Integral elements in K-theory and products of modular curves

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Introduction

This paper has two aims. The primary one is to clarify the relation between results of Beilinson [1] and Flach [7]. We begin by briefly recalling the relevant parts of their papers.

Suppose S is a connected smooth projective surface over $\mathbb{Q}$. Beilinson's conjectures relate the motivic cohomology groups $H^i_{\mathcal{M}}(S,\mathbb{Q}(n)) = K^{(n)}_{2n-i}(S)$ of S and the L-function of the motive $h^{i-1}(S)$ at $s=n$. In what follows we will only be concerned with the motive $h^2(S)$ and the "near-central" point $s=2$. In this case:

- The motivic cohomology $H^3_{\mathcal{M}}(S,\mathbb{Q}(2))$ is equal to the K-cohomology group $H^1(S,\mathcal{K}_2)\otimes\mathbb{Q}$, and $H^1(S,\mathcal{K}_2)$ is the H^1 of the Gersten complex
$$K_2k(S) \xrightarrow{(\partial_C)} \coprod_{C\subset S} k(C)^* \xrightarrow{\text{div}} \coprod_{P\in S}\mathbb{Z}. \tag{0.1}$$
Here C runs over irreducible curves in S and P over closed points; ∂_C is (up to a sign) the tame symbol attached to the valuation ord_C of $k(S)$; and div is the divisor map.

- The Deligne cohomology group $H^3_{\mathcal{D}}(S_{/\mathbb{R}},\mathbb{R}(2))$ equals the cokernel of the composite map
$$F^2H^2_{\text{dR}}(S/\mathbb{R}) \hookrightarrow H^2_{\text{dR}}(S/\mathbb{R}) = H^2(S(\mathbb{C}),\mathbb{C})^+ \xrightarrow{\text{Im}} H^2(S(\mathbb{C}),\mathbb{R}(1))^+$$
where $+$ denotes the fixed part under the de Rham conjugation, which is the product of the maps on Betti cohomology induced by complex conjugation on $S(\mathbb{C})$ and complex conjugation on the coefficients $\mathbb{C}$.

- Beilinson's conjectures predict that the regulator (Chern character) and cycle class maps induce an isomorphism
$$H^3_{\mathcal{M}/\mathbb{Z}}(S,\mathbb{Q}(2))\otimes_{\mathbb{Q}}\mathbb{R} \oplus NS(S)\otimes_{\mathbb{Z}}\mathbb{R} \xrightarrow{\sim} H^3_{\mathcal{D}}(S_{\mathbb{R}},\mathbb{R}(2))$$
where $H_{\mathcal{M}/\mathbb{Z}} \subset H_{\mathcal{M}}$ is the image of $K_1(\mathcal{S})\otimes\mathbb{Q} \to H^3_{\mathcal{M}}(S,2)$, for any regular proper model $\mathcal{S}$ over $\mathbb{Z}$ of S.

Now suppose that $S = X \times X'$ is a product of modular curves. The graph of a Hecke operator is a divisor on S, and its normalisation is itself a modular curve, or union of modular curves. Beilinson's construction is to take a collection of modular units u_α on Hecke correspondences Z_α such that $\sum \operatorname{div} u_\alpha = 0$ (as a 0-cycle on S). Then $\{u_\alpha\}_{Z_\alpha}$ defines a class in $H^3_{\mathcal{M}}(S, 2)$ by (0.1).

For the purposes of this introduction we consider only the simplest nontrivial case. Take $X = X_0(N)$, $X' = X_0(N')$ and let $\varphi\colon X \to E$, $\varphi'\colon X' \to E'$ be Weil parameterisations of modular elliptic curves over $\mathbb{Q}$. The proper pushforward $(\varphi \times \varphi')_*\colon H^3_{\mathcal{M}}(X \times X', 2) \to H^3_{\mathcal{M}}(E \times E', 2)$ maps $\{u_\alpha\}$ to a class $c \in H^3_{\mathcal{M}}(E \times E', 2)$.

The Deligne cohomology group $H^3_{\mathcal{D}}(E \times E'_{\mathbb{R}}, \mathbb{R}(2))$ has dimension 3. The classes of the algebraic cycles $E \times \{0\}$, $\{0\} \times E'$ span a 2-dimensional subspace. If there is a $\mathbb{Q}$-isogeny between E and E' then its graph gives a further algebraic cycle. Thus Beilinson's conjectures predict that $H^3_{\mathcal{M}/\mathbb{Z}}(E \times E', 2)$ is trivial if E and E' are $\mathbb{Q}$-isogenous, and that it has dimension 1 otherwise.

In [1, §6], Beilinson showed that if E and E' are not isogenous, then the images of the classes c in $H^3_{\mathcal{D}}(E \times E'_{\mathbb{R}}, \mathbb{R}(2))$, as the Hecke correspondence and units are varied, span a 1-dimensional $\mathbb{Q}$-subspace, in agreement with his conjectures. Theorem 2.3.4 below, applied to this situation, completes the picture by proving that these elements are indeed in $H^3_{\mathcal{M}/\mathbb{Z}}(E \times E', 2)$.

The first part of the paper addresses two problems that arise in formulating this result in general. The first is that, in order to define $H^3_{\mathcal{M}/\mathbb{Z}}(E \times E', 2)$, one needs a regular model for $E \times E'$ over $\operatorname{Spec} \mathbb{Z}$. (Conjecture 2.4.2.1 of [1], which would circumvent this requirement, turns out to be overoptimistic, see Remark 1.1.7 below.) The natural candidate, the fibre product of the minimal regular models of E and E', will have singularities if the conductors of E and E' have a common factor. If E and E' have semi-stable reduction, these singularities are ordinary double points (locally for the étale topology, of the form $xy = x'y' = p$) and can be resolved with a single blowup, but in general the existence of the resolution is open.

The second problem is that (*pace* [1, §6]) the integrality statement does not hold on the level of the product of modular curves — we can have $c \notin H^3_{\mathcal{M}/\mathbb{Z}}(X \times X', 2)$. Indeed, if E and E' are isogenous, Flach [7] has shown that Beilinson's elements generate an infinite-dimensional subspace of $H^3_{\mathcal{M}}(E \times E', 2)/H^3_{\mathcal{M}/\mathbb{Z}}(E \times E', 2)$. Therefore the desired statement can only be true after performing a motivic decomposition of $X \times X'$.

Both of these difficulties are resolved by the construction of an (unconditional!) theory of $H_{\mathcal{M}/\mathbb{Z}}$ for Chow motives. This is done in §1, using de Jong's results on alterations [6].

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mere armchair mathematics. I also would like to thank Rob de Jeu, Matthias Flach and Dinakar Ramakrishnan for helpful discussions. A related problem to this is the question of the integrality of the elements of K_1 of Hilbert modular surfaces considered by Ramakrishnan [10], and it is reasonable to hope that similar methods may help. The main difficulty seems to be to show that the action of Hecke algebra on the Chow motive of a Hilbert modular surface is semisimple.

1 Integral motivic cohomology

1.1 Statement of results

1.1.1. Throughout this section, $\mathfrak{o}$ will denote a Dedekind domain, and k its field of fractions. We have in mind the two cases:

(i) k a number field, $\mathfrak{o}$ the ring of S-integers of k for a (possibly empty, possibly infinite) set of primes S of k.

(ii) k a local field, $\mathfrak{o}$ its ring of integers.

1.1.2. Let $\mathcal{V}_k$ be the category of smooth projective k-schemes (which we simply call *k-varieties*), and $\mathcal{V}_{\mathfrak{o}}$ the category of all projective and flat $\mathfrak{o}$-schemes X which are regular (which we shall call $\mathfrak{o}$-varieties). The morphisms of $\mathcal{V}_k$, $\mathcal{V}_{\mathfrak{o}}$ are k- and $\mathfrak{o}$-morphisms. We denote the basechange functor $\mathcal{V}_{\mathfrak{o}} \to \mathcal{V}_k$ by the subscript $_{/k}$.

1.1.3. Fix a field E of characteristic zero, and let $\mathcal{M}_k \otimes E$ be the category of Chow motives over k with coefficients in E. When there is no ambiguity we write $\mathcal{M}_k$ for $\mathcal{M}_k \otimes E$. We use the conventions and notations of [12] regarding motives. In particular, if X_k, Y_k belong to $\mathcal{V}_k$ then by definition we have

$$\begin{aligned}\mathrm{Corr}^r(X_k, Y_k) &= \mathrm{Hom}_{\mathcal{M}_k}(h(X_k), h(Y_k) \otimes \mathbb{L}^{-r}) \\ &= CH^{\dim X_k + r}(X_k \times Y_k) \otimes E\end{aligned}$$

if X_k is equidimensional. An object of $\mathcal{M}_k$ is a triple (X_k, p, m) where $p = p^2 \in \mathrm{Corr}^0(X_k, X_k)$ and $m \in \mathbb{Z}$; the *category of graded correspondences* $\mathcal{CV}_k^{\mathrm{gr}}$ is the full subcategory of $\mathcal{M}_k$ whose objects are of the form $(X_k, 1, m)$. The motive $\mathbb{L} = (\mathrm{Spec}\, k, 1, -1)$ is the Lefschetz motive. We recover $\mathcal{M}_k$ as the Karoubian envelope of $\mathcal{CV}_k^{\mathrm{gr}}$.

1.1.4. As well as the contravariant functor $h\colon \mathcal{V}_k^{\mathrm{opp}} \to \mathcal{M}_k$, we have by transposition of the graph the covariant functor $h^\vee\colon \mathcal{V}_k \to \mathcal{M}_k$. The composites of these functors with the evident functor $\mathcal{V}_{\mathfrak{o}} \to \mathcal{V}_k$ will be denoted h_k, $h_k^\vee$.

1.1.5. We recall the definition of motivic cohomology with rational coefficients: for any X_k in $\mathcal{V}_k$ and n, $i \in \mathbb{Z}$,

$$H^i_{\mathcal{M}}(X, E(n)) = K^{(n)}_{2n-i} X_k \otimes E \subset K_{2n-i} X_k \otimes E$$

where as usual $K^{(n)}_q$ denotes the weight n Adams eigenspace of $K_q \otimes \mathbb{Q}$. Motivic cohomology extends uniquely to an additive covariant functor on $\mathcal{M}_k$ (we recall the construction in 1.3 below). Since the coefficient field E will be fixed in what follows we will generally write simply $H^i_{\mathcal{M}}(X, n)$.

For varieties which admit regular models over $\mathfrak{o}$, Beilinson has defined "integral motivic cohomology". We show that this has an (unconditional) extension to arbitrary motives over k:

1.1.6. Theorem. *There is a unique way to define subspaces*

$$H^i_{\mathcal{M}/\mathfrak{o}}(M, n) \subset H^i_{\mathcal{M}}(M, n)$$

for every Chow motive M over k, satisfying:

(i) If $c\colon M \to N$ is a morphism in $\mathcal{M}_k$ then $c(H^i_{\mathcal{M}/\mathfrak{o}}(M, n)) \subset H^i_{\mathcal{M}/\mathfrak{o}}(, n)$.

(ii) $M \mapsto H^i_{\mathcal{M}/\mathfrak{o}}(M, n)$ is additive in M.

(iii) If $X \in \mathrm{Ob}\,\mathcal{V}_{\mathfrak{o}}$ then

$$H^i_{\mathcal{M}/\mathfrak{o}}(h_k(X), n) = \mathrm{Im}(K^{(n)}_{2n-i} X \to K^{(n)}_{2n-i} X_{/k}) \otimes E.$$

1.1.7. Remark. In [1, 2.4.2.1], Beilinson conjectures that one could work with proper and flat models over $\mathfrak{o}$ instead of regular models, replacing K-theory by K'-theory. However, it is not in general true that for proper flat $\mathfrak{o}$-schemes X with smooth generic fibre, the image of $K'_* X \to K_* X_{/k}$ depends only on $X_{/k}$ (even ignoring torsion). Rob de Jeu has observed that this fails even in the case of elliptic curves; see his paper [5] in this volume.

1.2 Alterations and motives

1.2.1. We recall the following theorem of de Jong [6, Theorem 4.5 and 8.2]. By definition, an *alteration* is a proper surjective generically finite morphism of integral noetherian schemes.

1.2.2. Theorem. *Let X be a proper flat $\mathfrak{o}$-scheme which is integral. There exists a finite extension k'/k, an integral scheme X', projective over $\mathfrak{o}'$, the integral closure of $\mathfrak{o}$ in k', and an $\mathfrak{o}$-morphism $f\colon X' \to X$ such that:*

(i) X' is regular, and is semistable over $\mathfrak{o}'$;

(ii) f is an alteration.

1.2.3. Let $\mathcal{V}'_{\mathfrak{o}}$ be the full subcategory of $\mathcal{V}_{\mathfrak{o}}$ comprising all $\mathfrak{o}$-varieties X for which the structural morphism admits a Stein factorisation

$$X \xrightarrow{g} \operatorname{Spec} \mathfrak{o}' \longrightarrow \operatorname{Spec} \mathfrak{o}$$

where g is semistable and $\mathfrak{o}'$ is the integral closure of $\mathfrak{o}$ in a finite extension of k. We shall show that de Jong's theorem implies that the category of motives over k is generated by $\mathcal{V}'_{\mathfrak{o}}$, in a very strong sense.

1.2.4. Definition. $\mathcal{CV}^{\mathrm{gr}}_{\mathfrak{o}}$ is the category whose objects are pairs (X, m), where $X \in \operatorname{Ob} \mathcal{V}'_{\mathfrak{o}}$ and $m \in \mathbb{Z}$, and whose morphisms are

$$\begin{aligned} \operatorname{Hom}_{\mathcal{CV}^{\mathrm{gr}}_{\mathfrak{o}}}((X,m),(Y,n)) &= \operatorname{Hom}_{\mathcal{CV}^{\mathrm{gr}}_{k}}((X_{/k},1,m),(Y_{/k},1,n)) \\ &= \operatorname{Hom}_{\mathcal{M}_k}(h_k(X), h_k(Y) \otimes \mathbb{L}^{m-n}). \end{aligned}$$

There is an obvious functor $\mathcal{CV}^{\mathrm{gr}}_{\mathfrak{o}} \to \mathcal{CV}^{\mathrm{gr}}_{k}$ given by $(X,m) \mapsto (X_{/k}, 1, m)$ on objects, and the identity on morphisms; it is fully faithful by definition.

1.2.5. Lemma. *Every morphism $c\colon (X,m) \to (Y,n)$ in $\mathcal{CV}^{gr}_{\mathfrak{o}}$ is an E-linear combination of correspondences of the form $g_{/k*}f^*_{/k}$, where $f\colon Z \to X$, $g\colon Z \to Y$ are morphisms in $\mathcal{V}'_{\mathfrak{o}}$.*

Proof. We can assume that $c = [Z'_{/k}]$ for some integral closed subscheme $Z'_{/k} \subset X_{/k} \times Y_{/k}$. Let Z' be the closure of $Z'_{/k}$ in $X \times_{\mathfrak{o}} Y$, and let $Z \xrightarrow{p} Z'$ be an alteration, with $Z \in \mathcal{V}'_{\mathfrak{o}}$ (the existence of p follows from de Jong's theorem). We have a commutative diagram:

$$\begin{array}{ccccc} & & Z & & \\ & \overset{f}{\swarrow} & \downarrow{\scriptstyle p} & \overset{g}{\searrow} & \\ & & Z' & & \\ & & \downarrow & & \\ X & \longleftarrow & X \times Y & \longrightarrow & Y \end{array}$$

and $g_{/k*}f^*_{/k} = [p_{/k*}Z_{/k}] = \deg(p)c$ in $\operatorname{Corr}^*(X_{/k}, Y_{/k})$. ☐

1.2.6. Lemma. *Let $\mathcal{A}$, $\mathcal{A}'$ be additive categories and $\lambda\colon \mathcal{A}' \to \mathcal{A}$ a fully faithful additive functor. Suppose:*

- *for every object T of $\mathcal{A}$, there exists an object T' of $\mathcal{A}'$ and $\mathcal{A}$-morphisms $T \xrightarrow{a} \lambda T' \xrightarrow{b} T$ with $ba = id_T$.*

Let $\widetilde{\mathcal{A}'}$, $\widetilde{\mathcal{A}}$ be the Karoubian envelopes of $\mathcal{A}'$, $\mathcal{A}$. Then the canonical functor $\tilde{\lambda}\colon \widetilde{\mathcal{A}'} \to \widetilde{\mathcal{A}}$ is an equivalence.

Proof. $\tilde{\lambda}$ is also fully faithful, so it is enough to show that every object of $\mathcal{A}$ lies in the essential image; but if $T \in \operatorname{Ob}\mathcal{A}$ then the objects $(\lambda T', ab)$ and (T, id_T) of $\widetilde{\mathcal{A}}$ are isomorphic, and $(\lambda T', ab) = \tilde{\lambda}(T', \lambda^{-1}(ab))$. □

1.2.7. Corollary. *The functor*

$$\begin{aligned} \mathcal{CV}^{gr}_{\mathfrak{o}} &\to \mathcal{CV}^{gr}_{k} \\ (X, m) &\mapsto (X_{/k}, 1, m) \end{aligned}$$

induces an equivalence of categories between $\widetilde{\mathcal{CV}^{gr}_{\mathfrak{o}}}$ and $\mathcal{M}_k$.

Proof. We just have to check the condition of the lemma. If $X_k \in \operatorname{Ob}\mathcal{V}_k$ is irreducible, let $X'_k \xrightarrow{h} X_k$ be an alteration with $X'_k = X' \otimes_{\mathfrak{o}} k$ for some integral $X' \in \operatorname{Ob}\mathcal{V}'_{\mathfrak{o}}$. Then $(a, b) = (h^*, \deg(h)^{-1} h_*)$ satisfy the condition. □

1.2.8. Corollary. *Let $\mathcal{C}$ be an E-linear Karoubian category whose* Hom*-groups are $\mathbb{Z}$-graded. Suppose we have functors*

$$\begin{aligned} H &\colon \mathcal{V}'^{\,opp}_{\mathfrak{o}} \to \mathcal{C} \\ H' &\colon \mathcal{V}'_{\mathfrak{o}} \to \mathcal{C} \end{aligned}$$

satisfying:

(i) *For every $X \in \operatorname{Ob}\mathcal{V}'_{\mathfrak{o}}$, $H(X) = H'(X)$.*

(ii) *H and H' are additive for disjoint unions.*

(iii) *For $f\colon X \to Y$, Hf is graded of degree 0; $H'f$ is graded of degree $\dim Y - \dim X$ if X and Y are integral.*

(iv) *For any finite collection of diagrams $X \xleftarrow{f_\alpha} Z_\alpha \xrightarrow{g_\alpha} Y$ in $\mathcal{V}'_{\mathfrak{o}}$ and $c_\alpha \in E$, the morphism*

$$\sum_\alpha c_\alpha\, H'g_\alpha \circ Hf_\alpha \colon H(X) \to H(Y)$$

depends only on the class of $\sum c_\alpha g_{\alpha/k*} f^*_{\alpha/k}$ in $\operatorname{Corr}^*(X_{/k}, Y_{/k})$.

Then there is an additive functor $\bar{H}\colon \mathcal{M}_k \to \mathcal{C}$ such $\bar{H} \circ h_k = H$ and $\bar{H} \circ h_k^{\vee} = H'$.

Proof. First define the restriction of $\bar{H}$ to $\mathcal{CV}^{\mathrm{gr}}_{\mathfrak{o}}$. On objects, put $\bar{H}(X,m) = H(X)$ with grading $\mathrm{Gr}^i\,\bar{H}(X,m) = \mathrm{Gr}^{i+m}\,H(X)$. If $X \xleftarrow{f} Z \xrightarrow{g} Y$ is a diagram in $\mathcal{V}'_{\mathfrak{o}}$ with $\dim Y - \dim Z = n - m$, write

$$\begin{aligned} H(g_{/k*}f^*_{/k}) = H'g \circ Hf &\in \mathrm{Hom}^{n-m}_{\mathcal{C}}(H(X), H(Y)) \\ &= \mathrm{Hom}^0_{\mathcal{C}}(\bar{H}(X,m), \bar{H}(Y,n)). \end{aligned}$$

Extend this definition by E-linearity to the group of formal linear combinations

$$\widetilde{\mathrm{Hom}}((X,m),(Y,n)) = \left\{ \sum_\alpha c_\alpha g_{\alpha/k*} f^*_{\alpha/k} \;\middle|\; \begin{array}{l} X \xleftarrow{f_\alpha} Z_\alpha \xrightarrow{g_\alpha} Y,\ c_\alpha \in E, \\ \dim Y - \dim Z_\alpha = n - m \end{array} \right\}$$

By the hypotheses and lemma 1.2.5, it factors through the quotient

$$\begin{array}{ccc} \widetilde{\mathrm{Hom}}((X,m),(Y,n)) & \longrightarrow & \mathrm{Hom}_{\mathcal{CV}^{\mathrm{gr}}_{\mathfrak{o}}}((X,m)(Y,n)) \\ & \searrow^{\bar{H}} & \Big\downarrow{\scriptstyle \bar{H}} \\ & & \mathrm{Hom}_{\mathcal{C}}(\bar{H}(X,m), \bar{H}(Y,n)) \end{array}$$

and this gives a functor on $\mathcal{CV}^{\mathrm{gr}}_{\mathfrak{o}}$ with the required properties. Since $\widetilde{\mathcal{CV}^{\mathrm{gr}}_{\mathfrak{o}}}$ is equivalent to $\mathcal{M}_k$ by Corollary 1.2.7 and $\mathcal{C}$ is Karoubian, $\bar{H}$ factors through $\mathcal{M}_k$. □

1.3 Motivic cohomology

1.3.1. We briefly recall how the pullback and pushforward maps in motivic cohomology are defined; see [13] or [14] for details. If $f\colon X \to Y$ is a morphism of schemes, the pullback $f^*\colon K_*Y \to K_*X$ is a λ-ring homomorphism, hence induces a map on the Adams eigenspaces, which are motivic cohomology.

1.3.2. If X, Y are smooth over a field and equidimensional, and f is proper, then we have a proper pushforward map in K-theory

$$f_*\colon K_*X \xleftarrow{\sim} K'_*X \to K'_*Y \xrightarrow{\sim} K_*Y$$

by composing the pushforward map in K'-theory with the Poincaré duality isomorphism $K_*(-) \xrightarrow{\sim} K'_*(-)$ for regular schemes. This composite map respects the γ-filtration up to a shift of $\dim Y - \dim X$ (by the Riemann-Roch theorem), inducing a map

$$\mathrm{Gr}_\gamma(f_*)\colon \mathrm{Gr}^\bullet_\gamma K_*X \to \mathrm{Gr}^{\bullet - \dim X + \dim Y}_\gamma K_*Y.$$

Composing this with the isomorphism $K^{(n)}_*(-) \xrightarrow{\sim} \mathrm{Gr}^n_\gamma K_*(-) \otimes \mathbb{Q}$ (given by the formal Chern character) this defines the pushforward map for a proper morphism of smooth varieties.

1.3.3. Now fix $q \in \mathbb{N}$, and define graded E-vector spaces $H(X) = H'(X)$ for X in $\mathcal{V}_{\mathfrak{o}}$ by

$$\begin{aligned} H(X) &= \mathrm{Gr}^*_\gamma \operatorname{Im}(K_q X \to K_q X_{/k}) \otimes E \\ &\simeq \bigoplus_n \operatorname{Im}(K_q^{(n)} X \to K_q^{(n)} X_{/k}) \otimes E \subset \bigoplus_n H^{2n-q}_{\mathcal{M}}(X_{/k}, n) \end{aligned}$$

Let $f \colon X \to Y$ be any morphism in $\mathcal{V}_{\mathfrak{o}}$. The pullback map

$$f^*_{/k} \colon \bigoplus_n H^{2n-q}_{\mathcal{M}}(Y_{/k}, n) \to \bigoplus_n H^{2n-q}_{\mathcal{M}}(X_{/k}, n)$$

maps $H(Y)$ into $H(X)$ (since f is a morphism of the underlying $\mathfrak{o}$-schemes), and we take this to be Hf. For $H'f$, we consider the Cartesian diagram

$$\begin{array}{ccc} X_{/k} & \xhookrightarrow{j_X} & X \\ {\scriptstyle f_{/k}}\downarrow & & \downarrow{\scriptstyle f} \\ Y_{/k} & \xhookrightarrow{j_Y} & Y \end{array}$$

The projection formula gives

$$f_{/k*} j_X^* K_q X = j_Y^* f_* K_q X \subset j_Y^* K_q Y$$

hence $f_{/k*}(H(X)) \subset H(Y)$. So if we define $Hf = f^*_{/k}$, $H'f = f_{/k*}$, then all the conditions of Corollary 1.2.8 are satisfied (taking $\mathcal{C}$ to be the category of graded $\mathbb{Q}$-vector spaces). This proves Theorem 1.1.6.

1.3.4. Corollary. *Let $E = \mathbb{Q}$. If $X_{/k}$ is smooth and proper over k and we are given a diagram*

$$\begin{array}{ccc} X'_{/k} & \xhookrightarrow{j} & X' \\ {\scriptstyle \phi}\downarrow & & \\ X & & \end{array}$$

where ϕ is an alteration and X' is proper and flat over $\mathfrak{o}$ and regular, then

$$H^i_{\mathcal{M}/\mathfrak{o}}(h(X_{/k}), n) \xrightarrow[\phi^*]{\sim} \operatorname{Im}\Big[K^{(n)}_{2n-i}(X') \xrightarrow{j^*} K^{(n)}_{2n-i}(X'_{/k})\Big] \cap \phi^* K^{(n)}_{2n-i} X.$$

Proof. This follows from Theorem 1.1.6 because on motivic cohomology, $\phi_*\phi^*$ is multiplication by the generic degree of ϕ. □

1.3.5. We shall write down explicitly the (rather simple) relation between the local and global situations. Let k be a number field, v a finite place of k, and k_v the completion of k at v. There is a basechange functor $\mathcal{M}_k \to \mathcal{M}_{k_v}$, $M \mapsto M_v$, inducing natural maps

$$\rho_v \colon H^i_{\mathcal{M}}(M,n) \to H^i_{\mathcal{M}}(M_v,n)$$

and likewise in K-theory.

1.3.6. Proposition.

$$H^i_{\mathcal{M}/\mathfrak{o}}(M,n) = \bigcap_{v\notin S} \rho_v^{-1} H^i_{\mathcal{M}/\mathfrak{o}_v}(M_v,n) \subset H^i_{\mathcal{M}}(M,n).$$

Proof. It is enough to check this for $M = h_k(X)$. In this case it follows at once from the diagram of localisation sequences:

$$\begin{array}{ccccc}
K_qX & \longrightarrow & K_qX_{/k} & \longrightarrow & \bigoplus_{v\notin S} K_{q-1}X \otimes k(v) \\
\downarrow & & \downarrow & & \downarrow \\
\prod_{v\notin S} K_qX \otimes \mathfrak{o}_v & \longrightarrow & \prod_{v\notin S} K_qX \otimes k_v & \longrightarrow & \prod_{v\notin S} K_{q-1}X \otimes k(v)
\end{array}$$

in which the right hand vertical arrow is an injection. □

2 K_1 of products of modular curves

2.1 Notations and conventions

2.1.1. We recall some basic facts about modular curves (see [4] or [9]). For a positive integer n, M_n denotes the modular curve over $\mathbb{Q}$ parameterising elliptic curves with full level n structure. It is the complement in the proper curve $\overline{M}_n$ of the cusps M_n^∞ (a finite union of copies of $\operatorname{Spec}\mathbb{Q}(\boldsymbol{\mu}_n)$).

2.1.2. If n is the product of two coprime integers each ≥ 3, then the curve M_n has a standard regular model $M_{n/\mathbb{Z}}$ over $\mathbb{Z}$ which parameterises elliptic curves with a Drinfeld level n structure, and which is the complement in a proper curve $M_{n/\mathbb{Z}}$ of the cuspidal subscheme $M^\infty_{n/\mathbb{Z}}$, a union of copies of $\operatorname{Spec}\mathbb{Z}[\boldsymbol{\mu}_n]$. Apart from the fact that $\overline{M}_{n/\mathbb{Z}}$ is regular, we need to know that the structural morphism $\overline{M}_{n/\mathbb{Z}} \to \operatorname{Spec}\mathbb{Z}$ factors through $\operatorname{Spec}\mathbb{Z}[\boldsymbol{\mu}_n]$, and that $\overline{M}_{n/\mathbb{Z}} \to \operatorname{Spec}\mathbb{Z}[\boldsymbol{\mu}_n]$ is smooth away from the supersingular points in characteristic $p|n$.

2.1.3. We need also to consider the modular curves at infinite level: $M = \varprojlim M_n$, $\overline{M} = \varprojlim \overline{M}_n$. These are schemes over $\mathbb{Q}(\boldsymbol{\mu}_\infty)$ which are not of finite type. M is regular, but $\overline{M}$ is not (the local rings at the cusps are non-discrete valuation rings since the coverings $\overline{M}_{n'} \to \overline{M}_n$ are ramified at the cusps).

2.1.4. Let G be the algebraic group GL_2, and write $G_p = G(\mathbb{Q}_p)$, $G_f = G(\mathbb{A}_f)$ (finite adelic points). Then G_f acts on M and $\overline{M}$, and this action extends to the models over $\mathbb{Z}$. (We assume that our level structures are defined so that this is a right action). If

$$K_n = \ker\Big(G(\hat{\mathbb{Z}}) \to G(\mathbb{Z}/n\mathbb{Z})\Big)$$

is the standard level n open compact subgroup of G_f then M_n is the quotient M/K_n and $\overline{M}_n = \overline{M}/K_n$. For any open compact subgroup $K \subset G_f$ write $M_K = M/K$, $\overline{M}_K = \overline{M}/K$.

2.1.5. We write $\mathcal{H}_f = \mathcal{H}(G_f)$ for the Hecke algebra of locally constant $\overline{\mathbb{Q}}$-valued functions of compact support on G_f. It is an algebra under convolution and has the involution $\phi \mapsto \phi^t$, where $\phi^t(x) = \phi(x^{-1})$. Write $\mathcal{H}_n = \mathcal{H}(G_f, K_n)$ for the level n subalgebra of $\mathcal{H}_f$ (the subalgebra of K_n-biinvariant functions).

2.1.6. We can regard $\mathcal{H}_f$ as a module for the product $G_f \times G_f$ by

$$((g_1, g_2)\phi)(x) = \phi(g_1^{-1} x g_2) \tag{2.1.6.1}$$

and in the usual way it then becomes an $\mathcal{H}_f \otimes \mathcal{H}_f$-module, given by

$$(\psi_1 \otimes \psi_2)\phi = \psi_1 * \phi * \psi_2^t$$

if ψ_i, $\phi \in \mathcal{H}_f$. We then have the following consequence of Frobenius reciprocity:

2.1.7. Lemma. *Let σ_i $(1 \le i \le 3)$ be smooth $\overline{\mathbb{Q}}$-representations of G_f, with σ_2, σ_3 admissible. Then*

$$\mathrm{Hom}_{G_f \times G_f}(\sigma_1 \otimes \mathcal{H}_f, \sigma_2 \otimes \sigma_3) = \mathrm{Hom}_{G_f}(\sigma_1 \otimes \tilde{\sigma}_2 \otimes \tilde{\sigma}_3, \overline{\mathbb{Q}}).$$

Here $G_f \times G_f$ acts on $\sigma_2 \otimes \sigma_3$ by the tensor product action, and on $\sigma_1 \otimes \mathcal{H}_f$ by $(g_1, g_2)(v \otimes \phi) = g_1 v \otimes (g_1, g_2)\phi$, cf. (2.1.6.1).

Proof. More generally, let H be a group of t-d type [2], and $K \subset H$ a closed subgroup. Let σ, τ be smooth representations of K and H respectively over a field F of characteristic zero, and $\tilde{\tau}$ the H-contragredient of τ. Then (cf. [2] section 1)

$$\mathrm{Hom}_H(c\text{-}\mathrm{Ind}_K^H \sigma, \tilde{\tau}) \xrightarrow{\sim} \mathrm{Hom}_K(\sigma \otimes \tau, F).$$

Taking $H = G_f \times G_f$, K to be the the diagonal, and $\tau = \sigma_2 \times \sigma_3$ gives the lemma. □

The obvious analogous statements hold for the local Hecke algebra $\mathcal{H}_p = \mathcal{H}(G_p)$.

2.1.8. If $H = \varprojlim H_i$ is any profinite set and S is a scheme, we can form the product scheme $H \times S$, which is the inverse limit of the finite disjoint unions $H_i \times S$. If H is locally profinite (for example, $H = G_f$) we can similarly define $H \times S$ by gluing. We will use constructions of this kind without further comment and leave the (elementary) justifications to the reader.

2.2 Motivic decomposition

2.2.1. We now review the decomposition of the motive of a modular curve. We will work in the category $\mathcal{M}_{\mathbb{Q}} \otimes \overline{\mathbb{Q}}$ of Chow motives over $\mathbb{Q}$ with coefficients in $\overline{\mathbb{Q}}$ — an object of this category is a triple $V = (X, p, m)$ where X is a smooth projective $\mathbb{Q}$-scheme, $m \in \mathbb{Z}$ and $p = p^2 \in \mathrm{Corr}^0(X, X) \otimes \overline{\mathbb{Q}}$. (We use the letter V to avoid confusion with modular curves.) If χ is a Dirichlet character, we write $V(\chi)$ for the twist of V by the Artin motive attached to χ.

2.2.2. There is the usual Chow-Künneth decomposition

$$h(\overline{M}_n) = h^0(\overline{M}_n) \oplus h^1(\overline{M}_n) \oplus h^2(\overline{M}_n)$$

which depends on the choice of a 0-cycle of nonzero degree on $\overline{M}_n$. We take the decomposition determined by a cusp (or any sum of cusps); since all cusps are linearly equivalent modulo torsion by Manin-Drinfeld, this decomposition is canonical, and is respected by the change of level maps $h(\overline{M}_n) \to h(\overline{M}_{n'})$ for $n|n'$. We have $h^0(\overline{M}_n) = h(\mathrm{Spec}\,\mathbb{Q}(\boldsymbol{\mu}_n))$ and $h^2(\overline{M}_n) = \mathbb{L} \otimes h(\overline{M}_n)$.

2.2.3. The Hecke algebra $\mathcal{H}_n$ acts on the motive $h(\overline{M}_n)$ by correspondences. Since Hecke operators take cusps to cusps, this action preserves the Chow-Künneth decomposition. The action of $\mathcal{H}_n$ on $h^1(\overline{M}_n)$ is semisimple, since the map

$$\mathrm{End}\, h^1(\overline{M}_n) \to \mathrm{End}\, \Omega^1(\overline{M}_n/\mathbb{Q}) \otimes_{\mathbb{Q}} \overline{\mathbb{Q}}$$

is injective. Moreover, if p is prime and $p \equiv 1 \pmod{n}$, the Hecke operator T_p acts on $h^0(\overline{M}_n)$ and $h^2(\overline{M}_n)$ by multiplication by $p+1$. Since $p+1$ cannot be an eigenvalue of T_p on $\Omega^1(\overline{M}_n/\mathbb{Q})$ (since $p+1 > 2\sqrt{p}$), it follows that there exists an element of $\mathcal{H}_n$ (even a polynomial in T_p) which is the identity on $h^1(\overline{M}_n)$ and annihilates h^0 and h^2.

2.2.4. According to the multiplicity one theorem, there is a decomposition

$$\Omega^1(\overline{M}_n/\mathbb{Q}) \otimes_{\mathbb{Q}} \overline{\mathbb{Q}} = \bigoplus_{\pi} [\pi]^{K_n}$$

into pairwise non-isomorphic $\mathcal{H}_n$-modules. Each $[\pi]^{K_n}$ is the space of K_n-invariants of K_n in an irreducible admissible representation $\pi\colon G_f \to GL([\pi])$, and

$$\Omega^1(\overline{M}/\mathbb{Q}) \otimes_{\mathbb{Q}} \overline{\mathbb{Q}} \stackrel{\mathrm{def}}{=} \varinjlim \Omega^1(\overline{M}_n/\mathbb{Q}) \otimes_{\mathbb{Q}} \overline{\mathbb{Q}} = \bigoplus_{\pi} [\pi]$$

We therefore get the decomposition of the motive:

$$h^1(\overline{M}_n) = \bigoplus_{\pi} V_\pi \otimes_{\overline{\mathbb{Q}}} [\pi]^{K_n}$$

where the sum is over those π occurring in $\Omega^1(\overline{M}) \otimes \overline{\mathbb{Q}}$ such that $[\pi]^{K_n} \neq 0$, and where

$$V_\pi = \mathrm{Hom}_{\mathcal{H}_n}([\pi]^{K_n}, h^1(\overline{M}_n)) \in \mathcal{M}_{\mathbb{Q}} \otimes \overline{\mathbb{Q}}$$

The motives V_π are simple of rank 2, and V_π, $V_{\pi'}$ are isomorphic if and only if $\pi \simeq \pi'$. We also know that

$$V_\pi^\vee \simeq V_{\tilde{\pi}} \otimes \mathbb{L}^{\otimes -1}.$$

2.2.5. It is convenient to work in the Ind-category Ind-$\mathcal{M}_{\mathbb{Q}} \otimes \overline{\mathbb{Q}}$, whose objects are inductive systems of objects of $\mathcal{M}_{\mathbb{Q}} \otimes \overline{\mathbb{Q}}$. In Ind-$\mathcal{M}_{\mathbb{Q}} \otimes \overline{\mathbb{Q}}$ we can simply write

$$h^1(\overline{M}) \stackrel{\mathrm{def}}{=} \lim_{\rightarrow} h^1(\overline{M}_n) = \bigoplus_{\pi} V_\pi \otimes [\pi].$$

2.2.6. By the Künneth formula $h(\overline{M}_n^2) = h(\overline{M}_n) \otimes h(\overline{M}_n)$ we can decompose the motive of $\overline{M}_n^2$ in the limit as

$$h(\overline{M}^2) = \bigoplus_{0 \leq i,j \leq 2} h^i(\overline{M}) \otimes h^j(\overline{M})$$

in which the most interesting part is

$$h^{1,1}(\overline{M}^2) = h^1(\overline{M})^{\otimes 2} = \bigoplus_{\pi,\pi'} V_{\pi\times\pi'} \otimes [\pi \times \pi']..$$

Here for each pair (π, π') of irreducible admissible representations of G_f occurring in $\Omega^1(\overline{M}) \otimes \overline{\mathbb{Q}}$, we have written

$$V_{\pi\times\pi'} = V_\pi \otimes V_{\pi'}$$

and $[\pi \times \pi']$ is the space of the exterior tensor product of π and π'. We get a corresponding decomposition of the motivic cohomology:

$$\begin{aligned} H^3_{\mathcal{M}}(\overline{M}^2, 2) &= \bigoplus_{i,j} H^3_{\mathcal{M}}(h^{i,j}(\overline{M}_n^2), 2) \\ &\supset H^3_{\mathcal{M}}(h^{1,1}(\overline{M}^2), 2) = \bigoplus_{\pi,\pi'} H^3_{\mathcal{M}}(V_{\pi\times\pi'}, 2) \otimes [\pi \times \pi']. \end{aligned} \quad (2.2.6.1)$$

2.2.7. We need to recall what are the rank one submotives of $V_{\pi\times\pi'}$. These exist if and only if there is a Dirichlet character χ such that

$$\pi' \simeq \tilde{\pi} \otimes \chi \circ \det . \tag{2.2.7.1}$$

To each such character corresponds a unique submotive of $V_{\pi\otimes\pi'}$ isomorphic to $\mathbb{L}(\chi)$, and we write

$$V_{\pi\times\pi'} = V^{\mathrm{trans}}_{\pi\times\pi'} \oplus V^{\mathrm{alg}}_{\pi\times\pi'} \tag{2.2.7.2}$$

where $V^{\mathrm{alg}}_{\pi\times\pi'}$ is the sum of the rank one submotives associated to that characters satisfying (2.2.7.1). If no such χ exists then of course $V^{\mathrm{trans}}_{\pi\times\pi'} = V_{\pi\times\pi'}$.

Suppose then that there is a χ satisfying (2.2.7.1). Assume first that π does not have complex multiplication. Then χ is unique, and $V^{\mathrm{trans}}_{\pi\times\pi'}$ is a simple motive of rank 3; in fact

$$V^{\mathrm{trans}}_{\pi\times\pi'} = \mathrm{Ad}^2\, V_\pi \otimes \mathbb{L}(\chi)$$

where if V is any motive, $\mathrm{Ad}^2\, V$ is its adjoint square, which is the kernel of the projector $V \otimes V^\vee \to \mathbb{1} \to V \otimes V^\vee$.

If π has complex multiplication, and ϵ is the quadratic character attached to the CM field, then if (2.2.7.1) holds we also will have

$$\pi' \simeq \tilde{\pi} \otimes \epsilon\chi \circ \det .$$

In this case $V_{\pi\times\pi'} = \mathbb{L}(\chi) \oplus \mathbb{L}(\epsilon\chi)$, and $V^{\mathrm{trans}}_{\pi\times\pi'}$ is a simple rank 2 motive.

2.2.8. Returning to finite level, consider now the motivic cohomology of $\overline{M}^2_n \setminus M^{\infty 2}_n$. We have a diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & H^3_{\mathcal{M}}(\overline{M}^2_n, 2) & \longrightarrow & H^3_{\mathcal{M}}(\overline{M}^2_n \setminus M^{\infty 2}_n, 2) & \longrightarrow & \mathbb{Q}[M^{\infty 2}_n] \\ & & \uparrow & & & & \\ & & H^3_{\mathcal{M}}(h^{1,1}(\overline{M}^2_n), 2) & & & & \end{array}$$

where the vertical arrow has a unique $\mathcal{H}_n \otimes \mathcal{H}_n$-equivariant splitting, given by the Chow-Künneth decomposition. Moreover, $\mathcal{H}_n \otimes \mathcal{H}_n$ acts on $H^3_{\mathcal{M}}(h^{1,1}(\overline{M}^2_n), 2)$ via the tensor product of representations of the form $[\pi]^{K_n}$, whereas on $\mathbb{Q}[M^{\infty 2}_n]$ it acts via representations occurring in Eisenstein series. So by the Manin-Drinfeld theorem, there is a unique $\mathcal{H}_n \otimes \mathcal{H}_n$-equivariant splitting of the composite inclusion

$$H^3_{\mathcal{M}}(h^{1,1}(\overline{M}^2_n), 2) \hookrightarrow H^3_{\mathcal{M}}(\overline{M}^2_n \setminus M^{\infty 2}_n, 2)$$

and it is induced by an element of $\mathcal{H}_n \otimes \mathcal{H}_n$. Passing to infinite level, we obtain a unique $G_f \times G_f$-equivariant splitting of the inclusion

$$H^3_{\mathcal{M}}(h^{1,1}(\overline{M}^2), 2) \hookrightarrow H^3_{\mathcal{M}}(\overline{M}^2 \setminus M^{\infty 2}, 2). \tag{2.2.8.1}$$

2.3 Beilinson's elements

2.3.1. Unless otherwise noted, the reader should interpret all products of schemes as absolute products (over $\operatorname{Spec}\mathbb{Z}$). Consider the map

$$\begin{gathered} M \times G_f \to M^2 = M \times M \\ (m, g) \mapsto (m, mg) \end{gathered}$$

which is $G_f \times G_f$-equivariant with respect to the action

$$(m, g)(g_1, g_2) = (mg_1, g_1^{-1} g g_2)$$

of $G_f \times G_f$ on $M \times G_f$. If $K \subset K' \subset G_f$ are open compact subgroups, then we get the diagram below in which the composite horizontal maps i_K, $i_{K'}$ are proper:

$$\begin{array}{ccccc} i_K\colon\ M \times G_f/K \times K & \longrightarrow & M_K^2 & \hookrightarrow & \overline{M}_K^2 \setminus M_K^{\infty 2} \\ \big\downarrow & & & & \big\downarrow \beta \\ i_{K'}\colon\ M \times G_f/K' \times K' & \longrightarrow & M_{K'}^2 & \hookrightarrow & \overline{M}_{K'}^2 \setminus M_{K'}^{\infty 2} \end{array} \tag{2.3.1.1}$$

2.3.2. The quotient $M \times G_f/K \times K$ can be written as a disjoint union of modular curves of finite level

$$\coprod_{KgK \in K\backslash G_f/K} M_{K \cap gKg^{-1}}$$

from which it is easy to check that the two squares in (2.3.1.1) are Cartesian. We have in K-theory $i_{K*}\alpha^* = \beta^* i_{K'*}$. Therefore the proper pushforward $i_{K*}\colon \mathcal{O}^*(M \times G_f/K \times K) \to H^3_{\mathcal{M}}(\overline{M}_K^2 \setminus M_K^{\infty 2}, 2)$ defines in the limit a map

$$\begin{array}{ccc} \varinjlim \mathcal{O}^*(M \times G_f/K \times K) & & \varinjlim H^3_{\mathcal{M}}(\overline{M}_K^2 \setminus M_K^{\infty 2}, 2) \\ \| & & \| \\ \mathcal{O}^*(M \times G_f) & \xrightarrow{i_*} & H^3_{\mathcal{M}}(\overline{M}^2 \setminus M^{\infty 2}, 2) \end{array}$$

We have $\mathcal{O}^*(M \times G_f) \otimes_Z \overline{\mathbb{Q}} = \mathcal{O}^*(M) \otimes_{\mathbb{Z}} \mathcal{H}_f$, and the induced action of $G_f \times G_f$ is via the first factor on $\mathcal{O}^*(M)$ and by the action (2.1.6.1) on $\mathcal{H}_f$.

2.3.3. Now we compose with the Manin-Drinfeld splitting of (2.2.8.1) and use the motivic decomposition (2.2.6.1) to get a $G_f \times G_f$-equivariant homomorphism

$$\mathcal{O}^*(M) \otimes_{\mathbb{Z}} \mathcal{H}_f \to H^3_{\mathcal{M}}(V_\pi \otimes V_{\pi'}, 2) \otimes [\pi \times \pi'].$$

Applying Frobenius reciprocity 2.1.7, we finally get the *Beilinson homomorphism*

$$\text{Б}(\pi \times \pi')\colon \big(\mathcal{O}^*(M) \otimes_{\mathbb{Z}} \tilde{\pi} \otimes_{\overline{\mathbb{Q}}} \tilde{\pi}'\big)_{G_f} \to H^3_{\mathcal{M}}(V_{\pi \times \pi'}, 2)$$

whose source is the maximal quotient of $\mathcal{O}^*(M) \otimes_{\mathbb{Z}} \tilde{\pi} \otimes_{\overline{\mathbb{Q}}} \tilde{\pi}'$ on which G_f acts trivially. We can then state the main result of this paper, which completes [1, Theorem 6.1.1]:

2.3.4. Theorem. *Suppose π is not isomorphic to any twist of π'. Then the image of* $\text{Б}(\pi \times \pi')$ *is contained in* $H^3_{\mathcal{M}/\mathbb{Z}}(V_{\pi\times\pi'}, 2)$.

2.3.5. Remark. Beilinson's conjectures predict that the space $H^3_{\mathcal{M}/\mathbb{Z}}(V_{\pi\times\pi'}, 2)$ is 1-dimensional. Beilinson's computations [1] of the composition of $\text{Б}(\pi \times \pi')$ and the regulator show that the dimension is at least one. In the other direction, in [8] M. Harris and the author prove that the source $(\mathcal{O}^*(M) \otimes_{\mathbb{Z}} \tilde{\pi} \otimes_{\overline{\mathbb{Q}}} \tilde{\pi}')_{G_f}$ of the map $\text{Б}(\pi \times \pi')$ is exactly 1-dimensional if (and only if) π' is not a twist of π. In other words, the image of $\text{Б}(\pi \times \pi')$ has dimension one.

2.3.6. We now consider the case where π, π' are twists of one another, so that $V_{\pi\times\pi'}$ has the decomposition (2.2.7.2). The composite of the Beilinson homomorphism with the projection onto the algebraic component:

$$\begin{aligned}\text{Б}(\pi \times \pi')\colon \left(\mathcal{O}^*(M) \otimes_{\mathbb{Z}} \tilde{\pi} \otimes_{\overline{\mathbb{Q}}} \tilde{\pi}'\right)_{G_f} &\to H^3_{\mathcal{M}}(V_{\pi\times\pi'}, 2) \\ &\to H^3_{\mathcal{M}}(V^{\text{alg}}_{\pi\times\pi'}, 2) = \begin{cases} H^1_{\mathcal{M}}(\mathbb{1}(\chi), 1) & \text{(no CM)} \\ H^1_{\mathcal{M}}(\mathbb{1}(\chi) + \mathbb{1}(\epsilon\chi), 1) & \text{(CM)} \end{cases}\end{aligned}$$

is not particularly interesting; it can be described explicitly, using Lemma 2.5.2 below. (The motivic cohomology $H^1_{\mathcal{M}}(\mathbb{1}(\chi), 1)$ is simply the χ-isotypical component of $\mathbb{Q}(\boldsymbol{\mu}_\infty)^* \otimes_{\mathbb{Z}} \overline{\mathbb{Q}}$.) The transcendental component is more interesting.

2.3.7. Theorem. *Assume that π' is isomorphic to a twist of π.*

(i) If $\pi' \not\simeq \tilde{\pi}$, the image of $\text{Б}(\pi \times \pi')$ *lies in* $H^3_{\mathcal{M}/\mathbb{Z}}(V^{\text{trans}}_{\pi\times\pi'}, 2)$.

(ii) (Flach [7]) If $\pi' \simeq \tilde{\pi}$, then the image of $\text{Б}(\pi \times \pi')$ *in* $H^3_{\mathcal{M}}/H^3_{\mathcal{M}/\mathbb{Z}}(V^{\text{trans}}_{\pi\times\pi'}, 2)$ *is infinite-dimensional.*

2.3.8. Remark. Beilinson's conjectures imply that $H^3_{\mathcal{M}/\mathbb{Z}}(V^{\text{trans}}_{\pi\times\pi'}, 2) = 0$ (as the associated Deligne cohomology group is zero). So in case (i) the Beilinson elements should be trivial.

2.3.9. Remark. In the case when $\pi' \simeq \tilde{\pi}$ does not have complex multiplication, Flach's argument shows that for every prime p at which π is unramified, the image of $\text{Б}(\pi \times \tilde{\pi})$ does not lie in $H^3_{\mathcal{M}/\mathbb{Z}_{(p)}}(V^{\text{trans}}_{\pi\times\tilde{\pi}}, 2)$. His proof (which we do not reproduce here) can be summarised as follows: for each such p the motive V_π can be reduced mod p to obtain a motive $V^{(p)}_\pi$ over $\mathbb{F}_p$, and the obstruction to the integrality of an element of $H^3_{\mathcal{M}}(V_{\pi\times\tilde{\pi}}, 2)$ can be computed from the exact localisation sequence

$$H^3_{\mathcal{M}/\mathbb{Z}_{(p)}}(V_{\pi\times\tilde{\pi}}, 2) \to H^3_{\mathcal{M}}(V_{\pi\times\tilde{\pi}}, 2) \xrightarrow{\partial_p} \text{Pic}(V^{(p)}_{\pi\times\tilde{\pi}}).$$

Now $\mathrm{Pic}(V^{(p)}_{\pi\times\tilde\pi}) = \mathrm{End}\, V^{(p)}_\pi$, and the subspace

$$\mathrm{Pic}(V^{\mathrm{alg}(p)}_{\pi\times\tilde\pi}) \subset \mathrm{Pic}(V^{(p)}_{\pi\times\tilde\pi}) \tag{2.3.9.1}$$

simply corresponds to those endomorphisms of $V^{(p)}_\pi$ which lift to characteristic zero; in other words, the one-dimension subspace generated by the identity endomorphism. Then Flach uses an explicit modular unit supported on the Hecke correspondence T_p to construct an element of $H^3_{\mathcal{M}}(V_{\pi\times\tilde\pi}, 2)$ whose image under ∂_p is the graph of Frobenius; hence its transcendental component is not integral.

The same argument works in the case of complex multiplication: in this case the subspace (2.3.9.1) consists of all endomorphisms which lift to characteristic zero over the CM field. For a good prime p which is inert in the CM field, the Frobenius endomorphism of $V^{(p)}_\pi$ does not lift, so for all such primes the Flach element gives an element of $H^3_{\mathcal{M}}(V^{\mathrm{trans}}_{\pi\times\tilde\pi}, 2)$ which is non-integral at p.

2.3.10. Because we are working with a product of curves, it is possible to prove that integral motivic cohomology is the same as the Bloch-Kato f-subspace. We only make this statement precise in the case of ℓ-adic cohomology over $\mathbb{Q}_p$, with $p \neq \ell$; if U is any continuous finite-dimensional ℓ-adic representation of $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$, then $H^1_f(\mathbb{Q}_p, U) = H^1_{\mathrm{nr}}(\mathbb{Q}_p, U)$ is the unramified Galois cohomology, which fits into the exact sequence

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H^1(\mathbb{Q}_p^{\mathrm{nr}}/\mathbb{Q}_p, U)^{\mathcal{I}_p} & \longrightarrow & H^1(\mathbb{Q}_p, U) & \longrightarrow & H^1(\mathbb{Q}_p^{\mathrm{nr}}, U)^{\mathrm{Frob}_p=1} & \longrightarrow & 0 \\ & & \| \! \therefore & & & & \| \! \therefore & & \\ & & H^1_{\mathrm{nr}}(\mathbb{Q}_p, U) & & & & H^1/H^1_{\mathrm{nr}}(\mathbb{Q}_p, U) & & \end{array}$$

For a smooth and proper variety X over $\mathbb{Q}$, one has an ℓ-adic Abel-Jacobi homomorphism

$$AJ(X,j)\colon H^{2j-1}_{\mathcal{M}}(X,j) \longrightarrow H^1(\mathbb{Q}, H^{2j-2}(\overline{X}, \mathbb{Q}_\ell)(j))$$

and localising at p one can show that for $j = 2$ and X a product of two curves, the sequence

$$0 \longrightarrow H^3_{\mathcal{M}/\mathbb{Z}_{(p)}}(X,2) \longrightarrow H^3_{\mathcal{M}}(X,2) \xrightarrow{AJ(X,2)} H^1/H^1_{\mathrm{nr}}(\mathbb{Q}_p, H^2(\overline{X}, \mathbb{Q}_\ell)(2))$$

is exact.

2.3.11. We can now consider in greater detail the case $\pi' \simeq \tilde\pi$. Assume that π has no complex multiplication. For a motive V, let V_ℓ denote its ℓ-adic realisation. We then get for every p and every $\ell \neq p$ an exact sequence:

$$0 \to H^3_{\mathcal{M}/\mathbb{Z}_{(p)}}(V^{\mathrm{trans}}_{\pi\times\tilde\pi}, 2) \otimes \overline{\mathbb{Q}}_\ell \to H^3_{\mathcal{M}}(V^{\mathrm{trans}}_{\pi\times\tilde\pi}, 2) \otimes \overline{\mathbb{Q}}_\ell \to H^1/H^1_{\mathrm{nr}}(\mathbb{Q}_p, V^{\mathrm{trans}}_{\pi\times\tilde\pi,\ell}(2)).$$

We can ask when the last group

$$H^1/H^1_{\mathrm{nr}}(\mathbb{Q}_p, V^{\mathrm{trans}}_{\pi\times\tilde{\pi},\ell}(2)) = \left(\mathrm{Ad}^2\, V_{\pi,\ell}\right)^{\mathrm{Frob}_p=1}_{\mathcal{I}_p}$$

is nonzero. We know that Frobenius acts semisimply on $V_{\pi,\ell}$ (since it is part of the H^1 of a curve), so

$$\dim \left(\mathrm{Ad}^2\, V_{\pi,\ell}\right)^{\mathrm{Frob}_p=1}_{\mathcal{I}_p} = \dim \mathrm{End}_{\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)}(V_{\pi,\ell}) - 1.$$

To proceed further, it is most natural to consider the possible types of the local factor π_p of π, in the rough classification of irreducible admissible representations of $GL_2(\mathbb{Q}_p)$:

2.3.12. If π_p is principal series, then there are quasi-characters $\mu_1, \mu_2\colon \mathbb{Q}_p^* \to \overline{\mathbb{Q}}^*$ with $|\mu_i(p)| = \sqrt{p}$, such that $V_{\pi,\ell} = [\mu_1] \oplus [\mu_2]$. Moreover, it is known [3] that one cannot have $\mu_1 = \mu_2$, so in this case H^1/H^1_{nr} has dimension 1.

- If π_p is unramified, then Flach's construction 2.3.7(ii) shows that the image of $\text{Б}(\pi \times \pi')$ maps onto H^1/H^1_{nr}. More generally, if π_p is a twist of an unramified representation (*i.e.* if μ_1/μ_2 is unramified) then it is not hard to show (although we do not give the details) that after twisting, the Flach elements map onto H^1/H^1_{nr}.
- If μ_1/μ_2 is ramified, it is not clear whether there are global elements of motivic cohomology whose images generate H^1/H^1_{nr}.

2.3.13. If π_p is special or supercuspidal then $V_{\pi,\ell}$ is indecomposable as a representation of $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ over $\overline{\mathbb{Q}}_\ell$. Hence in this case $H^1/H^1_{\mathrm{nr}} = 0$, and so every class is automatically integral at p.

2.3.14. The case of complex multiplication can be treated in the same way. Let F be the (imaginary quadratic) field of complex multiplication; then there is a Hecke character $\phi\colon\ \mathrm{Gal}(\overline{\mathbb{Q}}/F) \to \overline{\mathbb{Q}}_\ell^*$ such that $V_{\pi,\ell}$ is the representation obtained from ϕ by induction. Then

$$\mathrm{End}_{\overline{\mathbb{Q}}_\ell}\, V_{\pi,\ell} = V_{\pi\times\tilde{\pi},\ell}(1) = V^{\mathrm{trans}}_{\pi\times\tilde{\pi},\ell}(1) \oplus \overline{\mathbb{Q}}_\ell \oplus \overline{\mathbb{Q}}_\ell(\epsilon).$$

The restriction of $V_{\pi,\ell}$ to $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ is irreducible if p is ramified in F, and is the sum of two non-isomorphic characters otherwise. We therefore obtain

$$\dim H^1/H^1_{\mathrm{nr}}(\mathbb{Q}_p, V^{\mathrm{trans}}_{\pi\times\tilde{\pi},\ell}(1)) = \begin{cases} 1 & \text{if } p \text{ is inert in } F \\ 0 & \text{otherwise.} \end{cases}$$

So if p is split or ramified, there is no obstruction to integrality. If p is an inert prime of good reduction, the Flach element maps to a generator H^1/H^1_{nr}, by the discussion in Remark 2.3.9. Therefore the only ambiguity remaining is when p is an inert prime for which ϕ_p is ramified.

2.4 Modular units

2.4.1. We recall without proof the results of [11] about the representation theory of modular units. For a continuous character $\chi\colon \mathbb{A}_f^* \to \overline{\mathbb{Q}}^*$ (resp. $\chi\colon \mathbb{Q}_p^* \to \overline{\mathbb{Q}}^*$), define $\mathcal{S}(\chi)$ (resp. $\mathcal{S}_p(\chi)$) to be the space of locally constant $\overline{\mathbb{Q}}$-valued functions ϕ on G_f resp. G_p such that

$$\phi\left(\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} g\right) = \chi(d) \left\| \frac{a}{d} \right\| \phi(g) \quad \text{for all } \begin{pmatrix} a & b \\ 0 & d \end{pmatrix},\ g \in G_f \text{ resp. } G_p.$$

Here $\|-\|$ denotes finite idelic (resp. p-adic) modulus. Right translation make $\mathcal{S}(\chi)$, $\mathcal{S}_p(\chi)$ admissible representations of G_f and G_p and if $\chi = \otimes_p \chi_p$ for characters $\chi_p\colon \mathbb{Q}_p^* \to \overline{\mathbb{Q}}^*$, almost all unramified, then $\mathcal{S}(\chi)$ is a restricted tensor product

$$\mathcal{S}(\chi) = \bigotimes_p{}' \mathcal{S}_p(\chi_p)$$

with respect to the spherical vectors $\phi_p^o \in \mathcal{S}_p(\chi_p)$ for χ_p unramified, uniquely determined by the condition $\phi_p^o|_{G(\mathbb{Z}_p)} = 1$.

2.4.2. Define subspaces for $\chi = 1$:

$$\mathcal{S}(1)^0 = \ker\left(\int_{G(\hat{Z})} : \mathcal{S}(1) \to \overline{\mathbb{Q}}\right)$$
$$\mathcal{S}_p(1)^0 = \ker\left(\int_{G(\mathbb{Z}_p)} : \mathcal{S}_p(1) \to \overline{\mathbb{Q}}\right)$$

Then $\mathcal{S}_p(1)^0$ is an irreducible G_p-module (the Steinberg representation of G_p), and there is a short exact nonsplit sequence

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{S}_p(1)^0 & \longrightarrow & \mathcal{S}_p(1) & \xrightarrow{\lambda_p} & \overline{\mathbb{Q}} & \longrightarrow & 0 \\ & & & & \Cup & & \Cup & & \\ & & & & \phi_p^o & \longmapsto & 1. & & \end{array}$$

Moreover $\mathcal{S}(1)^0$ is the space spanned by all $\otimes' \phi_p \in \mathcal{S}(1)$ such that for at least one p, $\lambda_p(\phi_p) = 0$.

2.4.3. Write $\mathcal{S}(1)^{00}$ for the space spanned by all $\otimes' \phi_p \in \mathcal{S}(1)$ such that for at least two distinct p, $\lambda_p(\phi_p) = 0$. It fits into an exact sequence:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{S}(1)^{00} & \longrightarrow & \mathcal{S}(1)^0 & \xrightarrow{(\Phi_p)_p} & \bigoplus_p \mathcal{S}_p(1)^0 & \longrightarrow & 0 \\ & & & & \Cup & & \Cup & & \\ & & & & \otimes' \phi_q & \mapsto & \left(\prod_{q \neq p} \lambda_q(\phi_q) \phi_p\right)_p & & \end{array}$$

(note that Φ_p is well-defined!)

2.4.4. Recall from [11] the representation-theoretic description of the space of modular units and its integral subspace. The divisor map gives an exact sequence

$$0 \longrightarrow \mathbb{Q}(\boldsymbol{\mu}_\infty)^* \longrightarrow \mathcal{O}^*(M) \xrightarrow{\text{div}} \mathcal{S}(1)^0 \oplus \bigoplus_{\chi\neq 1} \mathcal{S}(\chi) \tag{2.4.4.1}$$

which becomes exact on the right when the first 2 groups are tensored with $\overline{\mathbb{Q}}$ (Manin-Drinfeld theorem). The integral units $\mathcal{O}^*(M_{/\mathbb{Z}})$ fit into an exact sequence

$$0 \longrightarrow \mathbb{Z}[\boldsymbol{\mu}_\infty]^* \longrightarrow \mathcal{O}^*(M_{/\mathbb{Z}}) \xrightarrow{\text{div}} \mathcal{S}(1)^{00} \oplus \bigoplus_{\chi\neq 1} \mathcal{S}(\chi) \tag{2.4.4.2}$$

which maps to (2.4.4.1) by the obvious maps in the three terms. More precisely, for any p the sequence

$$0 \longrightarrow \mathbb{Q}(\boldsymbol{\mu}_\infty)^* \mathcal{O}^*(M_{/\mathbb{Z}_{(p)}}) \otimes_{\mathbb{Z}} \overline{\mathbb{Q}} \longrightarrow \mathcal{O}^*(M) \otimes_{\mathbb{Z}} \overline{\mathbb{Q}} \xrightarrow{\Phi_p \circ \text{div}} \mathcal{S}_p(1)^0 \longrightarrow 0$$

is exact.

2.5 Proofs

2.5.1. Lemma. *For every (π, π') the image under the Beilinson homomorphism $Б(\pi, \pi')$ of $\mathcal{O}^*(M_{/\mathbb{Z}}) \otimes \tilde{\pi} \otimes \tilde{\pi}'$ lies in $H^3_{\mathcal{M}/\mathbb{Z}}(V_{\pi\times\pi'}, 2)$.*

Proof. Recall that the Beilinson homomorphism is obtained from the pushforward map $\mathcal{O}^*(M) \otimes_{\mathbb{Z}} \mathcal{H}_f \to H^3_{\mathcal{M}}(\overline{M}^2 \setminus M^{\infty 2}, 2)$ by composing with the projection (2.2.8.1), which at any finite level is given by an element of the Hecke algebra. Therefore it is enough to show that

$$\operatorname{Im}\left[\mathcal{O}^*(M_{/\mathbb{Z}}) \otimes_{\mathbb{Z}} \mathcal{H}_f \to H^3_{\mathcal{M}}(\overline{M}^2 \setminus M^{\infty 2}, 2)\right] \cap H^3_{\mathcal{M}}(\overline{M}^2, 2) \subset H^3_{\mathcal{M}/\mathbb{Z}}(\overline{M}^2, 2).$$

We shall work at some finite level n. Choose a finite extension $F/\mathbb{Q}(\boldsymbol{\mu}_n)$, Galois over $\mathbb{Q}$, over which $\overline{M}_n$ acquires semistable reduction; let $\overline{M}'_{n/\mathbb{Z}}$ be the semistable model thus obtained. Since $\overline{M}_{n/\mathbb{Z}}$ is smooth over $\mathbb{Z}[\boldsymbol{\mu}_n]$ away from supersingular points, we can assume that there is a birational morphism $\overline{M}'_{n/\mathbb{Z}} \to \overline{M}_{n/\mathbb{Z}} \otimes_{\mathbb{Z}[\boldsymbol{\mu}_n]} \mathfrak{o}_F$ which is an isomorphism away from the supersingular points in characteristic $p|n$.

To obtain a regular alteration of $\overline{M}^2_{n/\mathbb{Z}}$ it then suffices to take the normalisation of $\overline{M}'_{n/\mathbb{Z}} \times_{\mathbb{Z}} \overline{M}'_{n/\mathbb{Z}}$, which has only ordinary double points as singularities, and blow them up once. We write $\overline{M}'_{n/\mathbb{Z}} \hat{\times} \overline{M}'_{n/\mathbb{Z}}$ for the result.

Let $\tau_\alpha \colon \bar{Y}_\alpha \to \overline{M}_n \times \overline{M}_n$ be Hecke correspondences, and $Y_\alpha \subset \bar{Y}_\alpha$ the complements of all cusps. We can assume that $Y_\alpha = M_m$ for some large m (the same for each α). Suppose that $u_\alpha \in \mathcal{O}^*(M_{m/\mathbb{Z}})$ are modular units, such that $\sum \tau_{\alpha *}(u_\alpha) \in H^3_{\mathcal{M}}(\overline{M}^2_n, 2)$. We are going to show that the pullback of $\sum \tau_{\alpha *}(u_\alpha)$

to $(\overline{M}_n \otimes F)^2$ extends to an element of $K_1(\overline{M}'_{n/\mathbb{Z}} \hat{\times} \overline{M}_{n/\mathbb{Z}}) \otimes \mathbb{Q}$ — by Corollary 1.3.4 this will prove the lemma.

Because it is a Hecke correspondence, the morphism

$$\tau_\alpha \colon \overline{M}_m \to \overline{M}_n \times \overline{M}_n$$

extends to a correspondence $\overline{M}_{m/\mathbb{Z}} \to \overline{M}_{n/\mathbb{Z}} \times \overline{M}_{n/\mathbb{Z}}$. By resolution of singularities for arithmetic surfaces, we can find some regular model $\overline{M}'_{m/\mathbb{Z}}$ for $\overline{M}_{m/F} := \overline{M}_m \otimes_{\mathbb{Q}(\boldsymbol{\mu}_n)} F$ over $\mathfrak{o}_F$ such that each τ_α extends to a correspondence

$$\tau'_\alpha \colon \overline{M}'_{m/\mathbb{Z}} \to \overline{M}'_{n/\mathbb{Z}} \hat{\times} \overline{M}'_{n/\mathbb{Z}}$$

and such that $\overline{M}'_{m/\mathbb{Z}} \to \overline{M}_{m/\mathbb{Z}}$ is finite away from supersingular points.

The divisor of cusps $M^\infty_{n/\mathbb{Z}} \subset \overline{M}_{n/\mathbb{Z}}$ pulls back to a divisor $M^{\infty\prime}_{n/\mathbb{Z}} \subset \overline{M}'_{n/\mathbb{Z}}$, which is a disjoint union of copies of $\operatorname{Spec} \mathfrak{o}_F$. Write $M'_{n/\mathbb{Z}} = \overline{M}'_{n/\mathbb{Z}} \setminus {M^\infty_{n/\mathbb{Z}}}'$ for its complement. Define similarly $M^{\infty\prime}_{m/\mathbb{Z}}$, $M'_{m/\mathbb{Z}}$. The inverse image of ${M^\infty_{n/\mathbb{Z}}}^2$ in $\overline{M}'_{n/\mathbb{Z}} \hat{\times} \overline{M}'_{n/\mathbb{Z}}$ is a finite union of copies of $\operatorname{Spec} \mathfrak{o}_F$, which we will simply denote $M^{\infty\prime}_{n/\mathbb{Z}} \hat{\times} M^{\infty\prime}_{n/\mathbb{Z}}$. The restriction

$$\tau'_\alpha \colon M'_{m/\mathbb{Z}} \to \overline{M}'_{n/\mathbb{Z}} \hat{\times} \overline{M}'_{n/\mathbb{Z}} \setminus M^{\infty\prime}_{n/\mathbb{Z}} \hat{\times} M^{\infty\prime}_{n/\mathbb{Z}}$$

is then proper.

The units u_α extend to units u'_α on $M'_{m/\mathbb{Z}}$ and so

$$\tau'_{\alpha*}(u'_\alpha) \in K_1(\overline{M}'_{n/\mathbb{Z}} \hat{\times} \overline{M}'_{n/\mathbb{Z}} \setminus M^{\infty\prime}_{n/\mathbb{Z}} \hat{\times} M^{\infty\prime}_{n/\mathbb{Z}})$$

We now have a commutative diagram of localisation sequences:

$$\begin{array}{ccccc} K_1(\overline{M}'_{n/\mathbb{Z}} \hat{\times} \overline{M}'_{n/\mathbb{Z}}) & \xrightarrow{\theta} & K_1(\overline{M}'_{n/\mathbb{Z}} \hat{\times} \overline{M}'_{n/\mathbb{Z}} \setminus M^{\infty\prime}_{n/\mathbb{Z}} \hat{\times} M^{\infty\prime}_{n/\mathbb{Z}}) & \to & K_0(M^{\infty\prime}_{n/\mathbb{Z}} \hat{\times} M^{\infty\prime}_{n/\mathbb{Z}}) \\ \downarrow & & \downarrow & & \downarrow \\ K_1((\overline{M}_{n/F})^2) & \to & K_1((\overline{M}_{n/F})^2 \setminus (M^\infty_{n/F})^2) & \to & K_0((M^\infty_{n/F})^2) \end{array}$$

and when tensored with $\mathbb{Q}$ the right-hand vertical arrow is an isomorphism. The middle vertical arrow maps $\tau'_{\alpha*}(u'_\alpha)$ to $\tau_{\alpha*}(u_\alpha)$. Since $\sum \tau_{\alpha*}(u_\alpha) \in H^3_{\mathcal{M}}(\overline{M}^2_n, 2)$, it follows from this diagram that

$$\sum \tau'_{\alpha*}(u'_\alpha) \in \operatorname{Im}(\theta)$$

as required. □

2.5.2. Lemma. *(i) If π' is not a twist of π, $Б(\pi,\pi')$ vanishes on $\mathbb{Q}(\boldsymbol{\mu}_\infty)^*\otimes\tilde\pi\otimes\tilde\pi'$.*

(ii) If π' is a twist of π, then

$$Б(\pi,\pi')(\mathbb{Q}(\boldsymbol{\mu}_\infty)^*\otimes\tilde\pi\otimes\tilde\pi')\subset H^3_{\mathcal{M}}(V^{\mathrm{alg}}_{\pi\times\pi'},2)\subset H^3_{\mathcal{M}}(V_{\pi\times\pi'},2).$$

Proof. (i) The action of G_f on $\mathbb{Q}(\boldsymbol{\mu}_\infty)^*$ factors through the determinant. If π and π' are not twists of one another, this implies that

$$\left(\mathbb{Q}(\boldsymbol{\mu}_\infty)^*\otimes\tilde\pi\otimes\tilde\pi'\right)_{G_f}=0.$$

(Alternatively one can use the argument in (ii) following.)

(ii) The homomorphism

$$\mathbb{Q}(\boldsymbol{\mu}_\infty)^*\otimes\mathcal{H}_f\to H^3_{\mathcal{M}}(\overline{M}^2,2)$$

can be described as follows: work at some finite level n, and let $a\in\mathbb{Q}(\boldsymbol{\mu}_n)^*$. For any Hecke correspondence $\tau\colon\overline{M}_m\to\overline{M}^2_n$, let $c_1(\tau)$ be the class of the cycle $\tau_*(\overline{M}_m)$ in $\mathrm{Pic}(\overline{M}^2_n)\otimes\overline{\mathbb{Q}}=H^2_{\mathcal{M}}(\overline{M}^2_n,1)$. Then the homomorphism is given by

$$a\otimes\tau\longmapsto pr_1^*(a)\cup c_1(\tau). \qquad (2.5.2.1)$$

In particular, it factors through $\mathbb{Q}(\boldsymbol{\mu}_\infty)*\otimes\mathrm{Pic}(\overline{M}^2)\otimes\overline{\mathbb{Q}}$, so that its image under the motivic decomposition lies in the algebraic part. □

2.5.3. Corollary. *The composite of the Beilinson homomorphism for $V^{\mathrm{trans}}_{\pi\times\pi'}$ and the quotient map $H_{\mathcal{M}}\to H_{\mathcal{M}}/H_{\mathcal{M}/\mathbb{Z}}$ factors as*

$$\begin{array}{ccc}
\left(\mathcal{O}^*(M)\otimes_{\mathbb{Z}}\tilde\pi\otimes_{\overline{\mathbb{Q}}}\tilde\pi'\right)_{G_f} & \xrightarrow{Б(\pi\times\pi')} & H^3_{\mathcal{M}}(V^{\mathrm{trans}}_{\pi\times\pi'},2)\\
\Big\downarrow{\scriptstyle(\Phi_p\circ\mathrm{div})_p} & & \Big\downarrow\\
\bigoplus_p\left(\mathcal{S}_p(1)^0\otimes_{\overline{\mathbb{Q}}}\tilde\pi\otimes_{\overline{\mathbb{Q}}}\tilde\pi'\right)_{G_f} & \longrightarrow & H^3_{\mathcal{M}}/H^3_{\mathcal{M}/\mathbb{Z}}(V^{\mathrm{trans}}_{\pi\times\pi'},2)
\end{array}$$

Proof. This follows from 2.5.1, 2.5.2 and the description of the modular units in §2.4. □

Proof of theorems 2.3.4 and 2.3.7(i). Both results follow as soon as we show that, if $\pi'\not\simeq\tilde\pi$, the bottom horizontal arrow in the above diagram is zero. For this, it is enough to show that for each p,

$$\left(\mathcal{S}_p(1)^0\otimes_{\overline{\mathbb{Q}}}\tilde\pi\otimes_{\overline{\mathbb{Q}}}\tilde\pi'\right)_{G_f}=0. \qquad (*)$$

Fix a prime p. For every $q\neq p$, the subgroup $G_q\subset G_f$ acts trivially on $\mathcal{S}_p(1)^0$, and by strong multiplicity one, there exist infinitely many q such that π'_q and π_q are not contragredient to one another. Choose one such $q\neq p$. Then $(\tilde\pi_q\otimes\tilde\pi'_q)_{G_q}=0$, hence $(*)$ holds. □

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