INTRODUCTION TO PLECTIC COHOMOLOGY

J. NEKOVÁŘ AND A. J. SCHOLL

Abstract. We formulate conjectures on the existence of extra symmetries of the cohomology of Shimura varieties whose defining group is a restriction of scalars from a totally real field. We discuss evidence in its favour and potential arithmetic applications.

1. Introduction

This article[1] is the first of a series of papers, in which we will examine the following phenomenon:

> **In the presence of real multiplication (by a totally real number field F of degree r), motives have a canonical and functorial additional structure (F-plectic structure).**

This statement is still largely conjectural – we call it the **Plectic Conjecture**[2]. More precise, but less general versions of this conjecture will be presented in Sections 6–7 below. In this paper we wish to give a survey of what the conjectures are, and what kind of consequences they have. In subsequent papers we will give more precise and general formulations of the conjectures, and details of the constructions and explicit computations outlined in the later part of this paper.

The geometric objects of interest are Shimura varieties and stacks (both pure and mixed) attached to Shimura data of the form $(G, \mathscr{X})$, where $G = R_{F/\mathbb{Q}}(H)$ is obtained by restriction of scalars from an algebraic group H defined over F, and diagrams consisting of such Shimura stacks and morphisms between them given in group-theoretical terms.

For example, the groups H appearing in the following diagram

$$
\begin{array}{ccc}
GL(2)_F \ltimes \mathbb{G}^2_{\mathrm{a},F} & \longleftarrow & R_{L/F}(GL(1)_L) \ltimes \mathbb{G}^2_{\mathrm{a},F} \\
\downarrow & & \downarrow \\
GL(2)_F & \longleftarrow & R_{L/F}(GL(1)_L)
\end{array}
\tag{1.1}
$$

Date: February 18, 2015.

The first author's research was supported in part by the grant ANR-BLAN-0114.

[1] The text that follows is an expanded version of the talk given by the second author at the conference.

[2] The terminology comes from the Greek $\pi\lambda\epsilon\kappa\tau\acute{o}\varsigma$, meaning "twisted, wreathed" [12], and is intended to reflect the wreath product structure of the plectic Galois group.

(where L is a totally imaginary quadratic extension of F) give rise to Shimura stacks

$$\begin{array}{ccc} \mathscr{A} & \longleftarrow & \mathscr{A}_\tau \\ \downarrow & & \downarrow \\ Y & \longleftarrow & \{\tau\}, \end{array}$$

where Y is an open Hilbert modular variety attached to $GL(2)_F$ (and a fixed level structure), $\tau \in Y$ is a CM point in Y and $\mathscr{A}$ is the universal object over Y. Morally, $\mathscr{A}$ is the quotient $[\Delta\backslash A]$, where A is the non-existent universal Hilbert-Blumenthal abelian scheme over Y, and $\Delta \subset O_{F,+}^{\times}$ is a subgroup (depending on the level structure) of finite index in the group of totally positive units of F. As we shall see in Section 12 below, it will be useful to consider also the stack $\mathscr{Y} = [\Delta\backslash Y]$, with Δ acting trivially on Y, which fits into the larger diagram

$$\begin{array}{ccc} \mathscr{A} & \longleftarrow & \mathscr{A}_\tau \\ \downarrow & & \downarrow \\ \mathscr{Y} & \longleftarrow & [\Delta\backslash\{\tau\}] \\ \downarrow & & \downarrow \\ Y & \longleftarrow & \{\tau\}, \end{array} \tag{1.2}$$

The Plectic Conjecture makes sense not only on the level of motives, but also for various realisations. If we consider, for example, ℓ-adic étale realisations, then the notion of a plectic structure (for fixed F, which will be dropped from the notation) is straightforward. If M is a mixed motive (in the old-fashioned sense) over a field k, then its ℓ-adic realisation M_ℓ is a representation of the absolute Galois group $\Gamma_k = \mathrm{Gal}(k^{\mathrm{sep}}/k)$ of k. This Galois group is contained in a larger **plectic Galois group** $\Gamma_k^{plec} \supset \Gamma_k$ and a plectic structure on M_ℓ will be an action of Γ_k^{plec} extending the Galois action of Γ_k.

The fundamental example is that of

$$\Gamma_{\mathbb{Q}} = \mathrm{Aut}(\overline{\mathbb{Q}}/\mathbb{Q}) \subset \mathrm{Aut}(F \otimes \overline{\mathbb{Q}}/F) =: \Gamma_{\mathbb{Q}}^{plec} \simeq S_r \ltimes \Gamma_F^r$$

(the latter isomorphism is non-canonical; it depends on the choice of r elements of $\Gamma_{\mathbb{Q}}$ extending the r embeddings of F to $\overline{\mathbb{Q}}$).

In the best of all possible worlds one would expect the following picture. If X is a diagram of Shimura stacks for groups of the form $G_i = R_{F/\mathbb{Q}}(H_i)$ which is defined over a field k, then the structure map $X \to \mathrm{Spec}(k)$ can be "plectified" to a cartesian diagram

$$\begin{array}{ccc} X & \dashrightarrow & X_{plec} \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi_{plec}} \\ \mathrm{Spec}(k) & \overset{\iota}{\dashrightarrow} & \mathrm{Spec}(k)_{plec}, \end{array} \tag{1.3}$$

in which the exotic "plectic" objects in the right column have the following property: if we put everywhere (pro-)étale topology, then sheaves on $\mathrm{Spec}(k)$ (resp. on $\mathrm{Spec}(k)_{plec}$) will be Γ_k-modules (resp. Γ_k^{plec}-modules). Furthermore, the base change morphism

$$\iota^* \mathbf{R}\pi_{plec,*}\mathbb{Q}_\ell \longrightarrow \mathbf{R}\pi_*\mathbb{Q}_\ell$$

will be an isomorphism, which means that $\mathbf{R}\pi_{plec,*}\mathbb{Q}_\ell$ will be a canonical object of the derived category of Γ_k^{plec}-modules with cohomology groups canonically isomorphic to the Γ_k-modules

$$H^i(\mathbf{R}\pi_*\mathbb{Q}_\ell) = H^i_{et}(X \otimes_k k^{\mathrm{sep}}, \mathbb{Q}_\ell) = h^i(X)_\ell,$$

namely, to the ℓ-adic realisations of the cohomology motives $h^i(X)$. In other words, $H^i(\mathbf{R}\pi_{plec,*}\mathbb{Q}_\ell)$ will define a canonical plectic structure on $h^i(X)_\ell$.

Acknowledgements. The second author would like to express his thanks to the Erwin Schrödinger Institute for their generous hospitality, when some of the work for this article was done. Both authors were guests of the Fields Institute in the early stages of this project, and the present paper was completed during an extended stay at MSRI, supported by the National Science Foundation under grant no. 0932078 000.

2. Analytic cohomology of compact pure Shimura varieties

Let $(G, \mathscr{X})$ be an arbitrary pure Shimura datum (with the minimal set of axioms [5, (2.1.1.1-3)]). Fix a point $h \in \mathscr{X}$ and write $\mathscr{X} = G(\mathbb{R})/K_\infty$, where $K_\infty = G(\mathbb{R})_h$. Consider $Y = Sh_K(G, \mathscr{X})$, for a fixed level structure[3] $K \subset G(\widehat{\mathbb{Q}})$. If the analytic space $Y^{an} = \coprod_i^k \Gamma_i \backslash \mathscr{X}$ is compact, then its cohomology can be written in terms of the Hilbert space decomposition

$$\bigoplus_i L^2(\Gamma_i \backslash G(\mathbb{R})/Z(\mathbb{R})) = \widehat{\bigoplus_{\pi \in Irr(G(\mathbb{R}))}} \pi^{\oplus m(\pi)}$$

using relative Lie cohomology of $\mathfrak{g} = \mathrm{Lie}(G(\mathbb{R}))_{\mathbb{C}}$:

$$H^*(Y^{an}, \mathbb{C}) = \bigoplus_\pi H^*(\mathfrak{g}, K_\infty; \pi)^{\oplus m(\pi)},$$

with only finitely many π contributing.

In the plectic case $G = R_{F/\mathbb{Q}}(H)$ there are real algebraic groups $H_v = H \otimes_{F,v} \mathbb{R}$ for all infinite primes v of F and product decompositions

$$G(\mathbb{R}) = \prod_{v|\infty} H_v(\mathbb{R}), \qquad K_\infty = \prod_{v|\infty} K_{\infty,v}, \qquad \mathscr{X} = \prod_{v|\infty} \mathscr{X}_v.$$

Moreover, $\pi = \bigotimes_{v|\infty} \pi_v$ with $\pi_v \in Irr(H_v(\mathbb{R}))$. The Künneth formula for relative Lie algebra cohomology (with $\mathfrak{h}_v = \mathrm{Lie}(H_v(\mathbb{R}))_{\mathbb{C}}$)

$$H^*(\mathfrak{g}, K_\infty; \pi) = \bigotimes_{v|\infty} H^*(\mathfrak{h}_v, K_{\infty,v}; \pi_v)$$

then yields a "weak Künneth decomposition" (with finitely many terms)

$$H^*(Y^{an}, \mathbb{C}) = \bigoplus_\pi \left(\bigotimes_{v|\infty} H^*(\mathfrak{h}_v, K_{\infty,v}; \pi_v) \right)^{\oplus m(\pi)} \tag{2.1}$$

The plectic conjecture in this particular case asserts that the appearance of $\bigotimes_{v|\infty}$ in (2.1) should be of a motivic origin. In particular, it should manifest itself in

[3] Here and elsewhere, $\widehat{L} = L \otimes \widehat{\mathbb{Z}}$ denotes the ring of finite adeles of a number field L.

every cohomological realisation. In the Hodge-de Rham realisation this amounts to suitable period relations (such as those conjectured by Oda [17] and Yoshida [25]). In the real Hodge realisation, since each of the individual factors $H^*(\mathfrak{h}_v, K_{\infty,v}; \pi_v)$ has a natural Hodge decomposition induced by the Hodge torus $\mathbb{C}^* \to H_v(\mathbb{R})$, the total cohomology of Y has a "plectic Hodge structure": it has a canonical $\mathbb{Z}^r \times \mathbb{Z}^r$-grading

$$H^*_{\mathbb{C}} = \bigoplus_{\mathbf{p},\mathbf{q}\in\mathbb{Z}^r} H^{\mathbf{pq}}$$

with $\overline{H^{\mathbf{pq}}} = H^{\mathbf{qp}}$ (see in Section 16 below).

3. Interlude: induction and tensor induction

For any Γ_F-module N, the wreath product $S_r \ltimes \Gamma_F^r$ naturally acts on both $N^{\oplus r}$ and $N^{\otimes r}$; the restrictions of these module structures to $\Gamma_{\mathbb{Q}}$ yield, respectively, the induced module and the tensor induction of N:

$$\mathrm{Ind}_{\Gamma_F}^{\Gamma_{\mathbb{Q}}}(N) \simeq N^{\oplus r}\big|_{\Gamma_{\mathbb{Q}}}, \qquad \otimes\text{-}\mathrm{Ind}_{\Gamma_F}^{\Gamma_{\mathbb{Q}}}(N) \simeq N^{\otimes r}\big|_{\Gamma_{\mathbb{Q}}}.$$

A more canonical version of this wreath product is the **plectic group**

$$\Gamma_{\mathbb{Q}}\#\Gamma_F = \mathrm{Aut}_{(Sets)\text{-}\Gamma_F}(\Gamma_{\mathbb{Q}}),$$

which is canonically isomorphic to $\mathrm{Aut}(F \otimes \overline{\mathbb{Q}}/F)$ and which acts canonically on the intrinsically defined $\mathrm{Ind}_{\Gamma_F}^{\Gamma_{\mathbb{Q}}}(N)$ and $\otimes\text{-}\mathrm{Ind}_{\Gamma_F}^{\Gamma_{\mathbb{Q}}}(N)$. The inclusion $\Gamma_{\mathbb{Q}} \subset \Gamma_{\mathbb{Q}}\#\Gamma_F$ is given by the action of $\Gamma_{\mathbb{Q}}$ on itself by left translations.

4. Etale cohomology of quaternionic Shimura varieties

A Shimura variety $Y = Sh_K(G, \mathscr{X})$ (pure or mixed) is defined over its reflex field $E \subset \overline{\mathbb{Q}} \subset \mathbb{C}$, which does not change if we replace $(G, \mathscr{X})$ by the corresponding pure Shimura datum $(G_{red}, \mathscr{X}_{red})$. The étale cohomology groups $H^* = H^*_{et}(Y \otimes_E \overline{\mathbb{Q}}, \overline{\mathbb{Q}}_\ell)$ are then ℓ-adic representations of Γ_E.

Consider pure quaternionic Shimura varieties: for these $G = R_{F/\mathbb{Q}}(H)$, where $H = B^\times$ is the multiplicative group of a quaternion division algebra B over F (so that Y is compact).

In the totally indefinite case $B\otimes\mathbb{R} \simeq M_2(\mathbb{R})^r$ we have $\mathscr{X} \simeq (\mathbb{C}-\mathbb{R})^r$, $\dim(Y) = r$ and $E = \mathbb{Q}$. The cohomology then decomposes as

$$H^* = H^*_{int} \oplus H^*_{rest},$$

with the interesting part $H^*_{int} = H^r_{int}$ coming from cuspidal Hilbert modular eigenforms f (of parallel weight 2) occurring only in degree r. The semi-simplification of the $\overline{\mathbb{Q}}_\ell[\Gamma_{\mathbb{Q}}]$-module H^r_{int} was determined in [9, 20] (see also [13, 14]). Together with the recent proof of semi-simplicity of H^r_{int} [16, Thm. 5.20(3)] this yields

$$H^*_{int} \simeq \bigoplus_f V_f^{\otimes r}\big|_{\Gamma_{\mathbb{Q}}}^{\oplus m(f)},$$

where V_f is the two-dimensional ℓ-adic representation of Γ_F attached [18, 24, 22] to f. The remaining part of H^* is isomorphic to

$$H^*_{rest} \simeq \bigoplus_\chi (\chi \oplus \chi(-1))^{\otimes r}\big|_{\Gamma_{\mathbb{Q}}}^{\oplus m(\chi)},$$

where $\chi : \Gamma_F \to \overline{\mathbb{Q}}_\ell^\times$ runs through characters of finite order, and χ (resp. $\chi(-1) := \chi \otimes \overline{\mathbb{Q}}_\ell(-1)$) occurs in H^0 (resp. in H^2).

In particular, H^* is a direct sum of tensor inductions of certain two-dimensional representations of Γ_F, and therefore has an (in general noncanonical) action of the plectic group $\Gamma_\mathbb{Q} \# \Gamma_F$. The Plectic Conjecture described in the following sections both refines and generalises this action.

In the general case we have $B \otimes \mathbb{R} \simeq M_2(\mathbb{R})^t \times \mathbb{H}^{r-t}$ with $t \geq 1$, $\mathscr{X} \simeq (\mathbb{C} - \mathbb{R})^t$, $\dim(Y) = t$ and the reflex field E is determined by

$$\Gamma_E = \Gamma_\mathbb{Q} \cap ((S_t \ltimes \Gamma_F^t) \times (S_{r-t} \ltimes \Gamma_F^{r-t})) \subset S_r \ltimes \Gamma_F^r.$$

Thanks to [9, 20, 16], the étale cohomology of Y again decomposes as $H^* = H^*_{int} \oplus H^*_{rest}$, where

$$H^*_{int} = H^t_{int} \simeq \bigoplus_f V_f^{\otimes t}\big|_{\Gamma_E}^{\oplus m(f)}, \quad H^*_{rest} \simeq \bigoplus_\chi (\chi \oplus \chi(-1))^{\otimes t}\big|_{\Gamma_E}^{\oplus m(\chi)},$$

hence is a direct sum of "partial" tensor inductions.

5. Plectic reflex Galois group

Let $(G, \mathscr{X})$ be an arbitrary pure Shimura datum. Recall that, if

$$\mu = \mu_h : \mathbb{G}_{\mathrm{m},\mathbb{C}} \longrightarrow G_\mathbb{C}$$

is the cocharacter attached to a point $h : \mathbb{S} \longrightarrow G_\mathbb{R}$ of $\mathscr{X}$, then its conjugacy class

$$[\mu] \in \mathrm{Hom}(\mathbb{G}_{\mathrm{m},\mathbb{C}}, G_\mathbb{C})/\mathrm{int}(G(\mathbb{C})) = \mathrm{Hom}(\mathbb{G}_{\mathrm{m},\overline{\mathbb{Q}}}, G_{\overline{\mathbb{Q}}})/\mathrm{int}(G(\overline{\mathbb{Q}}))$$

depends only on $(G, \mathscr{X})$. The set of conjugacy classes of cocharacters has a natural action of $\Gamma_\mathbb{Q}$, and the stabiliser of $[\mu]$ is the absolute Galois group of the reflex field: $\Gamma_E = (\Gamma_\mathbb{Q})_{[\mu]}$.

In the plectic case $G = R_{F/\mathbb{Q}}(H)$, since $G_{\overline{\mathbb{Q}}} = H \otimes_F (F \otimes \overline{\mathbb{Q}})$, the set of conjugacy classes of cocharacters admits an action of the plectic Galois group $\Gamma_\mathbb{Q}^{plec} = \mathrm{Aut}(F \otimes \mathbb{Q}/F) = \Gamma_\mathbb{Q} \# \Gamma_F$, extending that of $\Gamma_\mathbb{Q}$.

Definition 5.1. *The* plectic reflex Galois group *of $(R_{F/\mathbb{Q}}(H), \mathscr{X})$ is the stabiliser $(\Gamma_\mathbb{Q}^{plec})_{[\mu]}$ of $[\mu]$ in $\Gamma_\mathbb{Q}^{plec}$. More generally, the plectic reflex Galois group of a mixed Shimura datum of the form $(R_{F/\mathbb{Q}}(H), \mathscr{X})$ is defined to be the plectic reflex Galois group of the corresponding pure Shimura datum $(R_{F/\mathbb{Q}}(H_{red}), \mathscr{X}_{red})$.*

Proposition 5.2. *There exists an isomorphism $\Gamma_\mathbb{Q}^{plec} \simeq S_r \ltimes \Gamma_F^r$ under which $(\Gamma_\mathbb{Q}^{plec})_{[\mu]}$ corresponds to $\prod_i (S_{r_i} \ltimes \Gamma_{F_i}^{r_i})$, for suitable finite extensions F_i/F and $r_i \geq 1$ such that $\sum_i r_i = r$.*

For example, in the quaternionic case $H = B^\times$ the plectic reflex Galois group is isomorphic to $(S_t \ltimes \Gamma_F^t) \times (S_{r-t} \ltimes \Gamma_F^{r-t}) \subset S_r \ltimes \Gamma_F^r$.

6. ℓ-adic plectic conjecture for pure Shimura varieties

Let $(G, \mathscr{X})$ be a pure Shimura datum for $G = R_{F/\mathbb{Q}}(H)$ with reflex field E. For every open compact subgroup $K \subset G(\widehat{\mathbb{Q}}) = H(\widehat{F})$, the subgroup of central elements $Z_H(\widehat{F})$ acts on $Y = Sh_K(G, \mathscr{X})$ by right multiplication, with the discrete subgroup $\Delta = \Delta(K) := Z_H(F)^+ \cap K$ (which is a finitely generated abelian group) acting trivially. In the quaternionic case $H = B^\times$, Δ is a subgroup of finite index in $O_{F,+}^\times$.

One can attach to any algebraic representation $\xi : G_{\overline{\mathbb{Q}}_\ell} \longrightarrow GL(N)_{\overline{\mathbb{Q}}_\ell}$ satisfying $\xi(\Delta) = 1$ a lisse étale $\overline{\mathbb{Q}}_\ell$-sheaf $\mathscr{L}_\xi$ on Y (if K is small enough). Its geometric étale cohomology is related to the corresponding equivariant cohomology (for the trivial action of Δ on Y)

$$\mathbf{R}\Gamma_{et}(Y \otimes_E \overline{\mathbb{Q}}, \Delta; \mathscr{L}_\xi) = \mathbf{R}\Gamma_{et}([\Delta\backslash Y] \otimes_E \overline{\mathbb{Q}}, \mathscr{L}_\xi) \in D^+(\overline{\mathbb{Q}}_\ell[\Gamma_E])$$

by

$$\mathbf{R}\Gamma_{et}([\Delta\backslash Y] \otimes_E \overline{\mathbb{Q}}, \mathscr{L}_\xi) = \mathbf{R}\Gamma_{et}(Y \otimes_E \overline{\mathbb{Q}}, \mathscr{L}_\xi) \otimes_{\mathbb{Q}} \mathbf{R}\Gamma_{et}(\Delta, \mathbb{Q}). \tag{6.1}$$

In fact, one can make sense of $\mathbf{R}\Gamma_{et}([\Delta\backslash Y] \otimes_E \overline{\mathbb{Q}}, \mathscr{L}_\xi)$ for arbitrary ξ (viewing $\mathscr{L}_\xi$ as a sheaf on the stack $\mathscr{Y} = [\Delta\backslash Y]$), but its cohomology groups vanish if ξ is irreducible and $\xi(\Delta) \neq 1$.

We can now state the Plectic Conjecture in the ℓ-adic setting (for Y a pure Shimura variety).

Conjecture 6.1. *$\mathbf{R}\Gamma_{et}([\Delta\backslash Y] \otimes_E \overline{\mathbb{Q}}, \mathscr{L}_\xi)$ has a canonical and functorial lift to an object $\mathbf{R}\Gamma_{et,plec}([\Delta\backslash Y] \otimes_E \overline{\mathbb{Q}}, \mathscr{L}_\xi)$ of $D^+(\overline{\mathbb{Q}}_\ell[(\Gamma_{\mathbb{Q}}^{plec})_{[\mu]}])$.*

Remark 6.2. In view of (6.1), this is equivalent (for ξ with $\xi(\Delta) = 1$) to the corresponding statement for the complex $\mathbf{R}\Gamma_{et}(Y_{\overline{\mathbb{Q}}}, \mathscr{L}_\xi)$.

We expect (at least) functoriality with respect to Hecke correspondences and to morphisms of Shimura data (of the same type), and compatibility with products.

A weaker form of this conjecture involves only cohomology groups:

Conjecture 6.3. *The Galois action of $\Gamma_E = (\Gamma_{\mathbb{Q}})_{[\mu]}$ on étale cohomology groups $H^* = H^*_{et}([\Delta\backslash Y] \otimes_E \overline{\mathbb{Q}}, \mathscr{L}_\xi)$ extends to a canonical and functorial action of the plectic reflex Galois group $(\Gamma_{\mathbb{Q}}^{plec})_{[\mu]}$.*

Proposition 6.4. *Conjecture 6.3 holds for H^0_{et}, with the action of $(\Gamma_{\mathbb{Q}}^{plec})_{[\mu]}$ factoring through $\prod_i \Gamma_{F_i}^{ab}$.*

Proof. This follows from the explicit formula [5, 2.6] for the Galois action on $\pi_0(Y \otimes_E \overline{\mathbb{Q}})$. $\square$

Proposition 6.5. *If Y is compact, then the expected expression for the Euler characteristic*

$$\sum_{k \geq 0} (-1)^k [H^k] \in G_0(\overline{\mathbb{Q}}_\ell[\Gamma_E])$$

lies in the image of $G_0(\overline{\mathbb{Q}}_\ell[(\Gamma_{\mathbb{Q}}^{plec})_{[\mu]}])$.

Sketch proof. The predicted Euler characteristic formula is a linear combination of partial tensor inductions

$$\Big[\bigotimes_i \rho_i^{\otimes r_i}\Big]\Big|_{\Gamma_E}, \qquad \rho_i : \Gamma_{F_i} \longrightarrow GL_{n_i}(\overline{\mathbb{Q}}_\ell). \qquad \square$$

Proposition 6.6. *In the quaternionic case $H = B^\times$ (including the case $H = GL(2)_F$ of open Hilbert modular varieties) the Galois action of Γ_E extends to an action (in general, to many actions) of $(\Gamma_{\mathbb{Q}}^{plec})_{[\mu]}$ on cohomology, commuting with all Hecke correspondences.*

Proof. We can assume that ξ is irreducible and $\xi(\Delta) = 1$; then $\xi = \bigotimes_{v|\infty} \xi_v$, where ξ_v is an irreducible representation of $GL(2)_{\overline{\mathbb{Q}}_\ell}$ of the form $\xi_v = Sym^{k_v-2}(\overline{\mathbb{Q}}_\ell^2) \otimes \det^{(w-k_v)/2}$, $w \in \mathbb{Z}$ and $k_v \geq 2$ are integers such that $k_v \equiv w \pmod 2$. In the compact case $B \neq M_2(F)$ the description of H^* given in Section 4 still holds, modulo a suitable Tate twist and the fact that f will be a Hilbert modular form of weight $\underline{k} = (k_v)_{v|\infty}$ (moreover, H^*_{rest} will be non-zero only if $k_v = 2$ for all v). In the case $B = M_2(F)$ the Galois representation H^*_{rest} need not be semi-simple thanks to a contribution from Eisenstein series (see [7], [4] in the case $r = 2$), but the corresponding extension class extends to the plectic Galois group. □

Remark 6.7. The *canonical* action of $(\Gamma_{\mathbb{Q}}^{plec})_{[\mu]}$ whose existence is predicted by Conjecture 6.3 should be characterised in this case by a suitable local-global compatibility. In the case where B is totally indefinite, and p is a good prime, the factorisation of p determine a factorisation of the geometric Frobenius endomorphism of the special fibre at p of the Shimura variety. The characterisation in this case should amount to the compatibility of the action of these "partial Frobenii" on ℓ-adic cohomology with the action of the corresponding elements of the plectic group.

Proposition 6.8. *If $H = GL(2)_F$, then $\Gamma_{\mathbb{Q}}^{plec}$ acts canonically on the set of cusps of the Hilbert modular variety Y and on its set of CM points.*

Proof. The action on cusps is easy. The action on CM points follows from the fact that Tate's half-transfer is plectic; cf. [15, 2.2.5], where a slightly different case is treated. □

7. ℓ-adic plectic cohomology: the pure case

Recall that geometric and arithmetic (absolute) étale cohomology groups of a scheme of finite type $X \to \mathrm{Spec}(k)$ are related by

$$\mathbf{R}\Gamma_{et}(X, \mathbb{Q}_\ell(n)) = \mathbf{R}\Gamma(\Gamma_k, \mathbf{R}\Gamma_{et}(X \otimes_k k^{\mathrm{sep}}, \mathbb{Q}_\ell(n))),$$

hence by a Hochschild-Serre spectral sequence

$$E_2^{ij} = H^i(\Gamma_k, H^j_{et}(X \otimes_k k^{\mathrm{sep}}, \mathbb{Q}_\ell(n)) \Longrightarrow H^*_{et}(X, \mathbb{Q}_\ell(n)). \tag{7.1}$$

Let $Y = Sh_K(G, \mathscr{X})$ and Δ be as in Section 6.

Definition 7.1. *If Conjecture 6.1 holds and if the representation $\overline{\mathbb{Q}}_\ell(n)$ $(n \in \mathbb{Z})$ of Γ_E extends to a natural representation $\overline{\mathbb{Q}}_\ell(n)_{plec}$ of $(\Gamma_{\mathbb{Q}}^{plec})_{[\mu]}$ (for example, if $H = B^\times$ and $t \mid n$, then $\overline{\mathbb{Q}}_\ell(n)_{plec} = \overline{\mathbb{Q}}_\ell(n/t)^{\otimes t}$ as a representation of $S_t \ltimes \Gamma_F^t$), then the* plectic étale cohomology $\mathbf{R}\Gamma_{et,plec}([\Delta\backslash Y], \mathscr{L}_\xi(n))$ *of $[\Delta\backslash Y]$ with coefficients in $\mathscr{L}_\xi(n)$ is defined as*

$$\mathbf{R}\Gamma((\Gamma_{\mathbb{Q}}^{plec})_{[\mu]}, \mathbf{R}\Gamma_{et,plec}([\Delta\backslash Y] \otimes_E \overline{\mathbb{Q}}, \mathscr{L}_\xi) \otimes \overline{\mathbb{Q}}_\ell(n)_{plec}).$$

Likewise (when $\xi(\Delta) = 1$) define corresponding groups with $[\Delta\backslash Y]$ replaced by Y.

The Hochschild-Serre spectral sequence (7.1) will then be replaced by its plectic analogue

$$E_2^{ij} = H^i((\Gamma_{\mathbb{Q}}^{plec})_{[\mu]}, H^j_{et}([\Delta\backslash Y] \otimes_E \overline{\mathbb{Q}}, \mathscr{L}_\xi)(n)) \Longrightarrow H^*_{et,plec}([\Delta\backslash Y], \mathscr{L}_\xi(n)). \tag{7.2}$$

In the speculative framework of (1.3), the plectic cohomology of X with coefficients in an étale sheaf $\mathscr{F}$ on $\mathrm{Spec}(k)$ admitting a natural extension to a plectic étale sheaf $\mathscr{F}_{plec}$ on $\mathrm{Spec}(k)_{plec}$ would simply be given by $\mathbf{R}\Gamma_{et}(X_{plec}, \mathscr{F}_{plec})$.

8. The mixed case

The notion of a mixed Shimura datum $(G, \mathscr{X})$ as defined in the literature [19] precludes[4] (for G genuinely mixed i.e. $G \neq G_{red}$) the case $G = R_{F/\mathbb{Q}}(H)$ of interest here, and for a good reason: the corresponding double coset space

$$G(\mathbb{Q})\backslash \mathscr{X} \times G(\widehat{\mathbb{Q}})/K$$

is not even an analytic space. Typically, its fibres over the corresponding pure Shimura variety are quotients of a semi-abelian variety (equipped with an O_F-action) by a subgroup of finite index in $O_{F,+}^{\times}$. However there should be a good notion of the *mixed Shimura stack* $\mathscr{S}h_K(G, \mathscr{X})$ in such a situation. Surprisingly, the general theory of such objects has not yet been developed. In the case $H = GL(2) \ltimes \mathbb{G}_{\mathrm{a}}^2$, such a stack does exist, and is just the stack $\mathscr{A}$ of pointed Hilbert-Blumenthal abelian varieties (without prescribed polarisation) and level structure K. The stack $\mathscr{Y}$ in diagram (1.2) is the base moduli stack of HBAVs with level structure K, provided K is sufficiently small.

We then expect Conjecture 6.1 to hold (and therefore Definition 7.1 to make sense) for the stacks $\mathscr{S}h_K(G, \mathscr{X})$ whenever $G = R_{F/\mathbb{Q}}(H)$ and, with appropriate modifications, for diagrams.

9. Motivation

Plectic cohomology (provided it exists) is interesting for the following reason.

> **Every construction involving usual arithmetic cohomology groups of elliptic curves or modular curves (and Kuga-Sato varieties over them) can be carried out with abelian varieties with real multiplication and Hilbert modular varieties using plectic cohomology.**

The output of the construction will be a usual (non-plectic) object, but in order to construct it one will have to pass through the plectic world.

10. Theta functions and classical zeta elements

Many classical zeta elements responsible for special values of L-functions of automorphic forms on $GL(1)$ and $GL(2)$ over $\mathbb{Q}$ (and on $GL(1)$ over imaginary quadratic fields) arise from a motivic version of a suitably normalised two-variable theta-function $\Theta(z, \tau)$ ([21, Thm. 1.2.1], [8]) or from a slightly modified function (which is a meromorphic function, rather than a section of a line bundle) ${}_d\Theta(z, \tau) = (-1)^{(d-1)/2}\Theta(z, \tau)^{d^2}/\Theta(dz, \tau)$, where $d > 1$ is an integer prime to 6.

For example, the functions

$$g(\tau) = {}_d\Theta((a\tau + b)/N, \tau) \in \mathscr{O}(Y(N))^{\times} \qquad (a, b \in \mathbb{Z}, (6N, d) = 1)$$

(with at least one of the integers a, b not divisible by N) are Siegel (or modular) units on the open modular curve $Y(N)$.

[4]The axiom 2.1(vii) of [19], that the maximal $\mathbb{R}$-split central torus is $\mathbb{Q}$-split, is violated.

Specialisations of g to CM points $\tau \in \mathbb{Q}(\sqrt{-D})$ (resp. to cusps) give elliptic units (resp. cyclotomic units) in abelian extensions of $\mathbb{Q}(\sqrt{-D})$ (resp. of $\mathbb{Q}$).

Cup products $g_1 \cup g_2 \in K_2(Y(N))$ of two Siegel units were related by Beilinson [1] (via the regulator map) to the values of $L'(f,0)$ for cusp forms f of weight 2, and by Kato [8] (via an Euler system arising from norm-compatible systems of elements $g_{1,n} \cup g_{2,n} \in K_2(Y(Np^n)))$ to the Iwasawa Main Conjecture for cusp forms.

Moreover, the motivic version of ${}_d\Theta(z,\tau)$ arises as the first term of the motivic elliptic polylogarithm.

11. Theta functions and cohomology

We recall the invariant cohomological definition of ${}_d\Theta(z,\tau)$, for arbitrary families of elliptic curves in any of the following absolute (= arithmetic) cohomology theories $H^i(-,j)$.

- Motivic cohomology $H^i_{\mathscr{M}}(-,\mathbb{Z}(j))$ or $H^i_{\mathscr{M}}(-,\mathbb{Q}(j))$.
- Arithmetic étale cohomology $H^i_{et}(-,\mathbb{Z}_\ell(j))$ or $H^i_{et}(-,\mathbb{Q}_\ell(j))$.
- Deligne-Beilinson absolute Hodge cohomology $H^i_{\mathscr{H}}(-,\mathbb{R}(j))$ (for varieties over $\mathbb{R}$ or $\mathbb{C}$).
- (Log-)syntomic cohomology $H^i_{syn}(-,s_{\mathbb{Q}_p}(j))$ (for varieties over a p-adic field).

Let $E \longrightarrow Y$ be an elliptic curve over a scheme Y. The divisor map

$$\mathrm{div} : H^1(E - E[d], 1) \longrightarrow H^0(E[d], 0)^{deg=0} \tag{11.1}$$

admits a canonical section (characterised by a suitable compatibility with norm maps) after tensoring with $\mathbb{Z}[1/6d]$. This gives, for each divisor D of degree zero supported on the d-torsion of E, a canonical element

$$\theta_D \in H^1(E - E[d], 1) \otimes \mathbb{Z}[1/6d]$$

with $\mathrm{div}(\theta_D) = D$. For $D = d^2(0) - \sum_{x\in E[d]}(x)$ we obtain an element ${}_d\Theta \in H^1(E - E[d], 1)$ (in this case there is no need to tensor by $\mathbb{Z}[1/6d]$) which is equal to the function ${}_d\Theta(z,\tau)$ in the universal case when $Y = Y(N)$ (since in motivic cohomology $H^1_{\mathscr{M}}(X,\mathbb{Z}(1)) = \mathscr{O}(X)^\times$).

For any nowhere-vanishing torsion section $x : Y \to E$ of order prime to d, the pullback $g = x^*({}_d\Theta) \in H^1(Y,1)$ is a "cohomological Siegel unit". The cup product of two such classes is an element $g_1 \cup g_2 \in H^2(Y,2)$. In motivic cohomology, this is one of Beilinson's elements, and its image in absolute Hodge cohomology is given by a product of two Eisenstein series. When $Y = Y(Np^n)$ for n varying, the image in p-adic étale cohomology is Kato's Euler system.

12. Towards zeta elements over totally real fields

Our goal is to generalise the constructions in Sections 10 and 11 from $\mathbb{Q}$ to an arbitrary totally real number field F of degree $r > 1$. One novelty is that we will obtain formulae for leading terms of L-functions at points where they have a derivative of order r.

Instead of elliptic curves, it is natural to consider Hilbert-Blumenthal abelian varieties (= abelian varieties of dimension r equipped with an action of O_F). If $A \to S$ is a family of HBAV over a scheme S, the divisor map (11.1) is replaced by a map

$$H^{2r-1}(A - A[d], r) \longrightarrow H^0(A[d], 0)^{deg=0},$$

which again admits a canonical section after tensoring by $\mathbb{Z}[1/d(2r+1)!]$. To simplify the notation, we assume that the ring of coefficients of our cohomology theory is a $\mathbb{Z}[1/d(2r+1)!]$-algebra. As in Section 11, we obtain canonical elements ${}_d\Theta \in H^{2r-1}(A - A[d], r)$ $(d > 1)$ and their pull-backs by torsion sections $x : S \to A$ of order prime to d, namely, $x^*({}_d\Theta) \in H^{2r-1}(S, r)$.

The problem is that the element $x^*({}_d\Theta)$ is not interesting. For example, its specialisation to any $\overline{\mathbb{Q}}$-valued point (in particular, any CM point) of S will lie in the image of the group $H^{2r-1}_{\mathscr{M}}(\mathrm{Spec}(\overline{\mathbb{Q}}), \mathbb{Z}(r)) \otimes \mathbb{Z}[1/d(2r+1)!]$, which for $r > 1$ is a torsion group.

One can try to remedy the situation by considering refined cohomology theories. As in the Introduction, we should replace the universal elliptic curve $E \to Y(N)$ by the stack $\mathscr{A}$ over $\mathscr{Y} = [\Delta\backslash Y]$, where Y is an open Hilbert modular variety attached to $R_{F/\mathbb{Q}}(GL(2)_F)$ and a suitable level structure, and where $\Delta \simeq \mathbb{Z}^{r-1}$ (a subgroup – depending on the level structure – of $O^\times_{F,+}$ of finite index) acts trivially on Y. This means that we need to rerun the previous constructions for stacks (i.e. for equivariant cohomology groups).

We then obtain elements ${}_d\Theta \in H^{2r-1}(\mathscr{A} - \mathscr{A}[d], r)$ and, for any nowhere zero torsion section $x : \mathscr{Y} \to \mathscr{A}$ of order prime to d,

$$x^*({}_d\Theta) \in H^{2r-1}([\Delta\backslash Y], r) = \bigoplus_{i+j=2r-1} H^i(Y, r) \otimes H^j(\Delta, \mathbb{Z}).$$

Taking the cap product with a fixed generator $[\Delta] \in H_{r-1}(\Delta, \mathbb{Z}) \simeq \mathbb{Z}$ we obtain elements

$$g = x^*({}_d\Theta) \cap [\Delta] \in H^r(Y, r).$$

The numerology is now more satisfactory, but any specialisation of g to a $\overline{\mathbb{Q}}$-valued point of Y (in particular, any CM point) of Y will lie in the image of the torsion group

$$H^r_{\mathscr{M}}(\mathrm{Spec}(\overline{\mathbb{Q}}), \mathbb{Z}(r)) \otimes \mathbb{Z}[1/d(2r+1)!] = K^M_r(\overline{\mathbb{Q}}) \otimes \mathbb{Z}[1/d(2r+1)!].$$

Moreover, the cup product $g_1 \cup g_2 \in H^{2r}(Y, 2r)$ will also be uninteresting. This can be seen in étale realisations: in the Hochschild-Serre spectral sequence

$$E_2^{ij} = H^i(\Gamma_{\mathbb{Q}}, H^j_{et}(Y \otimes_{\mathbb{Q}} \overline{\mathbb{Q}}, \mathbb{Q}_\ell(2r)) \Longrightarrow H^*_{et}(Y, \mathbb{Q}_\ell(2r)) \tag{12.1}$$

the groups $H^j_{et}(Y \otimes_{\mathbb{Q}} \overline{\mathbb{Q}}, \mathbb{Q}_\ell(2r))$ are interesting only for $j = r$ (as in Section 4), but $H^i(\Gamma_{\mathbb{Q}}, -) = 0$ for $i > 2$. In particular, $H^{2r}_{et}(Y, \mathbb{Q}_\ell(2r))$ is uninteresting if $r > 2$. If $r = 2$, the relevant Galois H^2 may be non-zero, but is not expected to contain any non-trivial images of elements of motivic cohomology (since motivic H^2 over number fields is expected to vanish).

To sum up, the elements $g \in H^r(Y, r)$ are no good, either. We need to do something else.

13. Plectic theta elements

The solution is to plectify the previous construction. In order to do that, we will assume in Sections 13–15 that Conjecture 6.1 holds for all objects appearing in the diagram (1.2) (see Section 6 for partial results in this direction), as well as for complements of torsion sections arising from the level structure. The plectic formalism from Section 7 (with $(\Gamma^{plec}_{\mathbb{Q}})_{[\mu]} = \Gamma^{plec}_{\mathbb{Q}} = \mathrm{Aut}(F \otimes \overline{\mathbb{Q}}/F) \simeq S_r \ltimes \Gamma^r_F$ for

Y, $\mathscr{Y} = [\Delta\backslash Y]$ and $\mathscr{A}$) then applies and we obtain plectic theta elements (more precisely, their ℓ-adic realisations)

$$_d\Theta_{plec} \in H^{2r-1}_{et,plec}(\mathscr{A} - \mathscr{A}[d], \mathbb{Q}_\ell(r)),$$

their pull-backs by torsion sections arising from the level structure

$$x^*(_d\Theta_{plec}) \in H^{2r-1}_{et,plec}([\Delta\backslash Y], \mathbb{Q}_\ell(r))$$

and plectic Siegel classes

$$g_{plec} = x^*(_d\Theta) \cap [\Delta] \in H^r_{et,plec}(Y, \mathbb{Q}_\ell(r)).$$

14. Specialisations of plectic Siegel classes and Stark's conjectures

The plectic reflex group of a point $\tau \in Y$ which has complex multiplication by a quadratic extension L of F is isomorphic to $S_r \ltimes \Gamma_L^r$. The specialisation $g_{plec}(\tau)$ of g_{plec} at τ will be contained in the group

$$\begin{aligned} H^r_{et,plec}(\{\tau\}, \mathbb{Q}_\ell(r)) &\simeq H^r(S_r \ltimes \Gamma_L^r, \mathbb{Q}_\ell(r) \otimes \mathbb{Z}[\mathrm{Gal}(L_\tau/L)]) \\ &\simeq \textstyle\bigwedge^r_{\mathbb{Q}_\ell[\mathrm{Gal}(L_\tau/L)]} H^1(\Gamma_{L_\tau}, \mathbb{Q}_\ell(1)), \end{aligned}$$

where L_τ/L is a finite abelian extension depending on τ and the isomorphisms come from the plectic Hochschild-Serre spectral sequence (7.2) and Shapiro's Lemma.

If the above construction has a motivic version, then $g_{plec}(\tau)$ will be the image of a motivic element

$$g_{\mathscr{M},plec}(\tau) \in \textstyle\bigwedge^r_{\mathbb{Q}[\mathrm{Gal}(L_\tau/L)]} H^1_{\mathscr{M}}(L_\tau, \mathbb{Q}(1)) \simeq \textstyle\bigwedge^r_{\mathbb{Q}[\mathrm{Gal}(L_\tau/L)]} (L_\tau^\times \otimes \mathbb{Q}),$$

whose existence is predicted by a variant of Stark's conjectures.

Similarly, specialisations of g_{plec} to cusps will yield elements of

$$\textstyle\bigwedge^r_{\mathbb{Q}_\ell[\mathrm{Gal}(F'/F)]} H^1(\Gamma_{F'}, \mathbb{Q}_\ell(1))$$

whose motivic versions will be contained in $\bigwedge^r_{\mathbb{Q}[\mathrm{Gal}(F'/F)]}(F'^\times \otimes \mathbb{Q})$, for finite abelian extensions F'/F.

15. Plectic constructions involving plectic Siegel classes

If $g_{plec,1}, g_{plec,2} \in H^r_{et,plec}(Y, \mathbb{Q}_\ell(r))$ are two plectic Siegel classes, we can consider their cup product

$$g_{plec,1} \cup g_{plec,2} \in H^{2r}_{et,plec}(Y, \mathbb{Q}_\ell(2r)).$$

The plectic Hochschild-Serre spectral sequence (7.2) gives a map

$$H^{2r}_{et,plec}(Y, \mathbb{Q}_\ell(2r)) \longrightarrow E_2^{rr} = H^r(S_r \ltimes \Gamma_F^r, H^r_{et}(Y \otimes_{\mathbb{Q}} \overline{\mathbb{Q}}, \mathbb{Q}_\ell)(2r)).$$

Geometric étale cohomology $H^r_{et}(Y \otimes_{\mathbb{Q}} \overline{\mathbb{Q}}, \overline{\mathbb{Q}}_\ell)$ contains as direct summands representations of $S_r \ltimes \Gamma_F^r$ of the form

$$V_f^{\otimes r} \otimes \mathbb{Z}[\mathrm{Gal}(F'/F)],$$

where f is a cuspidal Hilbert eigenform of parallel weight 2 over F, F'/F is a finite abelian extension (depending on the level structure) and S_r (resp. Γ_F^r) acts on $\mathbb{Z}[\mathrm{Gal}(F'/F)]$ trivially (resp. through the product map $\Gamma_F^r \to \Gamma_F^{ab}$). Projecting the image of $g_{plec,1} \cup g_{plec,2}$ on such a summand and using Shapiro's Lemma, we obtain an element

$$\begin{aligned} (g_{plec,1} \cup g_{plec,2})_f \in \; & H^r(S_r \ltimes \Gamma_F^r, (V_f(2))^{\otimes r} \otimes \mathbb{Z}[\mathrm{Gal}(F'/F)]) \\ & \simeq \textstyle\bigwedge^r_{\mathbb{Q}_\ell[\mathrm{Gal}(F'/F)]} H^1(\Gamma_{F'}, V_f(2)). \end{aligned}$$

Such elements (with $\mathbb{Z}_\ell$-coefficients) form an Euler system of rank r, which generalises Kato's Euler system.

Fix an embedding $i : Y \hookrightarrow Y_1 \times Y_2$ given by group-theoretical data, where Y_1 and Y_2 are also open Hilbert modular varieties (attached to the same field F). Let f_1, f_2 be cuspidal Hilbert eigenforms of parallel weight 2 over F occurring in geometric cohomology of Y_1 and Y_2, respectively.

If we apply to g_{plec} the plectic Gysin map

$$i_* : H^r_{et,plec}(Y, \mathbb{Q}_\ell(r)) \longrightarrow H^{3r}_{et,plec}(Y_1 \times Y_2, \mathbb{Q}_\ell(2r))$$

composed with the map

$$H^{3r}_{et,plec}(Y_1 \times Y_2, \overline{\mathbb{Q}}_\ell(2r)) \longrightarrow H^r(S_r \ltimes \Gamma^r_F, (V_{f_1} \otimes V_{f_2})(2))^{\otimes r} \otimes \mathbb{Z}[\mathrm{Gal}(F'/F)])$$

coming from plectic Hochschild-Serre spectral sequence, we obtain an element of

$$\textstyle\bigwedge^r_{\mathbb{Q}_\ell[\mathrm{Gal}(F'/F)]} H^1(\Gamma_{F'}, V_{f_1} \otimes V_{f_2}(2)),$$

for a certain finite abelian extension F'/F. This is a generalisation of [10].

One can also replace $Y_1 \times Y_2$ by a Hilbert modular variety attached to a totally real quadratic extension F_0 of F. In this case $V_{f_1} \otimes V_{f_2}$ is replaced by the tensor induction of V_{f_0}, for a cuspidal eigenform f_0 of weight 2 over F_0, yielding an element of

$$\textstyle\bigwedge^r_{\mathbb{Q}_\ell[\mathrm{Gal}(F'/F)]} H^1(\Gamma_{F'}, (\otimes\text{-}\mathrm{Ind}^{\Gamma_F}_{\Gamma_{F_0}}(V_{f_0}))(2)).$$

16. Plectic Hodge theory

The constructions of the previous two sections are for the moment conjectural, relying on the existence of the plectic struture on étale cohomology. However in the analogous setting of real Hodge structures, it is possible to give a fairly complete unconditional description of the plectic structure, and to construct the Hodge-theoretic version of the plectic theta elements of Section 13. Explicit computations then lead to formulae for special values of L-functions, as predicted by Stark's conjectures (for L-functions of abelian characters of CM-fields) and Beilinson's conjectures (for L-functions of Hilbert modular forms). In this and the following section we will give some indications as to how this works. Further details, including the foundations of plectic Hodge theory, will appear in a later paper in this series.

For cohomology with real coefficients, all that matters is the algebra $F \otimes \mathbb{R}$, so for the moment we will largely ignore F, and simply fix an integer $r \geq 1$. For $\mathbf{n} = (n_1, \dots, n_r) \in \mathbb{Z}^r$ we shall write $|\mathbf{n}| = \sum n_i$.

Definition 16.1. *A (pure) r-plectic real Hodge structure of weight $n \in \mathbb{Z}$ is a finite-dimensional real vector space V whose complexification carries a grading indexed by $\mathbb{Z}^{2r}$:*

$$V_\mathbb{C} := V \otimes_\mathbb{R} \mathbb{C} = \bigoplus_{\mathbf{p},\mathbf{q} \in \mathbb{Z}^r} V^{\mathbf{pq}}$$

such that $\overline{V^{\mathbf{pq}}} = V^{\mathbf{qp}}$, and $V^{\mathbf{pq}} = 0$ unless $|\mathbf{p} + \mathbf{q}| = n$.

Such things form the objects of weight n of an obvious Tannakian category, which is easily seen to be equivalent to the category of representations of the real algebraic group $\mathbb{S}^r$, where $\mathbb{S} = R_{\mathbb{C}/\mathbb{R}}\mathbb{G}_m$ is the Deligne torus. Restricting to the diagonal $\mathbb{S} \subset \mathbb{S}^r$ gives a forgetful functor from plectic to usual Hodge structures. One has the "plectic Tate object" $\mathbb{R}(r)_{plec}$, for which the only non-vanishing $V^{\mathbf{pq}}$

occurs for $\mathbf{p} = \mathbf{q} = (-1, \dots, -1)$, and whose image under the forgetful functor is the usual Tate structure $\mathbb{R}(r)$.

Given a plectic Hodge structure V one can define r "partial Hodge filtrations" $(F_j^\bullet)_{1\le j\le r}$ on $V_\mathbb{C}$, given by

$$F_j^p V_\mathbb{C} = \sum_{\substack{\mathbf{p},\mathbf{q}\in\mathbb{Z}^r \\ p_j \ge p}} V^{\mathbf{pq}}.$$

The discussion of Section 2 shows that the real cohomology of a compact pure Shimura variety Y associated to a group of the form $R_{F/\mathbb{Q}}H$ has a canonical plectic Hodge structure.

When Y is not compact, we need the notion of a plectic mixed Hodge structure. Recall [6] that the category $\mathbb{R}$MHS of real mixed Hodge structures is equivalent to the category of representations of a real pro-algebraic group $\mathscr{G}_\mathscr{H}$, which is the semidirect product of $\mathbb{S}$ and a pro-unipotent group U, whose Lie algebra is described in *loc. cit.* Explicitly, a representation V of $\mathscr{G}_\mathscr{H}$ gives rise to two bigradings $V_\mathbb{C} = \bigoplus V_F^{pq} = \bigoplus V_{\bar F}^{pq}$, the first of which comes from the action of $\mathbb{S} \subset \mathscr{G}_\mathscr{H}$, together with a nilpotent endomorphism $\delta \in \mathrm{End}(V_\mathbb{C})$ such that $V_{\bar F}^{pq} = \exp(\delta)(V_F^{pq})$. It satisfies $\delta(V_F^{pq}) \subset \sum_{p'<p,q'<q} V_F^{p'q'}$, and the graded components of δ are the images of the (free) generators of $\mathrm{Lie}\, U_\mathbb{C}$. The filtrations F, $\bar F$ and W can be recovered from the bigradings as

$$F^p V_\mathbb{C} = \sum_{p'\ge p} V_F^{p'q}, \quad \bar F^q V_\mathbb{C} = \sum_{q'\ge q} V_{\bar F}^{pq'}, \quad W_n V_\mathbb{C} = \sum_{p+q\le n} V_F^{pq} = \sum_{p+q\le n} V_{\bar F}^{pq}.$$

We may then simply define the category $\mathbb{R}\mathrm{MHS}_{plec}$ of *plectic real mixed Hodge structures* to be the category $Rep_\mathbb{R}(\mathscr{G}_\mathscr{H}^r)$.

If V is an object of $\mathbb{R}\mathrm{MHS}_{plec}$ then from the action of $\mathbb{S}^r \subset \mathscr{G}_\mathscr{H}^r$ it acquires a $\mathbb{Z}^{2r}$-grading, and therefore a weight filtration $W_\bullet$ (defined over $\mathbb{R}$) and r partial Hodge filtrations $F_j^\bullet$ (defined over $\mathbb{C}$) in such a way that $gr_n^W V$ becomes a pure plectic Hodge structure of weight n. Conversely, suppose one is given a real vector space V together with a real filtration $W_\bullet$ on V and r complex filtrations $F_j^\bullet$, such that on each $gr_n^W V$ the F_j induce a pure plectic Hodge structure. Then, provided the filtrations satisfy a further technical compatibility which we will not write down here, V is a plectic mixed Hodge structure.

It is natural to expect that the real cohomology of a Shimura variety/stack (pure or mixed), whose associated group is a restriction of scalars from F, carries a natural plectic mixed Hodge structure. It seems hard to see this directly from the Lie algebra-theoretic description of cohomology in the non-compact case (it is not even known how to detect the weight filtration). However in simple cases, one can construct the plectic structure directly, using toroidal compactifications. In particular, one has the following result.

Theorem 16.2. *Let $G = R_{F/\mathbb{Q}}GL(2)$ and $\mathscr{X} = (\mathbb{C}-\mathbb{R})^r$. Then the real cohomology of $Y = Sh_K(G, \mathscr{X})$, for $K \subset G(\widehat{\mathbb{Q}})$ a sufficiently small open compact subgroup, carries a canonical plectic mixed Hodge structure. The same is true for the mixed Shimura stack $\mathscr{A}$ attached to $G = R_{F/\mathbb{Q}}(GL(2) \ltimes \mathbb{G}_a^2)$.*

In fact, slightly more is true: by the methods of [3], there exists a complex $\underline{\mathbf{R}\Gamma}(Y, \mathbb{R})$ equipped with filtrations $W_\bullet$ and (after tensoring with $\mathbb{C}$) $F_j^\bullet$, whose cohomology is $H^*(Y, \mathbb{R})$ together with its plectic Hodge structure. Then a similar procedure to [2] gives *plectic absolute Hodge cohomology* groups $H^i_{\mathscr{H},plec}(Y,\mathbb{R}(rn))$, together with a spectral sequence

$$E_2^{ij} = \mathrm{Ext}^i_{\mathbb{R}\mathrm{MHS}_{plec}}(\mathbb{R}(-nr)_{plec}, H^j(Y,\mathbb{R})) \Longrightarrow H^{i+j}_{\mathscr{H},plec}(Y,\mathbb{R}(rn)).$$

Similar statements hold for the cohomology of $\mathscr{A}$ and $\mathscr{A} - \mathscr{A}[d]$.

17. An arithmetic application

The formal procedure described in Section 13 may now be carried through in plectic absolute Hodge cohomology. One thus obtains canonical elements

$${}_d\Theta_{\mathscr{H},plec} \in H^{2r-1}_{\mathscr{H},plec}(\mathscr{A} - \mathscr{A}[d], \mathbb{R}(r)),$$

together with their pull-backs by torsion sections arising from the level structure

$$x^*({}_d\Theta_{\mathscr{H},plec}) \in H^{2r-1}_{\mathscr{H},plec}([\Delta\backslash Y], \mathbb{R}(r))$$

and, by taking the cap-product with a generator of $H_{r-1}(\Delta,\mathbb{Z})$, classes

$$g_{\mathscr{H},plec} = x^*({}_d\Theta_{\mathscr{H},plec}) \cap [\Delta] \in H^r_{\mathscr{H},plec}(Y,\mathbb{R}(r)).$$

which are the plectic analogues of the functions $\log|(\text{Siegel unit})|$.

There is an explicit formula for the form representing the class $g_{\mathscr{H},plec}$ in terms of coordinates. For simplicity we give this in the case $Y = \Gamma\backslash\mathfrak{H}^r$ for $\Gamma \subset SL_2(O_F)$, parametrising abelian varieties $A_\tau = \Lambda_\tau\backslash\mathbb{C}^r$ for $\tau = (\tau_j) \in \mathfrak{H}^r$, where $\mathfrak{H}$ is the upper half-plane and $\Lambda_\tau = \mathfrak{d}_F^{-1} + O_F\tau$. The variety A_τ carries a canonical principal polarisation associated to the form $\sum dz_j \wedge d\bar{z}_j/\operatorname{Im}(\tau_j)$. Let $(-,-)$ be the associated Hermitian form on $\mathbb{C}^r$. For $\gamma \in \Lambda_\tau$, let $\chi_\gamma\colon A_\tau \to U(1)$ be the character given by Pontryagin duality and the polarisation form. The real analytic function $g(\tau)$ representing the class $g_{\mathscr{H},plec}$ is then given by

$$g(\tau) = (\text{constant}) \sum_{y\in A_\tau[d]} \sum_{\gamma}{}' (\chi_\gamma(x+y) - \chi_\gamma(x)) \prod_{j=1}^{r} \frac{1}{\gamma_j\bar{\gamma}_j},$$

the sum being taken over nonzero $\gamma \in \Lambda_\tau$ modulo Δ. When τ is a CM-point, taking finite Fourier transforms of these series one obtains special values $L(\chi,1)$, where χ is an abelian character of the Galois group of the CM field $F(\tau)$.

Space precludes describing in detail here the Hodge-theoretic version of the cup product constructions sketched in Section 15, which lead to a formula for the r-th derivative $L^{(r)}(f,0)$, where f is a cuspidal Hilbert eigenform of parallel weight 2.

The plectic theta elements constructed above form the first of a hierarchy of cohomology classes, coming from a plectic analogue of the abelian polylogarithm classes of Wildeshaus [23]. There are explicit formulae for these higher plectic classes, related to those described by Levin [11]. Restricting to torsion sections, these polylogarithmic classes define *plectic Eisenstein classes* in the plectic absolute Hodge cohomology of Y (now with non-trivial coefficients), and evaluating these at CM points gives a cohomological interpretation for the derivatives $L^{(r)}(\chi,0)$, for all algebraic Hecke characters χ of the CM field for which $s=0$ is a non-critical value with $\mathrm{ord}_{s=0}L(\chi,s) = r$.

18. Final speculations

Suppose X is a Shimura variety or stack of the type considered, or a diagram of such objects. In the presence of Conjecture 6.1, we have defined (7.1) the plectic absolute étale cohomology groups $H^*_{et,plec}(X, \mathbb{Q}_\ell(n))$ of X, for appropriate $n \in \mathbb{Z}$. We have also (in many cases unconditionally) plectic absolute Hodge cohomology groups $H^*_{\mathscr{H},plec}(X, \mathbb{R}(n))$. In view of the universal nature of (usual) motivic cohomology, one is led to speculate:

Question 18.1. Do there exist "plectic motivic cohomology groups" for X: $H^*_{\mathscr{M},plec}(X, \mathbb{Z}(n))$ or $H^*_{\mathscr{M},plec}(X, \mathbb{Q}(n))$, with good functorial properties, together with functorial maps to the plectic étale and absolute Hodge groups?

Suppose such groups did exist, and were sufficiently functorial that the constructions of Sections 13–14 could be carried out. As indicated there, one could then prove versions of Stark's conjectures for abelian characters of CM and totally real fields, as well as special cases of Beilinson's conjectures for non-critical L-values of Hilbert modular forms.

The discussion in Sections 11–16 involved generalisations of "cyclotomic" zeta elements. It is tempting to ask:

Question 18.2. Is there a plectic analogue of anticyclotomic zeta-elements, such as Heegner points and (generalised) Heegner cycles?

Let f be a Hilbert modular eigenform over F and χ an algebraic Hecke character of a totally imaginary quadratic extension L of F (suitably compatible with the central character of f). It seems that a hypothetical geometric version (1.3) of the plectic formalism would yield interesting elements responsible for the special values of anticyclotomic Rankin-Selberg L-functions $L(f \times \theta_\chi, s)$ at the central point, in the case when the order of vanishing a of the L-function satisfies $1 \leq a \leq r$. This will be discussed in more detail in a later paper in this series.

References

[1] A. A. Beilinson: Higher regulators and values of L-functions. J. Soviet Math. **30** (1985), 2036–2070

[2] A. A. Beilinson: Notes on absolute Hodge cohomology. Applications of algebraic K-theory to algebraic geometry and number theory (Contemporary Mathematics **55** (1986)), 35–68

[3] J. I. Burgos: A C^∞ logarithmic Dolbealt complex. Compositio Math. **92** (1994), 61–86.

[4] A. Caspar: Realisations of Kummer-Chern-Eisenstein-motives, Manuscripta Math. **122** (2007), 23–57.

[5] P. Deligne: Variétés de Shimura: interprétation modulaire, et techniques de construction de modèles canoniques. Automorphic forms, representations and L-functions (Proc. Sympos. Pure Math., Oregon State Univ., Corvallis, Ore., 1977), Part 2, 247–289, Proc. Sympos. Pure Math. XXXIII, Amer. Math. Soc., Providence, R.I., 1979.

[6] P. Deligne: Structures de Hodge mixtes réelles. in: Motives. Proc. Symp. in Pure Math. 55 vol.1 (AMS 1994) 509-514.

[7] G. Harder: Eisensteinkohomologie und die Konstruktion gemischter Motive, Lecture Notes in Mathematics, 1562. Springer-Verlag, Berlin, 1993.

[8] Kazuya Kato, p-adic Hodge theory and values of zeta functions of modular forms, Cohomologies p-adiques et application arithmétiques. III, Astérisque, vol. 295, Société Mathématique de France, Paris, 2004.

[9] R.P. Langlands, On the zeta functions of some simple Shimura varieties, Canad. J. Math. **31** (1979), no. 6, 1121–1216.

[10] A. Lei, D. Loeffler, S. Zerbes: Euler systems for Rankin-Selberg convolutions of modular forms, Annals of Mathematics, **180** (2), 2014, 653–771

[11] A. Levin: Polylogarithmic currents on Abelian varieties. In: Regulators in Analysis, Geometry and Number Theory Progress in Mathematics Volume 171, 2000, pp 207–222.
[12] H. G. Liddell, R. Scott: An Intermediate Greek-English Lexicon. Oxford. Clarendon Press. 1889.
[13] J. S. Milne: Points on Shimura varieties mod p. Automorphic forms, representations and L-functions (Proc. Sympos. Pure Math., Oregon State Univ., Corvallis, Ore., 1977), Part 2, 165–184, Proc. Sympos. Pure Math. XXXIII, Amer. Math. Soc., Providence, R.I., 1979.
[14] J. S. Milne: Etude d'une class d'isogénie. In Variétiés de Shimura et Fonctions L, Publications Mathématiques de l'Université Paris 7 (1979), 73–81.
[15] J. Nekovář: Hidden symmetries in the theory of complex multiplication, Algebra, arithmetic, and geometry: in honor of Yu. I. Manin, Vol. II, 399–437, Progr. Math. 270, Birkhäuser Boston, Boston, MA, 2009.
[16] J. Nekovář: Generalised Eichler-Shimura relations and semi-simplicity of étale cohomology of quaternionic Shimura varieties, in preparation.
[17] T. Oda: Periods of Hilbert modular surfaces. Progress in Mathematics **19**, Birkäuser (1982).
[18] M. Ohta: On ℓ-adic representations attached to automorphic forms, Japan. J. Math. **8** (1982), 1-47.
[19] R. Pink: Arithmetical compactification of mixed Shimura varieties. Bonner Mathematische Schriften **209**, Univ. Bonn (1990), 340pp.
[20] H. Reimann, The semi-simple zeta function of quaternionic Shimura varieties, Lecture Notes in Mathematics **1657**, Springer-Verlag, Berlin, 1997.
[21] A. J. Scholl, An introduction to Kato's Euler system. In: Galois representations in arithmetic algebraic geometry, ed. A. J. Scholl and R. L. Taylor. Cambridge University Press, 1988.
[22] R. Taylor: On Galois representations associated to Hilbert modular forms, Inv. Math. **98** (1989), 265-280.
[23] J. Wildeshaus: Realizations of polylogarithms. Lecture Notes in Mathematics **1650**, Springer-Verlag, Berlin, 1997.
[24] A. Wiles: On ordinary λ-adic representations associated to modular forms. Inv. Math. **94** (1988) 529–573
[25] H. Yoshida: On the zeta functions of Shimura varieties and periods of Hilbert modular forms. Duke Math. J., **75** (1):121-191, 1994.

Institut de Mathématiques de Jussieu, Université Pierre et Marie Curie, 4 place Jussieu, F-75252 Paris, France
E-mail address: jan.nekovar@imj-prg.fr

Department of Pure Mathematics and Mathematical Statistics, Centre for Mathematical Sciences, University of Cambridge, Wilberforce Road, Cambridge CB3 0WB, England
E-mail address: a.j.scholl@dpmms.cam.ac.uk