Integral elements of K-theory and products of modular curves II

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Abstract

We discuss the relationship between different notions of "integrality" in motivic cohomology/K-theory which arise in the Beilinson and Bloch-Kato conjectures, and prove their equivalence in some cases for products of curves, as well as obtaining a general result, first proved by Jannsen (unpublished), reducing their equivalence to standard conjectures in arithmetic algebraic geometry.

1 Introduction

This paper is a continuation of [18]. Its main aim is to give an unconditional proof of the following comparison between two different notions of integral motivic cohomology, which was (in the special case $i = 3$, $n = d = 2$) stated (and used) without proof in [18, 2.3.10]. (I am grateful to those who insisted to me that this gap be filled.)

Theorem 1.1. *Let F be a number field, with ring of integers $\mathfrak{o}$. Let $C_1, \ldots, C_d$ be smooth projective curves over F, and let $M \subset h(\prod C_j)$ be a submotive of the Chow motive of their product. Let $0 < i \leq 2n-1$. Then if $n \geq d$, the integral motivic cohomology $H^i_{\mathcal{M}/\mathfrak{o}}(M, n)$ and the unramified motivic cohomology $H^i_{\mathcal{M},nr}(M, n)$ coincide.*

(Of course, one expects this to hold for any Chow motive without the condition $n \geq d$, and even the stronger statement in which $H_{\mathcal{M},nr}$ is replaced by the Bloch-Kato $H_{\mathcal{M},f}$-subgroup.) We prove this using a rather general compatibility in étale cohomology (3.1), plus Soulé's bounds on K-groups of special varieties over finite fields [19].

We first review the definitions of the various objects in Theorem 1.1. More generally, let $(F, \mathfrak{o})$ be one of the following:

(i) F a number field, $\mathfrak{o}$ its ring of integers or a localisation of it;

(ii) $\mathfrak{o}$ a Henselian discrete valuation ring whose field of fractions F has characteristic 0, and whose residue field is finite.

Let U/F be a proper and smooth scheme. Then there are defined motivic cohomology groups $H^i_{\mathcal{M}}(U,n) = H^i_{\mathcal{M}}(U,\mathbb{Q}(n))$, for integers i, n. With rational coefficients, one has a K-theoretic interpretation (or, if one prefers, definition):

$$H^i_{\mathcal{M}}(U,n) = K^{(n)}_{2n-i}U \subset K_{2n-i}(U)\otimes\mathbb{Q}$$

the eigenspace on which Adams operators ψ^q act as multiplication by q^n.

If U extends to a regular scheme X, proper and flat over $\mathfrak{o}$, then the integral motivic cohomology is defined to be

$$H^i_{\mathcal{M}/\mathfrak{o}}(U,n) := \mathrm{im}\left[K^{(n)}_{2n-i}(X) \to H^i_{\mathcal{M}}(U,n)\right]$$

If M is an effective Chow motive, then $X = e\cdot h(U)$ for some U and some idempotent $e \in \operatorname{End} h(U)$. One may choose U in such a way that it has a regular proper model X, and the subspaces

$$H^i_{\mathcal{M}}(M,n) = e\cdot H^i_{\mathcal{M}}(U,n), \quad H_{\mathcal{M}/\mathfrak{o}}(M,n) = e\cdot H^i_{\mathcal{M}/\mathfrak{o}}(U,n)$$

of $H^i_{\mathcal{M}}(U,n)$ depend functorially only on M (this is the main result of [18, §1]). The integral motivic cohomology groups $H^*_{\mathcal{M}/\mathfrak{o}}(M,*)$ feature in Beilinson's conjectures on special values of L-functions [1, 3, 14].

There is defined an ℓ-adic regulator map, with values in continuous ℓ-adic cohomology [8]

$$\mathrm{reg}_\ell\colon H^i_{\mathcal{M}}(U,n) \to H^i(U,\mathbb{Q}_\ell(n)).$$

If $i \neq 2n$ then one knows that the composite

$$H^i_{\mathcal{M}}(U,n) \to H^i(U,\mathbb{Q}_\ell(n)) \to H^i(U\otimes\overline{F},\mathbb{Q}_\ell(n))$$

is zero, so that the Hochschild-Serre spectral sequence in continuous ℓ-adic cohomology induces a homomorphism, the ℓ-adic Abel-Jacobi map

$$AJ_\ell\colon H^i_{\mathcal{M}}(U,n) \to H^1(F,V_\ell)).$$

Here we have written $V_\ell = H^{i-1}(U\otimes\overline{F},\mathbb{Q}_\ell(n))$ for the ℓ-adic cohomology of the geometric fibre. Let v be a prime of F not dividing ℓ, with residue field k_v, and F_v the completion of F at v. Let $G_v = \mathrm{Gal}(\overline{F}_v/F_v)$, $I_v = \mathrm{Gal}(\overline{F}_v/F_v^{nr})$ the inertia group, and $\Gamma_v = \mathrm{Gal}(\bar{k}_v/k_v) = G_v/I_v$.[1] Recall the exact sequence

[1] In case (ii), we mean that v is the canonical place of F, so $G_v = G$, and that the residue characteristic of F is different from ℓ.

of ramified and unramified cohomology

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H^1(\Gamma_v, V_\ell^{I_v}) & \longrightarrow & H^1(G_v, V_\ell) & \longrightarrow & H^1(I_v, V_\ell)^{\Gamma_v} & \longrightarrow & 0 \\ & & \| & & & & \| & & \\ & & H^1_{nr}(F_v, V_\ell) & & & & H^1_{ram}(F_v, V_\ell) & & \end{array}$$

Let $\mathrm{res}_v \colon H^1(F, V_\ell) \to H^1(F_v, V_\ell)$ be the restriction map. Bloch and Kato [4] define a subspace $H^1_f(F_v, V_\ell)$ which coincides with $H^1_{nr}(F_v, V_\ell)$ if $\ell \neq p_v$, and use this to define a subspace of motivic cohomology by

$$H^i_{\mathcal{M},f}(X, n) = \bigcap_{v,\ell} (\mathrm{res}_v \circ AJ_\ell)^{-1}(H^1_f(F_v, V_\ell)) \tag{1}$$

— in the notation of Bloch-Kato and Fontaine–Perrin-Riou, V_ℓ is the realisation of the motive $V = h^{i-1}(U)(n)$, and they write $H^1_{\mathcal{M},f}(V)$ for the group (1). Implicit in Bloch-Kato's generalisation of the Beilinson conjectures is part (i) of the following conjecture (and see already [2, 4.0.(b)] for the case $\ell \neq p_v$) — part (ii) is folklore:

Conjecture 1.2. *(i)* $H^i_{\mathcal{M},f}(U, n) = H^i_{\mathcal{M}/\mathfrak{o}}(U, n)$.

(ii) for fixed v the subspace

$$\ker \left[H^i_{\mathcal{M}}(U, n) \to H^1(F_v, V_\ell)/H^1_f(F_v, V_\ell) \right]$$

is independent of ℓ.

Let us from now on ignore the places v dividing ℓ (which, to be sure, are the most interesting ones) and define

$$H^i_{\mathcal{M},nr}(U, n) = \bigcap_{v,\ell \neq p_v} (\mathrm{res}_v \circ \mathrm{reg}_\ell)^{-1}(H^1_{nr}(F_v, V_\ell))$$

The ring $\mathrm{End}\, h(U)$ of correspondences on U (for rational equivalence) acts on everything in sight and so for a submotive $M \subset h(U)$ the groups $H^i_{\mathcal{M},f}(M, n)^0 \subset H^i_{\mathcal{M},nr}(M, n) \subset H^i_{\mathcal{M}}(M, n)$ are defined.

It is well known that one has $H_{\mathcal{M}/\mathfrak{o}} \subset H_{\mathcal{M},nr}$ (we recall the proof in the next section) and even that $H_{\mathcal{M}/\mathfrak{o}} \subset H_{\mathcal{M},f}$ under suitable hypotheses... (Similar statements hold for $\ell = char(k)$, see for example [12, 13]).

Jannsen showed (unpublished) that the equality of $H_{\mathcal{M}/\mathfrak{o}}$ and $H_{\mathcal{M},nr}$ would follow from two standard conjectures: the monodromy-weight conjecture on the action of inertia on ℓ-adic cohomology, and his generalisation of the Tate conjecture on algebraic cycles to arbitrary varieties over finite fields. See 2.4 below. After reviewing some of what is known in the next section, will prove a rather general compatibility in ℓ-adic cohomology, from which Jannsen's result will be a corollary.

For historical reasons I have kept to the old definition of motivic cohomology using K-theory, rather than higher Chow groups. It should not be hard to rewrite everything here in terms of higher Chow groups, using the localisation techniques of Levine [11]. However there are no new phenomena to be expected when working with $\mathbb{Z}$-coefficients, if only because, for a $\mathbb{Z}_\ell$-representation T of $\mathrm{Gal}(\overline{F}/F)$ (for F local or global) Bloch and Kato define $H^1_f(F,T)$ to be simply the preimage, via the natural map $H^1(F,T) \to H^1(F, T\otimes\mathbb{Q}_\ell)$ of the subspace $H^1_f(F,T\otimes\mathbb{Q}_\ell) \subset H^1(F,T\otimes\mathbb{Q}_\ell)$. Moreover, the integral groups "without denominators" are only meaningful in the presence of a regular model X of U, not just a regular alteration.

2 Preliminaries

For completeness, let us first recall what happens when $i = 2n$. In this case, the localisation sequence of K-theory shows that $H^{2n}_{\mathcal{M}/\mathfrak{o}}(U,n)$ and $H^{2n}_{\mathcal{M}}(U,n)$ are equal; this group is $CH^n(U)\otimes\mathbb{Q}$, the Chow group of codimension n cycles on U. In this case the cycle class map $H^{2n}_{\mathcal{M}}(U,n) \to H^{2n}(\overline{U},\mathbb{Q}_\ell(n))$ is non-zero, and its kernel is $H^{2n}_{\mathcal{M}}(U,n)^0 := CH^n(U)^0\otimes\mathbb{Q}$, the subgroup of cycles homologically equivalent to zero. The Abel-Jacobi homomorphism is a map from $H^{2n}_{\mathcal{M}}(U,n)^0$ to $H^1(F,V_\ell)$, and the obstruction to the equality $H^{2n}_{\mathcal{M},nr}(U,n)^0 = H^{2n}_{\mathcal{M}}(U,n)^0$ lies in the ramified cohomology groups

$$H^1(I_v,V_\ell)^{\Gamma_v} = \mathrm{Hom}_{\Gamma_v}(\mathbb{Q}_\ell(1-n), H^{2n-1}(\overline{U},\mathbb{Q}_\ell(n-1))_{I_v}). \qquad (2)$$

The monodromy-weight conjecture (recalled as 2.1 below) implies that the I_v-coinvariants of $H^{2n-1}(\overline{U},\mathbb{Q}_\ell)$ have weights $\geq 2n-1$, and therefore that the obstruction group (2) vanishes. In other words, $H^{2n}_{\mathcal{M},nr}(U,n)^0 \subset H^{2n}_{\mathcal{M}/\mathfrak{o}}(U,n)^0 = H^{2n}_{\mathcal{M}}(U,n)^0$, with equality if the monodromy-weight conjecture holds.

Since $H^i_{\mathcal{M}}(U,n) \subset K_{2n-i}U\otimes\mathbb{Q}$ vanishes for $i > 2n$, we assume henceforth that $q := 2n - i > 0$.

For the moment suppose that we are in setting (i). Write $\mathfrak{o}_{(v)}$ for the localisation of $\mathfrak{o}$ at v, $\mathfrak{o}_v$ for its completion, and k_v for its residue field. Assume that U has a regular and proper model X over $\mathfrak{o}$. Then from the localisation sequences

$$K_q X \to K_q U \to \coprod_v K'_{q-1} X \otimes k_v$$
$$K_q X \otimes \mathfrak{o}_{(v)} \to K_q U \to K'_{q-1} X \otimes k_v$$
$$K_q X \otimes \mathfrak{o}_v \to K_q U \otimes F_v \to K'_{q-1} X \otimes k_v$$

we see that

$$H^i_{\mathcal{M}/\mathfrak{o}}(U,n) = \ker \left[H^i_{\mathcal{M}}(U,n) \to \coprod_v \frac{H^i_{\mathcal{M}}(U \otimes F_v, n)}{H^i_{\mathcal{M}/\mathfrak{o}}(U \otimes F_v, n)} \right]$$

(cf. [18, 1.3.5–6]). Since by definition the corresponding identity holds for $H_{\mathcal{M},nr}$, the comparison between $H_{\mathcal{M}/\mathfrak{o}}$ and $H_{\mathcal{M},nr}$ is reduced to the local case.

We also recall that both the integrality and the unramified conditions are stable under finite extensions F'/F: under the inclusion $H^i_{\mathcal{M}}(U,n) \subset H^i_{\mathcal{M}}(U \otimes F', n)$ one has

$$H^i_{\mathcal{M}/\mathfrak{o}}(U,n) = H^i_{\mathcal{M}}(U,n) \cap H^i_{\mathcal{M}/\mathfrak{o}}(U \otimes F', n)$$
$$H^i_{\mathcal{M},nr}(U,n) = H^i_{\mathcal{M}}(U,n) \cap H^i_{\mathcal{M},nr}(U \otimes F', n)$$

which for $H_{\mathcal{M},nr}$ is clear from the definition, and for $H_{\mathcal{M}/\mathfrak{o}}$ follows from [18, §1].

For the rest of the paper we will assume that we are in the local case (ii): thus F is local, with valuation ring $\mathfrak{o}$ and finite residue field k, and write $S = \operatorname{Spec} \mathfrak{o} = \{\eta, s\}$ as usual. Let $f\colon X \to S$ be proper and flat, with special fibre $g\colon Y = X_s \to \operatorname{Spec} k$ and generic fibre $U = X \setminus Y = X_\eta$. Let $d = \dim U$, and write $G = \operatorname{Gal}(\overline{F}/F)$, I for the inertia subgroup of G and $\Gamma = \operatorname{Gal}(\bar{k}/k) = G/I$.

We consider the analogue of AJ_ℓ on X itself. By the proper base-change theorem

$$H^0(S, R^i f_* \mathbb{Q}_\ell(n)) = H^0(s, R^i g_* \mathbb{Q}_\ell(n)) = H^i(\overline{Y}, \mathbb{Q}_\ell(n))^\Gamma = 0$$

since by Deligne [6], the weights of $H^i(\overline{Y}, \mathbb{Q}_\ell(n)$ are $\leq (i-2n)$, hence nonzero. So from the Hochschild-Serre spectral sequence we obtain an edge homomorphism

$$e_1\colon H^i(X, \mathbb{Q}_\ell(n)) \to H^1(S, \mathcal{F}) \qquad \text{where } \mathcal{F} = R^{i-1} f_* \mathbb{Q}_\ell(n).$$

Composing with the Chern character $ch\colon K_qX \to H^i(X, \mathbb{Q}_\ell(n))$, we obtain a commutative diagram, in which the bottom row is exact:

$$
\begin{array}{ccccccc}
 & & K_qX & \longrightarrow & K_qU & & \\
 & & \big\downarrow{\scriptstyle e_1\circ ch} & & \big\downarrow{\scriptstyle AJ_\ell} & & \\
 & & H^1(S,\mathcal{F}) & \longrightarrow & H^1(\eta,\mathcal{F}_\eta) & & \\
 & & \big\downarrow & & \Big\| & & \\
 & & H^1(S, j_*\mathcal{F}_\eta) & & \Big\| & & \\
 & & \Big\| & & \Big\| & & \\
 & & H^1(s, i^*j_*\mathcal{F}_\eta) & & \Big\| & & \\
 & & \Big\| & & \Big\| & & \\
0 & \longrightarrow & H^1(\Gamma, \mathcal{F}_{\bar\eta}^I) & \longrightarrow & H^1(G, \mathcal{F}_{\bar\eta}) & \longrightarrow & H^1(I, \mathcal{F}_{\bar\eta})^\Gamma
\end{array}
\tag{3}
$$

This shows that $H^i_{\mathcal{M}/\mathfrak{o}}(X, n) \subset H^i_{\mathcal{M},\ell-f}(X, n)$ whenever $\ell \neq char(k)$, as mentioned in the introduction.

We next review when the obstruction group $H^1(I, \mathcal{F}_{\bar\eta})^\Gamma$ can be non-zero. First recall:

Conjecture 2.1 (Monodromy-weight conjecture)**.** *Let $W_\bullet$ denote the weight filtration on $H^j(\overline{U}, \mathbb{Q}_\ell)$, and let $N\colon H^j(\overline{U}, \mathbb{Q}_\ell) \to H^j(\overline{U}, \mathbb{Q}_\ell)(-1)$ denote the "logarithm of monodromy" operator. Then for each $r \geq 0$, N^r induces an isomorphism*

$$\bar{N}^r\colon gr^W_{j+r} H^j(\overline{U}, \mathbb{Q}_\ell) \xrightarrow{\sim} gr^W_{j-r} H^j(\overline{U}, \mathbb{Q}_\ell)(-r).$$

Assume that X is regular, and that Y is a reduced strict normal crossings divisor in X. Then the weight spectral sequence of Rapoport-Zink [15,] controls the weights of $H^j(\overline{U}, \mathbb{Q}_\ell)$; let $h = h(X)$ be the least positive integer such that no set of more than h components of Y has non-trivial intersection. Then

$$\mathrm{gr}^W_w\, H^j(\overline{U}, \mathbb{Q}_\ell) \neq 0 \;\Rightarrow\; \max\{0, j-h, 2d-j\} \leq w \leq \min\{2j, j+h, 2d\}.$$

In general we may replace U by an alteration U' for which such a model $U' \subset X'$ exists, and take $h = h(X')$.

Therefore if $H^1(I,\mathcal{F}_{\bar\eta})^\Gamma = \mathrm{Hom}_\Gamma(\mathbb{Q}_\ell(1-n), H^{i-1}(\overline{U},\mathbb{Q}_\ell)_I)$ is non-zero, the pair (i,n) must satisfy the inequalities

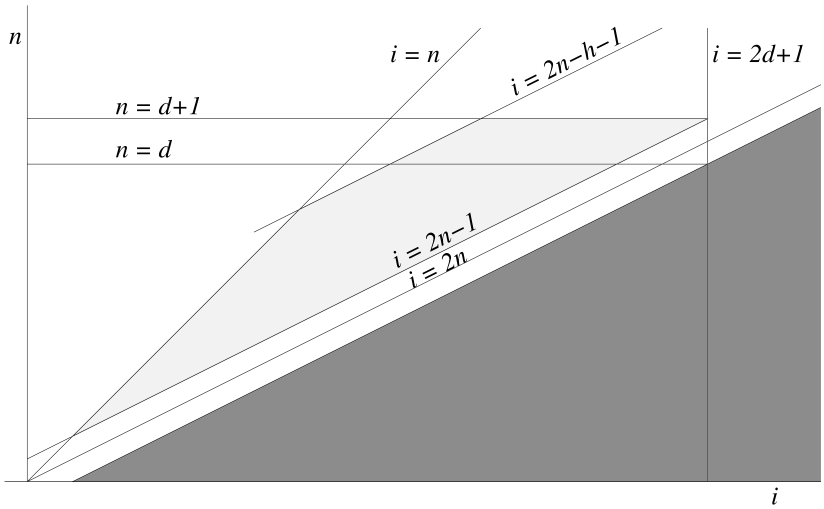
$$n \le d+1, \quad n \le i \le 2d \quad \text{and} \quad i \ge 2n-h-1.$$

We also have the obvious inequality $i \le 2n$. So far we have not used the monodromy-weight conjecture; if we assume it, then the weights of $H^j(\overline{U},\mathbb{Q}_\ell)_I$ are all $\ge j$, whence we have an additional inequality $i \le 2n-1$, which just excludes the case $i = 2n$ already considered at the beginning of this section.

For U a product of curves, Theorem 1.1 therefore shows that:

- in the region $n > d+1$, one has $H^i_{\mathcal{M}/\mathfrak{o}}(U,n) = H^i_{\mathcal{M}}(U,n)$ (for this the compatibility 4.1 is not needed, only the computations on the special fibre at the end of this section); and
- along the lines $n = d$ and $n = d+1$ the integrality conditions (which are in general non-trivial) coincide.

Notice also that over a number field one expects $H^i_{\mathcal{M}}(U,n) = 0$ as soon as $i > 2d+1$.



To go further we want to enlarge the diagram (3) to

$$\begin{array}{ccccccccc}
 & & K_qX & \longrightarrow & K_qU & \xrightarrow{\alpha} & K'_{q-1}Y & & \\
 & & \downarrow & & \downarrow & \searrow^{\beta} & \downarrow{\scriptstyle\phi} & & \\
0 & \longrightarrow & H^1(\Gamma,\mathcal{F}^I_{\bar\eta}) & \longrightarrow & H^1(G,\mathcal{F}_{\bar\eta}) & \longrightarrow & H^1(I,\mathcal{F}_{\bar\eta})^\Gamma & \longrightarrow & 0
\end{array} \tag{4}$$

for a suitable vertical map ϕ, where the top row is the localisation sequence in K'-theory, so as to compare the kernels of α and β. We recall (see §3) that under the boundary map ∂, the subspace $K_q^{(n)}U \subset K_qU \otimes \mathbb{Q}$ maps into the subspace $K'^{(n-d-1)}_{q-1}Y \subset K'_{q-1}Y \otimes \mathbb{Q}$, and that the Riemann-Roch transformation τ maps $K'^{(n-d-1)}_{q-1}Y$ to the space of Γ-invariants of the ℓ-adic homology group

$$\begin{aligned} H_{2d-i+1}(\overline{Y}, \mathbb{Q}_\ell(d-n+1)) &= H^{i-2d-1}(\overline{Y}, Rf_s^!\mathbb{Q}_\ell(n-d-1)) \\ &\simeq H^{2d-i+1}(\overline{Y}, \mathbb{Q}_\ell(d-n+1))^\vee \end{aligned}$$

(the isomorphism being given by Grothendieck-Verdier duality). In the bottom row, we have

$$H^1(I, \mathcal{F}_{\bar\eta}) = H^{i-1}(\overline{U}, \mathbb{Q}_\ell(n-1))_I \simeq \left[H^{2d-i+1}(\overline{U}, \mathbb{Q}_\ell(d-n+1))^I\right]^\vee$$

by Poincaré duality. Finally we have the specialisation map

$$sp\colon H^{2d-i+1}(\overline{Y}, \mathbb{Q}_\ell) \to H^{2d-i+1}(\overline{U}, \mathbb{Q}_\ell)^I$$

and we can therefore formulate the desired compatibility as:

Proposition 2.2. *The following diagram is commutative up to sign:*

$$\begin{array}{ccc} K_q^{(n)}U & \longrightarrow & K'^{(n-d-1)}_{q-1}Y \\ & & \downarrow \tau \\ \downarrow AJ_\ell & & H^{i-2d-1}(\overline{Y}, Rf_s^!\mathbb{Q}_\ell(n-d-1)) \\ & & \downarrow \simeq \\ H^1(G, H^{i-1}(\overline{U}, \mathbb{Q}_\ell(n))) & & H^{2d-i+1}(\overline{Y}, \mathbb{Q}_\ell(d-n+1))^\vee \\ \downarrow & & \uparrow sp^\vee \\ H^1(I, H^{i-1}(\overline{U}, \mathbb{Q}_\ell(n)))^\Gamma & \hookrightarrow & (H^{2d-i+1}(\overline{U}, \mathbb{Q}_\ell(d-n+1))^\vee)_I \end{array}$$

This will be reformulated in a more general setting in the next section. First, we draw some consequences from it. We recall that the monodromy-weight conjecture implies:

Conjecture 2.3 (Local invariant cycle "theorem")**.** *Suppose that X is regular. Then for every j the specialisation map*

$$sp\colon H^j(\overline{Y}, \mathbb{Q}_\ell) \to H^j(\overline{U}, \mathbb{Q}_\ell)^I$$

is a surjection.

From 2.2 one then obtains immediately:

Corollary 2.4 (Jannsen)**.** *Suppose that the Riemann-Roch transformation*

$$\tau\colon K'^{(d-n-1)}_{q-1} Y \otimes \mathbb{Q}_\ell \to H_{2d-2n+q+1}(\overline{Y}, \mathbb{Q}_\ell(d-n+1))$$

is injective, and that the local invariant cycle theorem 2.3 holds for $(X, i-1)$*. Then* $H^i_{\mathcal{M}/\mathfrak{o}}(U, n) = H^i_{\mathcal{M},nr}(U, n)$*.*

The hypothesis that τ is injective would be a consequence of Jannsen's generalisation of the Tate conjecture:

Conjecture 2.5 (Jannsen [9, 12.4(a)])**.** *If Y is proper over a finite field k, of dimension d, then Frobenius acts semisimply on the ℓ-adic homology of $\overline{Y}$, and for every q and m the Riemann-Roch transformation is an isomorphism*

$$\tau\colon K'^{(m)}_q Y \otimes \mathbb{Q}_\ell \xrightarrow{\sim} H_{q-2m}(\overline{Y}, \mathbb{Q}_\ell(-m))^\Gamma.$$

As is shown in [9, 12.7], this is equivalent to standard conjectures for K-theory of nonsingular varieties over finite fields:

Conjecture 2.6 (Tate, Parshin)**.** *Let Y be proper and smooth over a finite field k.*

- *The action of* $\mathrm{Gal}(\bar{k}/k)$ *on* $H^*(\overline{Y}, \mathbb{Q}_\ell)$ *is semisimple.*
- *The cycle class map* $CH^*(Y) \otimes \mathbb{Q}_\ell \to H^{2*}(\overline{Y}, \mathbb{Q}_\ell(*))^{\mathrm{Gal}(\bar{k}/k)}$ *is an isomorphism.*
- *If $q > 0$, then $K_q Y \otimes \mathbb{Q} = 0$.*

(Jannsen's proof that 2.6 implies 2.5 assumes resolution of singularities, but one can remove this by appealing instead to De Jong's alterations theorem [10].)

We now analyze the proof in more detail to obtain Theorem 1.1. Granted Proposition 2.2, It suffices to prove the following two Propositions.

Proposition 2.7. *Let $U = C_1 \times \cdots \times C_d$ be a product of smooth proper curves. Then for all j, the monodromy-weight conjecture holds for $H^j(\overline{U})$.*

Proof. The monodromy-weight conjecture is stable under products (by the Künneth formula and [6, (1.6.9)]), so in particular it holds if U is a product of curves (even for products of varieties of dimension at most 2, by [15]). □

Proposition 2.8. *Let $U = C_1 \times \cdots \times C_d$ be a product of smooth proper curves. Then after replacing F by a finite extension, U admits a proper regular model $X/\mathfrak{o}$ for which:*

(i) *Y is a strict normal crossings divisor on X, and for every intersection Z of components of Y, the Γ-module $H^*(\bar{Z}, \mathbb{Q}_\ell)$ is semisimple.*

(ii) *the Riemann-Roch transformation on the homology of the special fibre*

$$\tau \colon H^{\mathcal{M}}_{2m-j}(Y, m) \otimes \mathbb{Q}_\ell \to H_{2m-j}(\overline{Y}, \mathbb{Q}_\ell(m))^\Gamma$$

is an isomorphism for $m \leq 1$.

Proof. We first need to construct a suitable regular model for U. After passing to a finite extension of F we may assume that each factor C_μ has semistable reduction, and further has a semistable model D_μ whose special fibre is a reduced strict normal crossing divisor, whose components and singular points are all rational over the residue field. Let $X' = \prod D_\mu$. Then X' is regular apart from singularities which are locally smooth over a product of double points; that is, locally isomorphic, for the étale topology, to

$$\operatorname{Spec} \mathfrak{o}[x_1, y_1, \ldots, y_r, y_r, z_1, \ldots, z_s]/(x_1 y_1 - \pi_F, \ldots, x_r y_r - \pi_F).$$

Take $X \to X'$ to be the resolution given in [5, Lemme 5.5]. The special fibre $Y = \cup Y_\alpha$ is a normal crossings divisor in X. Write as usual

$$Y_J = \bigcap_{\alpha \in J} Y_\alpha \qquad \text{for } J \subset \{\alpha\}$$
$$Y_{\langle q \rangle} = \coprod_{\#J = q+1} Y_J \qquad \text{for } q \geq 0$$

Then the description of the desingularisation as an iterated blowup [16, §2] shows that each Y_J belongs to $\mathcal{C}_k$, the smallest class of smooth and proper schemes over k such that

(i) $\mathcal{C}_k$ contains all products of smooth proper geometrically connected curves;

(ii) If W is in $\mathcal{C}_k$ and $P \to W$ is a projective bundle, then P is in $\mathcal{C}_k$;

(iii) If $Z \subset W$ with W and Z both in $\mathcal{C}_k$, then the blowup of W along Z is in $\mathcal{C}_k$.

If W is in $\mathcal{C}_k$ and $\dim W = d$, then the Chow motive of W can be computed using the fomulae for the Chow motives of projective bundles and blowups, and it is a sum of Chow motives of the form $\otimes_{1 \leq j \leq s} h^1(D_j) \otimes \mathbb{L}^{\otimes t}$ for curves D_j and some $t \geq 0$ with $s + t \leq d$. From this it follows that the ℓ-adic cohomology of Y_J is semisimple.

Together with the inclusion maps $Y_{J'} \subset Y_J$ for $J' \supset J'$, the $Y_{\langle q \rangle}$ form a strict simplicial scheme

$$Y_{\langle \bullet \rangle} = \left[\cdots \overset{\longrightarrow}{\underset{\longrightarrow}{\vdots}} Y_{\langle 2 \rangle} \overset{\longrightarrow}{\underset{\longrightarrow}{\longrightarrow}} Y_{\langle 1 \rangle} \overset{\longrightarrow}{\longrightarrow} Y_{\langle 0 \rangle} \right]$$

and the homology, both ℓ-adic and motivic, of Y is computed by a spectral sequence:

$$\begin{aligned} {}^{\mathcal{M}}E_1^{rs} &= H^{\mathcal{M}}_{2m-s}(Y_{\langle -r \rangle}, m) \Rightarrow H^{\mathcal{M}}_{2m-r-s}(Y, m) \\ {}^{\ell}E_1^{rs} &= H_{2m-s}(\bar{Y}_{\langle -r \rangle}, m) \Rightarrow H_{2m-r-s}(\overline{Y}, m) \end{aligned} \tag{5}$$

In the ℓ-adic spectral sequence, since the Y_J are smooth and proper we can rewrite the E_1 terms as

$${}^{\ell}E_1^{rs} = H^{2d+2r-2m+s}(\bar{Y}_{\langle -r \rangle}, d + r - m)$$

which is pure of weight s, and semisimple by (i). So the term $({}^{\ell}E_1^{rs})^{\Gamma}$ vanishes unless $s = 0$, and so we may conclude that, after passing to Γ-invariants, the spectral sequence degenerates to an identity

$$H_{2m-j}(\overline{Y}, \mathbb{Q}_\ell(m))^{\Gamma} = H_j \left[H_{2m}(\bar{Y}_{\langle \bullet \rangle}, \mathbb{Q}_\ell(m))^{\Gamma} \right].$$

Consider now the motivic spectral sequence. Its E_1-terms may be computed as K-theory:

$${}^{\mathcal{M}}E_1^{rs} = H_{\mathcal{M}}^{2d+2r-2m+s}(Y_{\langle -r \rangle}, d + r - m) = K_{-s}^{(d+r-m)} Y_{\langle -r \rangle}.$$

We can then apply the following trivial extension of [19, Theorem 4].

Theorem 2.9 (Soulé)**.** *Let Z be in $\mathcal{C}_k$, of dimension $\leq d$. Then*

(i) for every $a > 0$ and every $b \geq d - 1$, $K_a^{(b)} Z = 0$; and

(ii) for $m = 0$, 1 the cycle class map $CH_m(Z) \otimes \mathbb{Q}_\ell \to H^{2(d-m)}(\bar{Z}, \mathbb{Q}_\ell(d - m))$ is an isomorphism.

Proof. As observed above, the Chow motive of Z is a submotive of the motive of the product of d curves, to which Soulé's result applies. □

In the present case, since $\dim Y_{\langle -r \rangle} = d + r$, part (i) gives ${}^{\mathcal{M}}E_1^{rs} = 0$ for all $s \neq 0$, provided $m \leq 1$. Therefore the spectral sequence also reduces to an identity

$$H^{\mathcal{M}}_{2m-j}(Y, m) = H_j \left[H^{\mathcal{M}}_{2m}(Y_{\langle \bullet \rangle}, m) \right] = H_j \left[CH_m(Y_{\langle \bullet \rangle}) \otimes \mathbb{Q} \right].$$

By (ii) we also have for every $m \leq 1$ an isomorphism of homological complexes

$$CH_m(Y_{\langle \bullet \rangle}) \otimes \mathbb{Q}_\ell \to H_{2m}(\bar{Y}_{\langle \bullet \rangle}, \mathbb{Q}_\ell(m))^\Gamma$$

(for $m < 0$ both complexes are obviously zero). Therefore by comparing homology we get that τ is an isomorphism. $\square$

3 Homological setting

In this section, $S = \operatorname{Spec} \mathfrak{o}$ is to be any Henselian trait (the spectrum of a Henselian discrete valuation ring), with generic and closed points η, s, of residue characteristic different from ℓ, and $f\colon X \to S$ any quasi-projective and flat morphism of relative dimension d. Label the morphisms:

$$\begin{array}{ccccc} Y & \xhookrightarrow{g} & X & \xhookleftarrow{h} & U \\ {\scriptstyle f_s}\downarrow & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f_\eta} \\ s & \xhookrightarrow[i]{} & S & \xhookleftarrow[j]{} & \eta \end{array}$$

We will replace K-theory by K'-theory and étale cohomology by homology. We review some facts from [20]. Recall that when U is smooth, the γ-filtration $F_\gamma^\bullet$ on K_qU satisfies

$$(F_\gamma^n K_q U) \otimes \mathbb{Q} = \bigoplus_{m \geq n} K_q^{(m)} U.$$

In general one has an increasing filtration $F_\bullet$ on $K'U \otimes \mathbb{Q}$ (defined by embedding U in a smooth scheme Z and taking a shift of the γ-filtration on $K^Z U = K'U$). There are modified Adams operators ϕ^k on K'-theory and, if $K_q'^{(n)} U \subset K_q' U \otimes \mathbb{Q}$ denotes the $(\phi^k = k^m)$-eigenspace, then

$$F_{-n}(K_q' U \otimes \mathbb{Q}) = \bigoplus_{m \geq n} K_q'^{(m)} U.$$

When U is smooth the isomorphism $K_q U \xrightarrow{\sim} K_q' U$ carries $F_\gamma^n (K_q U \otimes \mathbb{Q})$ to $F_{d-n}(K_q' U \otimes \mathbb{Q})$ and therefore induces isomorphisms $K_q(n) U \xrightarrow{\sim} K_q'^{(n-d)} U$.

In [7] there are defined ℓ-adic Riemann-Roch transformations

$$\tau\colon K_q' U \to H_{q-2m}(U, \mathbb{Q}_\ell(-m))$$

whose target is ℓ-adic homology, defined as

$$H_{-j}(U, \mathbb{Q}_\ell(-m)) = H^j(U, Rf_\eta^! \mathbb{Q}_\ell(m)).$$

When U is smooth, the Riemann-Roch theorem shows that for the Adams eigenspaces there is a commutative diagram

$$\begin{array}{ccc}
K_q^{(n)} & \xrightarrow{ch} & H^{2n-q}(U, \mathbb{Q}_\ell(n)) \\
 & & \big\downarrow (P.D.) \simeq \\
\simeq \big\downarrow & & H^{2n-2d-q}(U, Rf_\eta^! \mathbb{Q}_\ell(n-d)) \\
 & & \big\| \\
K_q'^{(n-d)} U & \xrightarrow{\tau} & H_{q+2d-2n}(U, \mathbb{Q}_\ell(d-n))
\end{array}$$

where the isomorphism labelled $(P.D.)$ is the "Poincaré duality" isomorphism given by $Rf_\eta^! \mathbb{Q}_\ell = \mathbb{Q}_\ell(d)[2d]$.

All this applies equally to Y. In étale homology there is a boundary map

$$\partial_\ell \colon H_{-i}(U, \mathbb{Q}_\ell(-m)) \to H_{-i+1}(Y, \mathbb{Q}_\ell(-m+1))$$

defined as the composite

$$\begin{aligned}
H_{-i}(U, \mathbb{Q}_\ell(-m)) = H^i(U, Rf_\eta^! \mathbb{Q}_\ell(m)) \xrightarrow{\partial} & H_Y^{i+1}(X, Rf^! \mathbb{Q}_\ell(m)) \\
= & H^{i+1}(Y, Rg^! Rf_s^! \mathbb{Q}_\ell(m)) \\
= & H^{i+1}(Y, Rf_s^! Ri^! \mathbb{Q}_\ell(m)) \\
= & H^{i-1}(Y, Rf_s^! \mathbb{Q}_\ell(m-1)) \\
= & H_{-i+1}(Y, \mathbb{Q}_\ell(-m+1))
\end{aligned}$$

using the purity $Ri^! \mathbb{Q}_\ell = \mathbb{Q}_\ell(-1)[-2]$ on S. The boundary maps $\partial_{\mathcal{M}}$ and ∂_ℓ in K'-theory and étale homology are compatible: the square

$$\begin{array}{ccc}
K_q'^{(m)} U & \xrightarrow{\tau} & H_{q-2m}(U, \mathbb{Q}_\ell(-m)) \\
\big\downarrow \partial_{\mathcal{M}} & & \big\downarrow \partial_\ell \\
K_{q-1}'^{(m-1)} Y & \xrightarrow{\tau} & H_{q-2m+1}(Y, \mathbb{Q}_\ell(-m+1))
\end{array} \tag{6}$$

is commutative, cf. [9, end of §8.1]. (The strange numbering of the homological boundary map comes from the equality of the dimensions of U and Y; by considering U as having dimension $(d+1)$ — as for example is done in [11] — would lead to a more natural numbering).

We have a Hochschild-Serre spectral sequence in homology:

$$E_2^{ab} = H^a(G, H_{-b}(U, \mathbb{Q}_\ell(\bullet))) \Rightarrow H_{-a-b}(U, \mathbb{Q}_\ell(\bullet))$$

and therefore, if Fil^n is the abutment filtration, so that

$$\mathrm{Fil}^1 H_*(U, \mathbb{Q}_\ell(\bullet)) = \ker \left[H_*(U, \mathbb{Q}_\ell(\bullet)) \to H_*(\overline{U}, \mathbb{Q}_\ell(\bullet)) \right]$$

there is an edge homomorphism

$$e_1\colon\ \mathrm{Fil}^1 H_j(U, \mathbb{Q}_\ell(\bullet)) \to H^1(G, H_{j+1}(U, \mathbb{Q}_\ell(\bullet))).$$

Let $(K_q'^{(m)}U)^0 = \tau^{-1}(\mathrm{Fil}^1 H_{q-2m}(U, \mathbb{Q}_\ell(-m))) \subset K_q'^{(m)}U$. We can then state the homological generalisation of 2.2. Let

$$sp'\colon H_*(\overline{U}, \mathbb{Q}_\ell(\bullet))_I \to H_*(\overline{Y}, \mathbb{Q}_\ell(\bullet))$$

be the transpose, for Grothendieck-Verdier duality, of the specialisation map

$$sp\colon H_c^*(\overline{Y}, \mathbb{Q}_\ell(\bullet)) \to H_c^*(\overline{U}, \mathbb{Q}_\ell(\bullet))^I.$$

Proposition 3.1. *The following diagram is commutative up to sign:*

$$\begin{array}{ccc}
(K_q'^{(m)}U)^0 & \xrightarrow{\quad\partial_{\mathcal{M}}\quad} & K_{q-1}'^{(m-1)}Y \\
\downarrow{\scriptstyle\tau} & & \downarrow{\scriptstyle\tau} \\
\mathrm{Fil}^1 H_{q-2m}(U, \mathbb{Q}_\ell(-m)) & & H_{q-2m+1}(Y, \mathbb{Q}_\ell(1-m)) \\
\downarrow{\scriptstyle e_1} & & \downarrow \\
H^1(G, H_{q-2m+1}(\overline{U}, \mathbb{Q}_\ell(-m))) & & H_{q-2m+1}(\overline{Y}, \mathbb{Q}_\ell(1-m)) \\
\downarrow & & \uparrow{\scriptstyle sp'} \\
H^1(I, H_{q-2m+1}(\overline{U}, \mathbb{Q}_\ell(-m))) & \hookrightarrow & H_{q-2m+1}(\overline{U}, \mathbb{Q}_\ell(1-m))_I
\end{array}$$

The compatibility of boundary maps (6) means that we can get rid of the K'-theory and express 3.1 as a purely cohomological compatibility. We shall state and prove this in the next section.

4 ℓ-adic compatibility

Since the target space in the diagram is the homology $H_{q-2m+1}(\overline{Y}, \mathbb{Q}_\ell(1-m))$ of the geometric special fibre, we may replace S by its strict Henselisation. Then we can remove the twists, and Proposition 3.1 will follow from the commutativity of the following diagram, for any $r \in \mathbb{Z}$:

$$\begin{array}{ccc}
\mathrm{Fil}^1 H^{r+1}(U, Rf_\eta^! \mathbb{Q}_\ell) & \xrightarrow{e_1} & H^1(I, H^r(\overline{U}, Rf_{\bar\eta}^! \mathbb{Q}_\ell)) \\
\big\downarrow & & \big\downarrow{=} \\
H^{r+1}(U, Rf_\eta^! \mathbb{Q}_\ell) & & H^r(\overline{U}, Rf_{\bar\eta}^! \mathbb{Q}_\ell)_I(-1) \\
\big\downarrow{\partial} & & \big\downarrow{=} \\
H^{r+2}(Y, Rg^! Rf^! \mathbb{Q}_\ell) & & \left[H_c^{-r}(\overline{U}, \mathbb{Q}_\ell^\vee)^I(1)\right]^\vee \\
\big\downarrow{=} & & \big\downarrow{sp^\vee} \\
H^r(Y, Rf_s^! \mathbb{Q}_\ell(-1)) & \xrightarrow{=} & [H_c^{-r}(Y, \mathbb{Q}_\ell^\vee)(1)]^\vee
\end{array}$$

We may push this down onto S, where it becomes the case $K = Rf_* Rf^! \mathbb{Q}_\ell$, $L = Rf_! \mathbb{Q}_\ell$ of the following statement.

Proposition 4.1. *Let S be a strictly Henselian trait, with generic and closed points η, s, whose residue characteristic is different from ℓ. Let K, $L \in \mathcal{D}_c^+(S, \mathbb{Q}_\ell)$ together with a pairing $K \otimes L \to \mathbb{Q}_\ell(1)$, inducing a cohomological pairing*

$$\beta \colon H_s^2(S, K) \otimes H^0(s, L_s) \to H_s^2(S, \mathbb{Q}_\ell(1)) = \mathbb{Q}_\ell$$

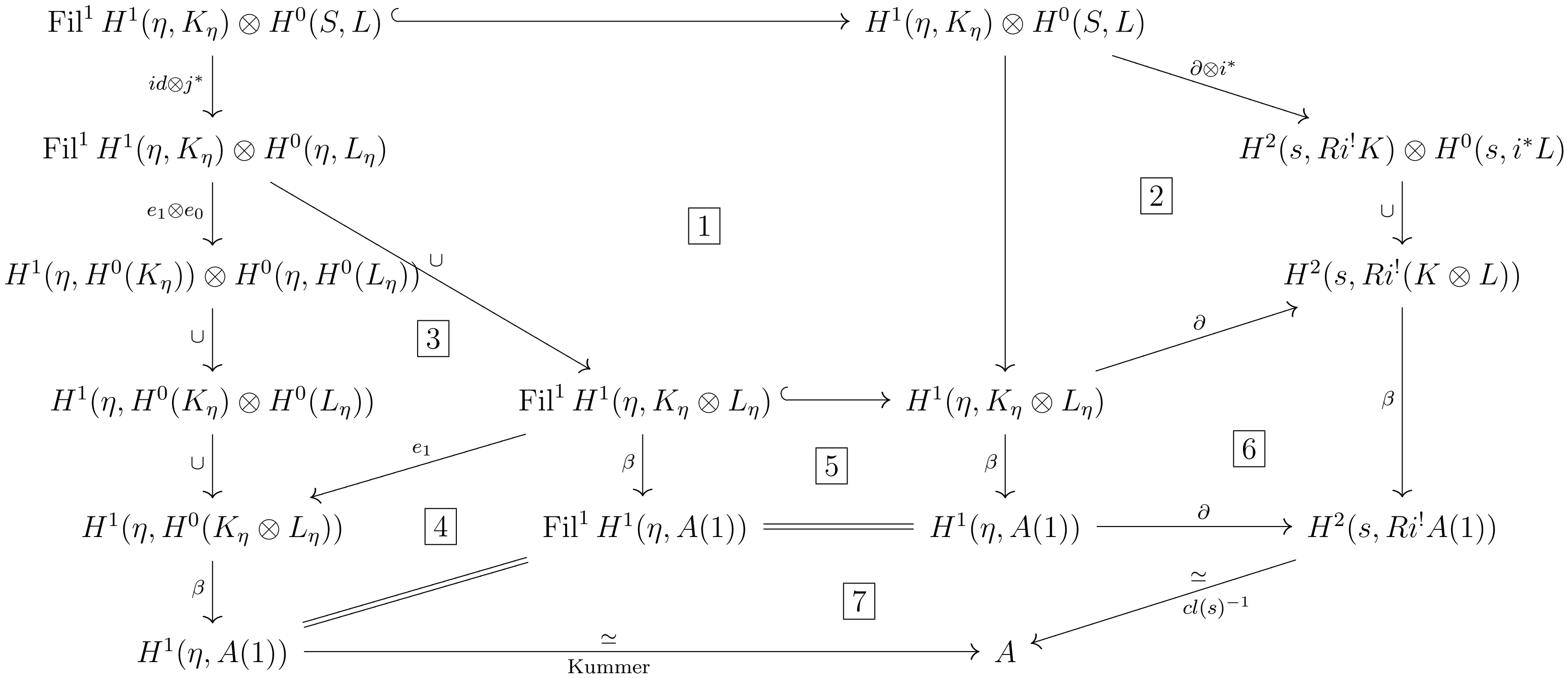
Then the following diagram is commutative up to sign:

$$\begin{array}{ccc}
\mathrm{Fil}^1 H^1(\eta, K_\eta) & \xrightarrow{e_1} & H^1(\eta, H^0(K_\eta)) \\
\big\downarrow & & \big\downarrow{=} \\
H^1(\eta, K_\eta) & & H^0(K_{\bar\eta})_I(-1) \\
\Big\downarrow{\partial} & & \big\downarrow{\beta} \\
 & & \left[H^0(L_{\bar\eta})^I\right]^\vee \\
 & & \big\downarrow{sp^\vee} \\
H_s^2(S, K) & \xrightarrow{\beta} & H^0(s, L_s)^\vee
\end{array}$$

Proof. We can check this by pairing the whole diagram with $H^0(S, L)$, and are therefore reduced to the commutativity of the diagram:

$$\begin{array}{ccc}
\mathrm{Fil}^1 H^1(\eta, K_\eta) \otimes H^0(S, L) & \xrightarrow{e_1 \otimes id} & H^1(\eta, H^0(K_\eta)) \otimes H^0(S, L) \\
\big\downarrow & & \big\downarrow \\
H^1(\eta, K_\eta) \otimes H^0(S, L) & & H^0(K_{\bar\eta})_I \otimes H^0(L_{\bar\eta})^I(-1) \\
\big\downarrow {\scriptstyle \partial \otimes id} & & {\scriptstyle \beta} \big\downarrow \\
H^2_s(S, K) \otimes H^0(S, L) & \xrightarrow{\beta} & H^2_s(S, \mathbb{Q}_\ell(1)) = \mathbb{Q}_\ell
\end{array}$$

To prove this we enlarge it to the enormous diagram below:


$\mathrm{Fil}^1 H^1(\eta, K_\eta) \otimes H^0(S, L)$
$H^1(\eta, K_\eta) \otimes H^0(S, L)$
$id \otimes j^*$
$\partial \otimes i^*$
$\mathrm{Fil}^1 H^1(\eta, K_\eta) \otimes H^0(\eta, L_\eta)$
$H^2(s, Ri^! K) \otimes H^0(s, i^* L)$
$e_1 \otimes e_0$
$\cup$
1
2
$H^1(\eta, H^0(K_\eta)) \otimes H^0(\eta, H^0(L_\eta))$
$H^2(s, Ri^!(K \otimes L))$
3
∂
$H^1(\eta, H^0(K_\eta) \otimes H^0(L_\eta))$
$\mathrm{Fil}^1 H^1(\eta, K_\eta \otimes L_\eta)$
$H^1(\eta, K_\eta \otimes L_\eta)$
β
e_1
5
6
$H^1(\eta, H^0(K_\eta \otimes L_\eta))$
4
$\mathrm{Fil}^1 H^1(\eta, A(1))$
$H^1(\eta, A(1))$
$H^2(s, Ri^! A(1))$
7
$\simeq$
$cl(s)^{-1}$
$H^1(\eta, A(1))$
Kummer
A

The commutativity of the various parts of this diagram are as follows:

Parts (1), (4) and (5) obviously commute, and (6) commutes by functoriality.

Part (2) commutes up to sign by [17, 0.1], and part (3) commutes by [17, 0.4]. The remaining compatibility is (7), which is anti-commutative by [22, "Cycle", 2.1.3]. □

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