonically isomorphic to the universal elliptic curve over $M_{n,p}$, hence has a canonical subgroup scheme C. Write $X^k_{n,p}$ for the fibre product $X^k_n \times_{M_n} M_{n,p}$.

4.0.1. Let Q be the quotient of $X_{n,p}$ by C, with level n structure coming from that on $X_{n,p}$, and let Q^k be its k-fold fibre product over $M_{n,p}$. Consider the diagram:

$$\begin{array}{ccccccc} X^k_n & \xleftarrow{\phi_1} & X^k_{n,p} & \xrightarrow{\psi} & Q^k & \xrightarrow{\phi_2} & X^k_n \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ M_n & \longleftarrow & M_{n,p} & = & M_{n,p} & \longrightarrow & M_n \end{array}$$

where the first and third squares are Cartesian, given by the classifying maps for the elliptic curves with level n-structure $X_{n,p}$ and Q over $M_{n,p}$.

4.0.2. Define the *Hecke correspondence* T_p on X^k_n by

$$T_p = \phi_{1*}\psi^*\phi_2^*.$$

The morphisms ϕ_i, ψ are finite and flat, and we therefore get induced endomorphism T_p of $H^*(X^k_n(\mathbf{C}), \mathbf{Q})$ and $H^*_c(X^k_n(\mathbf{C}), \mathbf{Q})$, and their l-adic analogues.

4.1.0. In section (3.12) of [**DeFM**], Deligne defines Hecke operators on the groups $H^1(M_n(\mathbf{C}), \mathrm{Sym}^k R^1 f_*\mathbf{Q})$, the groups with compact support, and the corresponding groups in l-adic cohomology. Provisionally denote these operators by T'_p.

4.1.1. Proposition. *The isomorphism*

$$H^*(X^k_n(\mathbf{C}), \mathbf{Q})(\epsilon_k) \xrightarrow{\sim} H^1(M_n(\mathbf{C}), \mathrm{Sym}^k R^1 f_*\mathbf{Q})$$

(given by the Leray spectral sequence for $X^k_n \to M_n$) identifies T_p and T'_p. The same is true for the cohomology with compact support.

This follows easily from Proposition 3.18 of [**DeFM**] and the functoriality of the Leray spectral sequence. ■

4.1.2. We now define the Hecke correspondence—still to be denoted T_p—on $\overline{\overline{X}}{}^k_n$ as the closure of the graph of T_p in $\overline{\overline{X}}{}^k_n \times \overline{\overline{X}}{}^k_n$.

4.1.3. Proposition. *The action of Γ_k on $H^*_B(\overline{\overline{X}}{}^k_n)$ commutes with T_p. The action of the transpose of T_p on the space of cusp forms $H^0(\overline{\overline{X}}{}^k_n, \Omega^{k+1})$ is the same as that of the classical Hecke operator.*

Proof. The first part is immediate from the definition. For the second, see [**Katz**] 1.11. ■

4.1.4. The preceding two propositions imply that T_p induces an endomorphism of the motive ${}^k_n\mathcal{W}$, and that on the realisation $H_B({}^k_n\mathcal{W}) = {}^k_nW_B$ it agrees with T'_p.

4.2.0. Write $k = w-2$, and assume first that $n \geq 3$. The operators T_p for $p \nmid n$ generate a semisimple algebra of endomorphisms of the parabolic cohomology $H_B({}^k_n\mathcal{W})$ which commutes with the action of $GL_2(\mathbf{Z}/n)$. Since f is a newform, the subspace of the space of cusp forms $F^{k+1}H_{DR}({}^k_n\mathcal{W}) \otimes L$ generated by f is the common eigenspace of the ${}^tT_p$ with eigenvalues a_p intersected with the invariants under the subgroup

$$\begin{pmatrix} * & * \\ 0 & 1 \end{pmatrix} \subset GL_2(\mathbf{Z}/n).$$

There is accordingly a projector Ψ_f in the endomorphism algebra of ${}^k_n\mathcal{W}\otimes L$—viewed here as a Grothendieck motive—whose kernel on the space of cusp forms is the subspace generated by f. We define $M(f)$ be the submotive of ${}^k_n\mathcal{W}\otimes L$ which is the kernel of Ψ_f.

If $n<3$ then we replace n by nd for some $d\geq 3$ and take invariants under $\ker\{GL_2(\mathbf{Z}/nd)\to GL_2(\mathbf{Z}/n)\}$.

4.2.1. The $\mathbf{Q}$-schemes M_n, X^k_n extend to smooth schemes over $\mathbf{Z}[1/n]$ which have a modular interpretation (see for example [**DR**] V.1.17), and the compactification $\overline{\overline{X}}{}^k_n$ extends to a smooth and proper scheme over $\mathbf{Z}[1/n]$. The motive $M(f)$ therefore has good reduction at any $p\not| n$, in the sense of [**GM**] B.3.8; its λ-adic realisations are unramified at p, and it has a crystalline realisation $H_{\mathrm{crys}}(M(f))$.

4.2.2. Identifying $H_l({}^k_n\mathcal{W})$ with ${}^k_nW_l$, we can then use the congruence relation ([**DeFM**] 4.9) to deduce 1.4.2(i), by a standard method: the relation $T_p=F+I_p^*V$ identifies the eigenvalues of F on $H_\lambda(M(f))$ as being roots of $\mathbf{T}_p(X)$; an additional relation as in [**Shimura**] (7.5.2) shows that $H_\lambda(M(f))\simeq H_\lambda(M(\bar f))\otimes\chi$, which is used to show that both roots of $\mathbf{T}_p(X)$ occur.

4.2.3. Now we prove 1.4.2(ii). Since $M(f)$ has good reduction at p and $p>\dim\overline{\overline{X}}{}^k_n=w-1$, by the main result of [**FM**] the representation $H_\pi(M(f))$ of $\mathrm{Gal}(\overline{\mathbf{Q}}_p/\mathbf{Q}_p)$ is crystalline, and the associated ϕ-filtered module is $H_{\mathrm{crys}}(M(f))$ together with its Hodge filtration. But if V is a smooth and proper $\mathbf{F}_p$-scheme, and Ψ an algebraic correspondence whose images in $\mathrm{End}(H_l^*(V))$ and $\mathrm{End}(H^*_{\mathrm{crys}}(V))$ are projectors, then the characteristic polynomials of the Frobenius endomorphism on $\Psi(H_l^i(V))$ and $\Psi(H^i_{\mathrm{crys}}(V))$ are equal, by [**KMess**] Theorem 2(2). Therefore the characteristic polynomial of ϕ on $H_{\mathrm{crys}}(M(f))$ is $\mathbf{T}_p(X)$. ■

4.2.4. *Remark.* It is possible to show that the ϕ-filtered module $H_{\mathrm{crys}}({}^k_n\mathcal{W})$ is isomorphic to the module $L_k(n,\mathbf{Q}_p)$ defined in [**S**] 2.7, 3.3 for $p\not| 2n$.

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