Plectic structures in number theory & geometry

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Shimura varieties

Class of algebraic varieties with rich arithmetic structure.

Numerous applications in number theory:

- Mazur: bounding torsion subgroups of elliptic curves over $\mathbb{Q}$
- Breuil–Conrad–Diamond–Taylor–Wiles: modularity theorem (and FTL. . .)
- Gross–Zagier–Kolyvagin: "Heegner points" (introduced by Birch) giving special cases of BSD conjectures

These all use 1-dimensional Shimura varieties (modular curves). Higher-dimensional Shimura varieties are harder to exploit (e.g. no explicit function theory).

But sometimes they appear to have extra symmetries — *plectic structure*. ("Almost" product structure)

Künneth formula

X_1, X_2 topological spaces, $X = X_1 \times X_2$.

Theorem (Künneth formula)

$$H^n(X, \mathbb{Q}) \simeq \bigoplus_{i+j=n} H^i(X_1, \mathbb{Q}) \otimes H^j(X_2, \mathbb{Q})$$

So

$$H^*(X, \mathbb{Q}) \stackrel{def}{=} \bigoplus_n H^n(X, \mathbb{Q}) \simeq H^*(X_1, \mathbb{Q}) \otimes H^*(X_2, \mathbb{Q})$$

Functorial, and if X_i have some extra structure, preserved by this isomorphism (e.g. automorphisms of X_i).

Künneth formula in Hodge theory

Suppose $X_i/\mathbb{C}$ are algebraic varieties. Then $H^n(X_i)$ has a (mixed) *Hodge structure*.

X_i projective and smooth. Then have *Hodge decomposition* ($\mathbb{Z}^2$-grading on H^*) given by forms of type (p, q)

$$H^n(X, \mathbb{C}) = \bigoplus_{p+q=n} H^{pq}(X)$$

and

$$H^{pq}(X) = \bigoplus_{\substack{p_1+p_2=p \\ q_1+q_2=q}} H^{p_1 q_1}(X_1) \otimes H^{p_2 q_2}(X_2)$$

So $H^*(X, \mathbb{C})$ has a $\mathbb{Z}^2 \times \mathbb{Z}^2$-grading ("plectic Hodge structure")

If X_i aren't smooth projective, get something more complicated ("plectic mixed Hodge structure").

Künneth formula in ℓ-adic cohomology

Suppose X_i are varieties over field k, separable closure $\bar{k}$.

Have ℓ-adic (étale) cohomology $H^*(X_{/\bar{k}}, \mathbb{Q}_\ell)$, which has action of $\Gamma_k = \mathrm{Gal}(\bar{k}/k)$.

Künneth formula is an isomorphism of representations of Γ_K:

$$H^*(X_{/\bar{k}}, \mathbb{Q}_\ell) \simeq H^*(X_{1/\bar{k}}, \mathbb{Q}_\ell) \otimes_{\mathbb{Q}_\ell} H^*(X_{2/\bar{k}}, \mathbb{Q}_\ell)$$

Gives more: from actions on two factors, get action of product $\Gamma_k \times \Gamma_k$ on $H^*(X_{/\bar{k}}, \mathbb{Q}_\ell)$.

"Usual" action is restriction to diagonal $\Gamma_k \subset \Gamma_k \times \Gamma_k$.

- Künneth theorem is actually on level of "cochain complexes": so $\mathrm{R}\Gamma(X_{/\bar{k}}, \mathbb{Q}_\ell)$ is a complex of vector spaces with action of $\Gamma_k \times \Gamma_k$.
- Need Künneth formula to define the $\Gamma_k \times \Gamma_k$-action (no natural action of product on $X_{/\bar{k}}$).

Beyond Künneth?

Varieties which aren't products?

Trivial example: G finite group acting (freely, say) on X_1 and X_2, $Y = (X_1 \times X_2)/G$.

Then Y is not (in general) a product, but $H^*(Y) = H^*(X_! \times X_2)^G$ behaves in the same way:

- plectic Hodge structure ($\mathbb{C}$ case)
- action of $\Gamma_k \times \Gamma_k$ (arithmetic case).

Quotients by infinite groups? $\longrightarrow$ Shimura varieties.

Shimura varieties

Shimura variety: an algebraic variety S (over number field) such that $S(\mathbb{C}) = \sqcup \Gamma\backslash D$ where

- D a Hermitian symmetric domain G/K
- $\Gamma \subset G$ arithmetic subgroup

Basic example: modular curves.

- $D = \mathcal{H}$ upper half plane $= SL_2(R)/SO(2)$.
- $\Gamma =$ congruence subgroup of $SL_2(\mathbb{Z})$, e.g.

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \;\middle|\; c \equiv 0 \mod N \right\}$$

$\Gamma_0(N)\backslash\mathcal{H} = Y_0(N)$, affine curve.
Projective closure $X_0(N) = Y_0(N) \bigcup$(finite) defined over $\mathbb{Q}$.

Cohomology of modular curves

In dimension 1 (as representation of $\Gamma_{\mathbb{Q}}$)

$$H^1(X_0(N)_{/\overline{\mathbb{Q}}}, \mathbb{Q}_\ell) = \begin{pmatrix} \text{sum of 2-dimensional pieces,} \\ \text{parametrised by modular forms} \end{pmatrix}$$

For every elliptic curve $E/\mathbb{Q}$, $\exists$ N such that

$$H^1(E_{/\overline{\mathbb{Q}}}, \mathbb{Q}_\ell) \subset H^1(X_0(N)_{/\overline{\mathbb{Q}}}, \mathbb{Q}_\ell).$$

(One formulation of the **modularity theorem**)

Plectic structure I

Plectic structure arises when $D = D_1 \times \cdots \times D_r$ but Γ is not commensurable with $\Gamma_1 \times \cdots \times \Gamma_r$.

Simplest example: **Hilbert modular surfaces**. $D = \mathcal{H} \times \mathcal{H}$, $F = \mathbb{Q}(\sqrt{d})$ real quadratic field.

Γ congruence subgroup:

$$\begin{array}{ccc} \Gamma \subset SL_2(F) \subset SL_2(\mathbb{R}) \times SL_2(\mathbb{R}) = G \\ \sqrt{d} \quad \mapsto \quad (\sqrt{d}, -\sqrt{d}) \end{array}$$

$S = \Gamma \backslash (\mathcal{H} \times \mathcal{H})$, algebraic surface defined over number field (often over $\mathbb{Q}$).

Can replace F by a totally real field of degree $r > 1$, $D = \mathcal{H}^r$ giving *Hilbert–Blumenthal modular varieties* (HBMV)

Plectic structure: Hodge theory

Theorem

Suppose $D = D_1 \times \cdots \times D_r$ and $X = \Gamma\backslash D$ is **compact**. *Then have canonical plectic Hodge structure*

$$H^*(X,\mathbb{C}) = \bigoplus_{\underline{p},\,\underline{q}\in\mathbb{Z}^r} H^{\underline{p},\underline{q}}(X)$$

Proof: Hodge structure on $H^*(X)$ depends mainly on G, not Γ:

$$H^*(X,\mathbb{C}) = \bigoplus_{\pi} H^*(\mathfrak{g},\mathfrak{k};\pi)^{m(\pi,\Gamma)} \qquad (\textit{Matsushima's formula})$$

Here π are irred. reps of $\mathfrak{g} = \mathrm{Lie}(G)$, and Lie algebra cohomology $H^*(\mathfrak{g},\mathfrak{k},\pi)$ has Hodge decomposition (Laplacian $\leftrightarrow$ Casimir) and

$$H^*(\mathfrak{g},\mathfrak{k},\pi_1\otimes\cdots\otimes\pi_r) = \bigotimes_i H^*(\mathfrak{g}_i,\mathfrak{k}_i,\pi_i)$$

Plectic structure: mixed Hodge theory

Case of noncompact quotient more complicated.

Theorem

S an HBMV (any dimension). Then $H^n(S, \mathbb{C})$ has a canonical plectic mixed Hodge structure.

This means it has $(2r+1)$ filtrations:

- $W_\bullet$ (weight, increasing) and
- $F_j^\bullet$, $\overline{F_j^\bullet}$ (Hodge and conjugate, decreasing) for $1 \leq j \leq r$

$+$ certain conditions — in particular, $\mathrm{gr}_*^W H^n(S)$ has a $\mathbb{Z}^{2r}$-grading.

ℓ-adic cohomology of HBMVs

S an HBMV associated to F of degree r. [Assume S defined $/\mathbb{Q}$.]

- most interesting cohomology is $H^r(S_{/\overline{\mathbb{Q}}}, \mathbb{Q}_\ell)$, representation of $\Gamma_{\mathbb{Q}}$.
- contains subrepresentations U_f of dimension 2^r, indexed by (Hilbert) modular forms f.
- To f is attached a 2-dimension representation V_f of Γ_F.
- If $f \leftrightarrow E/F$ (elliptic curve) then $V_f = H^1(E_{/\overline{\mathbb{Q}}}, \mathbb{Q}_\ell)$.

Langlands, Brylinski–Labesse:

$$\underbrace{V_f \otimes \cdots \otimes V_f}_{r \text{ times}} \quad \text{repr. of } \Gamma_F^r \rtimes \mathrm{Sym}(r) \overset{def}{=} \Gamma_F^{pl} = \text{"plectic Galois group"}$$

and permutation action on $\Gamma_{\mathbb{Q}}/\Gamma_F$ gives $\Gamma_{\mathbb{Q}} \hookrightarrow \Gamma_F^{pl}$ ("$\otimes$-induction")

Theorem (Brylinski-Labesse, Nekovář)

U_f, $V_f^{\otimes r}$ are isomorphic (as representations of $\Gamma_{\mathbb{Q}}$).

Plectic structure on ℓ-adic cohomology

So at least part of H^r has an action of Γ_F^{pl}. In fact all of H^* does:

Theorem ("Beyond Künneth")

There is a (non-unique) action of Γ_F^{pl} on $H^*(S_{/\overline{\mathbb{Q}}}, \mathbb{Q}_\ell)$ extending the action of $\Gamma_{\mathbb{Q}}$.

[here S really is a Shimura variety attached to GL_2/F, which is a disjoint union of HBMVs.]

Conjecture ("Plectic conjecture, ℓ-adic version")

There is a canonical complex of $\mathbb{Q}_\ell[\Gamma_F^{pl}]$-modules $\mathrm{R}\Gamma(S^{pl}, \mathbb{Q}_\ell)$ whose restriction of $\Gamma_{\mathbb{Q}}$ is $\mathrm{R}\Gamma(S_{/\overline{\mathbb{Q}}}, \mathbb{Q}_\ell)$.
(Also for SVs associated to any reductive G/F, functoriality. . .)

Consequences of the plectic conjecture

Plectic formalism $\implies$ constructions and theorems for modular curves (i.e. over $\mathbb{Q}$) can be extended to HBMVs (i.e. over F).

E.g. there exist plectic analogues of classical theta functions (which are related to L-values, and should give rise to generalisations of elliptic units). For modular curves, there are *Heegner points* associated to an imaginary quadratic field $K = \mathbb{Q}(\sqrt{-D})$:

$\tau \in \mathcal{H} \cap K$

$\longrightarrow$ points $Q_\tau \in X_0(N)(\overline{\mathbb{Q}})$

$\longrightarrow$ $P_\tau \in E(\overline{\mathbb{Q}})$ (even $E(\mathbb{Q})$) for elliptic curve $E/\mathbb{Q}$

$\implies$ (Gross–Zagier, Kolyvagin) special cases of BSD conjectures (when L-function has simple zero at $s = 1$)

Is there a plectic version of this?

Plectic Heegner points???

S an HBMV for F, degree r; E/F elliptic curve with $H^1(E)^{\otimes r} \subset H^r(S)$.

$\tau \in K$ = totally imaginary quadratic extension of F $\longrightarrow$ "CM points" $Q_\tau \in S(\overline{\mathbb{Q}})$.

Plectic formalism $\Longrightarrow$ $\exists$ cohomology class

$$\begin{array}{ccc} [Q_\tau] & \in & H^r(\Gamma_F^{pl}, H^r(S_{/\overline{\mathbb{Q}}}, \mathbb{Q}_\ell)(r)) \\ & & \downarrow \\ & & H^r(\Gamma_F^{pl}, \bigotimes^r H^1(E_{/\overline{\mathbb{Q}}}, \mathbb{Q}_\ell)(r)) \\ & & \| \text{ (Künneth)} \\ & & \Lambda^r H^1(\Gamma_F, H^1(E_{/\overline{\mathbb{Q}}}, \mathbb{Q}_\ell)(1)) \\ & & \bigcup \text{ (Kummer theory)} \\ & & \Lambda^r E(F) \otimes \mathbb{Q}_\ell \end{array}$$

Speculation: $[Q_\tau]$ maps into $\Lambda^r E(F)$ iff $\mathrm{ord}_{s=1} L(E,s) \geq r$, and if so, $\exists$ "plectic Gross–Zagier" formula relating $[Q_\tau]$ and $L^{(r)}(E,1)$.

THE END