Height pairings and special values of L-functions

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Introduction

The object of this paper is to give a "motivic" interpretation of the height pairings on algebraic cycles introduced by Beilinson [**2**], Bloch [**4**] and Gillet and Soulé [**14**]. The existence of such an interpretation was sketched in [**21**], §V.

Let X be a smooth projective variety over $\mathbf{Q}$, and suppose that a, $b \geq 0$ are integers satisfying $a+b=\dim X+1$. In the rest of this paper we shall use Beilinson's definition of the height pairing for cycles. Recall from [**2**] that (under suitable hypotheses) there are defined local pairings $<x,y>_p \in \mathbf{Q}$, $<x,y>_\infty \in \mathbf{R}$ for cycles x and y of codimensions a and b whose supports are disjoint. With these one constructs a global pairing $<x,y>_{\mathbf{Q}} \in \mathbf{R}$, which depends only on the rational equivalence classes of x and y. By the moving lemma this defines a pairing on Chow groups

$$<,>_{\mathbf{Q}}: CH^a(X)\otimes CH^b(X) \longrightarrow \mathbf{R}.$$

Our first result reinterprets the local height pairings. Fix disjoint closed subsets Y, Z of codimensions a, b, and let H, H' be the groups of cycles defined over $\overline{\mathbf{Q}}$ whose supports are contained in Y and Z, respectively. Then the local pairings can be described as follows. Consider the cohomology group $H^{2a-1}(\overline{X}-Y\operatorname{rel}Z,\mathbf{Q}_l(a))$. As a $\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$-module it has a canonical filtration (by weight) with graded pieces $H'\otimes\mathbf{Q}_l(0)$, $H^{2a-1}(\overline{X},\mathbf{Q}_l(a))$ and $H^\vee\otimes\mathbf{Q}_l(1)$. The essential observation is that (under suitable assumptions) the restriction of this representation to the inertia group at a finite prime $p\neq l$ is partially split; the constituent $H^{2a-1}(\overline{X},\mathbf{Q}_l(a))$ is a direct summand and one is left with an extension of $H'\otimes\mathbf{Q}_l(0)$ by $H^\vee\otimes\mathbf{Q}_l(1)$. By Kummer theory this is classified by a homomorphism $H\otimes H'\to\mathbf{Q}_l$, which turns out to be precisely the local pairing at p. For the infinite component one replaces the l-adic cohomology with cohomology with real coefficients, viewed as a mixed $\mathbf{R}$-Hodge structure.

In other words, there is an interpretation of the (local) height pairing in terms of the "mixed motive" $h^{2a-1}(X-Y\operatorname{rel}Z)(a)$. However in order to make this a truly motivic interpretation it is desirable to remove the dependence on the choice of supports Y, Z. The construction we adopt is suggested by the conjectures on periods and special values of L-functions, as reformulated in [**21**]; see also 7.8 below. These relate the behaviour at $s=0$ of the L-function of the motive $M=h^{2a-1}(X)(a)$ to the cycle class groups $CH^a(X)^0$, $CH^b(X)^0$. In sections 6–7 we attach to X a certain "mixed motive" $\widetilde{M}$ whose weight filtration has three non-trivial graded pieces, isomorphic to

$$\left(CH^b(X)^0\right)^\vee\otimes\mathbf{Q}(1),\quad M\quad\text{and } CH^a(X)^0\otimes\mathbf{Q}(0).$$

Associated to $\widetilde{M}$ is its period mapping $\widetilde{M}_B^+\otimes\mathbf{R}\to(\widetilde{M}_{dR}/F^0)\otimes\mathbf{R}$. We show that it can be described simply in terms of the period mapping for M and the (global) height pairing.

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This shows that the Birch-Swinnerton-Dyer-Beilinson-Bloch conjecture for the behaviour of the L-series of M at $s=0$ is essentially equivalent to the critical value conjectures for mixed motives ([**21**], conjectures A–C). At present we can only construct $\widetilde{M}$ under certain hypotheses; the most desirable (conjectural) situation is described in §6.2–6.7.

To perform these constructions in an unconditional way, the first requirement is a theory of mixed motives. A candidate for such a theory has been constructed independently by Deligne and Jannsen; we recall their construction in §1 below.

In order to compare the "motivic" and "geometric" pairings it is useful to work over an arbitrary number field, which we do up until section 6. However for the construction of a unique "universal extension" we need the ground field to be $\mathbf{Q}$.

One recurrent problem in the comparison of heights is that of signs. I have tried very hard to ensure consistency of signs (see section 0 for the necessary conventions), as the signature of the height pairing should be significant (see [**2**] for a precise conjecture).

Here are some related topics not covered in this account:

(i) A formulation of the motivic pairing for motives with arbitrary coefficients. However this should present no essential difficulty.

(ii) A description of the relation between the heights considered here and bi-extensions (see [**18**] and [**5**]). The canonical pairings of section 3 are none other than splittings of local biextensions, as was pointed out to me by Beilinson.

(iii) A precise description of Brylinski's height pairings [**6**] for local systems on curves in the motivic setting.

(iv) p-adic pairings. For this, see the forthcoming work of Nekovář[**19**], in which a related—but rather more sophisticated—p-adic theory is developed.

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0. Notations and signs

If k is a number field and v is a finite place v of k we write $\mathcal{G}_v \subset \mathrm{Gal}(\bar{k}/k)$ for a decomposition group at v, $\mathcal{I}_v \subset \mathcal{G}_v$ for the inertia subgroup, and $\mathrm{Frob}_v \in \mathcal{G}_v$ for a geometric Frobenius element. The completion of k at v is k_v and the residue field $k(v)$; we write q_v for its cardinality. If there is no risk of confusion, we write $\mathcal{O}$ for the ring of integers of k. If X is a k-scheme, we write $\overline{X}$ for $X \otimes_k \bar{k}$.

Cohomology groups of schemes are (unless otherwise indicated) étale cohomology; for schemes not defined over an algebraically closed we use continuous étale cohomology [**15**]. Likewise all Galois cohomology is continuous group cohomology [**22**].

For a (suitably good) scheme X, the group of codimension p cycles on X is denoted $\mathcal{Z}^p(X)$ and the Chow group $CH^p(X)$. If X is a scheme over a field k of characteristic zero, the subgroups of cycles and rational equivalence classes whose cohomology classes in $H^{2p}(\overline{X}, \mathbf{Q}_l(p))$ vanish are denoted $\mathcal{Z}^p(X)^0$ and $CH^p(X)^0$ respectively.

If A is an abelian group we often write $A_{\mathbf{Q}}$ in place of $A \otimes \mathbf{Q}$.

Signs. The definition of height pairings given in [**2**] involves exact sequences of cohomology and duality, and therefore gives rise to problems of signs. We follow the "usual" conventions for signs; to avoid any confusion, this means that in the derived category we take for distinguished triangles those coming from semi-split short exact sequences of complexes (in agreement with SGA4$\frac{1}{2}$ "C.D." and [**3**]). Recall also that if $A^\bullet$ and $B^\bullet$ are complexes and $A^\bullet \otimes B^\bullet$ is their tensor product (with usual differential $d_{A\otimes B} = d_A \otimes id_B + (-1)^{\deg_A} id_A \otimes d_B$) then the canonical isomorphisms

$$A[1] \otimes B \xleftarrow{\sim} (A \otimes B)[1] \xrightarrow{\sim} A \otimes (B[1]) \tag{0.0.1}$$

are given by

$$a \otimes b \leftarrow\!\shortmid a \otimes b \mapsto (-1)^{\deg(a)} a \otimes b.$$

Useful references for signs in connection with tensor products are SGA4, Éxp. XVII §1, and [**12**] (but note that the first reference takes the opposite convention for distinguished triangles in the derived category). We need the following compatibilities. Unless stated otherwise, ∂ denotes the connecting homomorphism in the long exact sequence for cohomology with supports.

0.1. Lemma. *Let X be a scheme, K_1, $K_2 \in D^b(X_{\text{ét}}, \mathbf{Z}/l^n)$, and Y_1, $Y_2 \subset X$ closed subsets. Then the diagram:*

$$\begin{array}{ccc} H^p_{Y_1}(X, K_1) \otimes H^q(X - Y_2, K_2) & \xrightarrow{\cup} & H^{p+q}_{Y_1 - Y_2}(X - Y_2, K_1 \overset{\mathbf{L}}{\otimes} K_2) \\ \big\downarrow \mathrm{id} \otimes \partial & & \big\downarrow \partial \\ H^p_{Y_1}(X, K_1) \otimes H^{q+1}_{Y_2}(X,, K_2) & \xrightarrow{\cup} & H^{p+q+1}_{Y_1 \cap Y_2}(X, K_1 \overset{\mathbf{L}}{\otimes} K_2) \end{array}$$

commutes with sign $(-1)^p$. (The second vertical arrow is the boundary in the long exact sequence

$$\ldots \to H^\bullet_{Y_1 \cap Y_2}(X) \to H^\bullet_{Y_1}(X) \to H^\bullet_{Y_1 - Y_2}(X) \to \cdots .)$$

Proof. See [**12**], Corollary 2.3. ■

0.2. Lemma. *Let X be a scheme and Y a closed subscheme. Let $F, G \in D^b(X_{\text{ét}}, \mathbf{Z}/n)$. Then the diagram*

$$\begin{array}{ccc}
H^p(Y, i^*F) \otimes H^q(X-Y, j^*G) & \xrightarrow{\mathrm{id}\otimes\partial} & H^p(Y, i^*F) \otimes H^{q+1}_Y(X, G) \\
\big\downarrow{\scriptstyle \partial\otimes\mathrm{id}} & & \big\downarrow{\scriptstyle \cup} \\
H^{p+1}(X, j_!j^*F) \otimes H^q(X-Y, j^*G) & \xrightarrow{\cup} & H^{p+q+1}(X, F \overset{\mathbf{L}}{\otimes} G)
\end{array}$$

commutes up to sign $(-1)^{p+1}$.

Proof. Compatibilities of this kind seem to be generally well-known, but I could not find a reference for this, even in the special case required here (proof of 7.5 below), and so for completeness will give a proof. The general result we need from homological algebra is:

Proposition. *Suppose that*

$$\begin{array}{ccccccccc}
0 & \longrightarrow & A & \xrightarrow{u} & B & \xrightarrow{v} & C & \longrightarrow & 0 \\
0 & \longleftarrow & A' & \xleftarrow{u'} & B' & \xleftarrow{v'} & C' & \longleftarrow & 0
\end{array}$$

are exact sequences of bounded-below complexes of R-modules, which have splittings $s : C^n \to B^n$, $s' : A'^n \to B'^n$ in every degree. Let $f : C \to A[1]$, $f' : A' \to C'[1]$ be the usual connecting homomorphisms: $f = ds - sd$, etc. Let $\beta : B \otimes B' \to D$ be a pairing into another complex whose restriction to $A \otimes C'$ is chain homotopic to zero. Then there are defined canonical pairings

$$\alpha : A \otimes A' \to D, \quad \gamma : C \otimes C' \to D$$

such that in the diagram

$$\begin{array}{ccccc}
C[-1] \otimes A' & \xrightarrow{f[-1]\otimes 1} & A \otimes A' & \xleftarrow{1\otimes u'} & A \otimes B' \\
 & & \big\downarrow{\scriptstyle \alpha} & & \big\downarrow{\scriptstyle u\otimes 1} \\
\big\downarrow{\scriptstyle \wr\,(0.0.1)} & & D & \xleftarrow{\beta} & B \otimes B' \\
 & & \big\uparrow{\scriptstyle \gamma} & & \big\uparrow{\scriptstyle 1\otimes v'} \\
C \otimes A'[-1] & \xrightarrow{1\otimes f'[-1]} & C \otimes C' & \xleftarrow{v\otimes 1} & B \otimes C'
\end{array}$$

the two right-hand squares commute up to homotopy, and the left-hand pentagon is anti-commutative up to homotopy.

Proof. We give the necessary formulae: if $\beta \circ (u \otimes v') = dH + Hd$ for a homotopy H, then α and γ are defined by

$$\alpha(x \otimes y) = \beta(ux \otimes s'y) - (-1)^p H(x \otimes w'y), \quad \gamma(x \otimes y) = \beta(sx \otimes v'y) - H(wx \otimes y)$$

and the sum of the two maps $C[-1] \otimes A' \to D$ coming from the left-hand pentagon equals $dK + Kd$ with

$$K = \beta \circ (s \otimes s') : C^p \otimes A'^q \to D^{p+q}. \qquad \blacksquare$$

In the present case we replace the triangle

$$j_!j_*F \to F \to i_*i^*F \to j_!j^*F[1]$$

by a short exact sequence of complexes of injective sheaves $0 \to \mathcal{A} \to \mathcal{B} \to \mathcal{C} \to 0$, and take $A = \Gamma(X, \mathcal{A})$, etc. For the second sequence we take

$$0 \longleftarrow \Gamma(U, j^*\mathcal{I}) \longleftarrow \Gamma(X, \mathcal{I}) \longleftarrow \Gamma_Y(X, \mathcal{I}) \longleftarrow 0$$

for an injective resolution $G \to \mathcal{I}$. Since the pairing

$$R\Gamma_Y(X, F) \overset{\mathbf{L}}{\otimes} R\Gamma(X, j_!j^*G) \to R\Gamma(X, F \overset{\mathbf{L}}{\otimes} G)$$

is zero in the derived category (it factors through $R\Gamma(X, i_*Ri^!F \overset{\mathbf{L}}{\otimes} j_!j^*G) = 0$) we can choose a complex D representing $R\Gamma(X, F \overset{\mathbf{L}}{\otimes} G)$ such that the hypotheses of the proposition are satisfied. ■

0.3. Consider a first quadrant spectral sequence $E_2^{ij} \Rightarrow E_\infty^*$. Denoting the filtration on the abutment by $\mathrm{Fil}^\bullet$, recall that the differentials of the spectral sequence give rise to edge-homomorphisms

$$\begin{aligned} e^0 &: E_\infty^n = \mathrm{Fil}^0 E_\infty^n \longrightarrow E_2^{0,n} \\ e^1 &: \ker(e^0) = \mathrm{Fil}^1 E_\infty^n \longrightarrow E_2^{1,n-1}. \end{aligned}$$

0.4. Lemma. *Let $\mathcal{T}$ be an abelian rigid F-linear tensor category, where F is a field of characteristic zero. Let $K^\bullet$, $L^\bullet \in D^b(\mathcal{T})$. Write $H^i(\mathcal{T}, -) = \mathrm{Ext}^i_{\mathcal{T}}(\mathbf{1}_{\mathcal{T}}, -)$, and consider the edge homomorphisms e^i attached to the spectral sequences*

$$E_2^{ij} = H^i(\mathcal{T}, \mathcal{H}^j(K^\bullet)) \Rightarrow H^{i+j}(\mathcal{T}, K^\bullet), \quad E_2^{ij} = H^i(\mathcal{T}, \mathcal{H}^j(K^\bullet \otimes L^\bullet)) \Rightarrow H^{i+j}(\mathcal{T}, K^\bullet \otimes L^\bullet)$$

Then the following diagram is commutative:

$$\begin{array}{ccc} \mathrm{Fil}^1 H^i(\mathcal{T}, K^\bullet) \otimes H^j(\mathcal{T}, L^\bullet) & \overset{\cup}{\longrightarrow} & \mathrm{Fil}^1 H^{i+j}(\mathcal{T}, K^\bullet \otimes L^\bullet) \\ \big\downarrow{\scriptstyle e^1 \otimes e^0} & & \big\downarrow{\scriptstyle e^1} \\ H^1(\mathcal{T}, \mathcal{H}^{i-1}(K^\bullet)) \otimes H^0(\mathcal{T}, \mathcal{H}^j(L^\bullet)) & \overset{\cup}{\longrightarrow} & H^1(\mathcal{T}, \mathcal{H}^{i+j-1}(K^\bullet \otimes L^\bullet)) \end{array}$$

(We have written $\otimes$ for $\underline{\otimes}$ since by hypothesis tensor product is exact.)

Proof. The edge homomorphisms can be described as follows: consider the distinguished triangle

$$\tau_{\leq n-1} K^\bullet \longrightarrow \tau_{\leq n} K^\bullet \overset{\gamma_n}{\longrightarrow} \mathcal{H}^n(K^\bullet)[-n].$$

Then e^0 and Fil^1 are given by the short exact sequence

$$\begin{array}{ccccccc} 0 & \longrightarrow & H^n(\mathcal{T}, \tau_{\leq n-1} K^\bullet) & \longrightarrow & H^n(\mathcal{T}, \tau_{\leq n} K^\bullet) & \overset{e^0 = H^n(\gamma_n)}{\longrightarrow} & H^n(\mathcal{T}, \mathcal{H}^n(K^\bullet)[-n]) \\ & & \| & & \| & & \| \\ & & \mathrm{Fil}^1 H^n(\mathcal{T}, K^\bullet) & & H^n(\mathcal{T}, K^\bullet) & & H^0(\mathcal{T}, \mathcal{H}^n(K^\bullet)) \end{array}$$

and e^1 is the map

$$e^1 = H^n(\gamma_{n-1})\colon \mathrm{Fil}^1 H^n(\mathcal{T}, K^\bullet) = H^n(\mathcal{T}, \tau_{\leq n-1} K^\bullet) \longrightarrow H^1(\mathcal{T}, \mathcal{H}^{n-1}(K^\bullet)).$$

We have a commutative diagram

$$\begin{array}{ccc} \tau_{\leq i-1} K^\bullet \otimes \tau_{\leq j} L^\bullet & \longrightarrow & \tau_{\leq i+j-1}(K^\bullet \otimes L^\bullet) \\ \downarrow & & \downarrow \\ \mathcal{H}^{i-1}(K^\bullet)[-i+1] \otimes \mathcal{H}^j(L^\bullet)[-j] & \xrightarrow{(*)} & \mathcal{H}^{i+j-1}(K^\bullet \otimes L^\bullet)[-i-j+1] \end{array}$$

where the map $(*)$ is given by the cup product $\mathcal{H}^{i-1}(K^\bullet) \otimes \mathcal{H}^j(L^\bullet) \to \mathcal{H}^{i+j-1}(K^\bullet \otimes L^\bullet)$ in degree $i+j-1$; this differs from the "correct" map (0.0.1) by the factor $(-1)^j$. Taking cohomology and using the functoriality of cup-product, the claim of the lemma follows. ■

1. A category of mixed motives

1.0. A manageable category of motives has been constructed by Deligne [**11**], in which the morphisms are defined by absolute Hodge cycles. This construction has been extended independently by Deligne ([**10**], §1) and Jannsen ([**16**], part I) to give a category of mixed motives. We recall here some of the properties of these categories. For simplicity we consider only motives with coefficients in **Q**.

1.1. Let k be a number field. The category $\mathcal{CV}_k$ is defined to be the category whose objects are symbols $h(X)$ for smooth and projective varieties X over k, and whose morphisms are homological correspondences defined by absolute Hodge cycles. The Tannakian category $\mathcal{M}_k$ of (unmixed) motives over k is constructed from $\mathcal{CV}_k$ by adjoining the kernels of projectors and the Tate motive $\mathbf{Q}(1)$, and modifying the commutativity constraint (see [**11**] Ch. 2.6 for details). Associated to an object M of $\mathcal{M}_k$ are its various realisations M_l, M_{dR} and M_σ (for each $\sigma\colon \bar{k} \to \mathbf{C}$) together with the comparison isomorphisms

$$I_{\sigma,l} : M_\sigma \otimes_{\mathbf{Q}} \mathbf{Q}_l \xrightarrow{\sim} M_l, \qquad I_{\sigma,\infty} : M_\sigma \otimes_{\mathbf{Q}} \mathbf{C} \xrightarrow{\sim} M_{dR} \otimes_\sigma \mathbf{C}.$$

A motive M over k is *pure of weight* w if the eigenvalues of an unramified Frobenius element $\mathrm{Frob}_v \in \mathrm{Gal}(\overline{k}/k)$ acting on M_l have absolute value $q_v^{w/2}$, and if for each σ the Hodge filtration induces a Hodge structure on M_σ which is pure of weight w. If X is smooth and projective over k, then $M = h^i(X)(m)$ is pure of weight $w = i - 2m$; and every unmixed motive is a direct sum of pure motives.

1.2. To define a category of mixed motives, Deligne and Jannsen first define a category of "mixed realisations" $\mathcal{MR}_k$. An object R of $\mathcal{MR}_k$ is given by the following data:

— For each $\sigma : \bar{k} \to \mathbf{C}$, a finite-dimensional **Q**-vector space R_σ, depending only on the restriction of σ to k;

— A finite-dimensional k-vector space R_{dR}, with a decreasing filtration $F^\bullet$ (Hodge) and an increasing filtration $W_\bullet$ (weight);

— For every prime l a finite-dimensional continuous representation R_l of $\mathrm{Gal}(\bar{k}/k)$;

— A family of isomorphisms ("comparison")

$$I_{\sigma,\infty} : R_\sigma \otimes \mathbf{C} \xrightarrow{\sim} R_{dR} \otimes_\sigma \mathbf{C}$$
$$I_{\sigma,l} : R_\sigma \otimes \mathbf{Q}_l \xrightarrow{\sim} R_l.$$

These are subject to certain conditions:

(i) For each σ the filtrations $F^\bullet$, $W_\bullet$ define (through $I_{\sigma,\infty}$) a mixed $\mathbf{Q}$-Hodge structure on R_σ;

(ii) If $\rho \in \mathrm{Gal}(\bar{k}/k)$ then $I_{\sigma\rho,l} = \rho \circ I_{\sigma,l}$;

(iii) The filtration $I_{\sigma,l_*} I^*_{\sigma,\infty} W_\bullet$ on R_l is $\mathrm{Gal}(\bar{k}/k)$-equivariant and independent of σ. (Note that the filtration $I^*_{\sigma,\infty} W_\bullet$ on $R_\sigma \otimes \mathbf{C}$ is defined over $\mathbf{Q}$, by (i)).

1.3. For any scheme X of finite type over k, the singular, de Rham and l-adic cohomology groups associate to X certain mixed realisations $h^i(X)$. (The same is true of the cohomology groups with support.) Jannsen then proves ([**16**], Theorem 4.4) that $\mathcal{M}_k$ is equivalent (by the obvious functor) to the smallest Tannakian subcategory of $\mathcal{MR}_k$ containing the realisations $h^i(X)$ for X smooth and projective over k. One may then define $\mathcal{MM}_k$, the category of mixed motives over k, as the smallest Tannakian subcategory of $\mathcal{MR}_k$ containing $h^i(X)$ for every X of finite type over k. The category $\mathcal{MM}_k$ contains $\mathcal{M}_k$ as a full, semisimple subcategory. Each mixed motive carries an increasing filtration (the weight filtration $W_\bullet$) whose graded pieces are pure motives.

It is also convenient to work with mixed realisations coming from cohomology with support or relative cohomology, as the comparison isomorphisms are defined in these cases also. See for example the discussion in [**8**] (for relative cohomology) as well as the treatment in [**16**], 6.11 where it is shown that all the long exact cohomology sequences are compatible with the comparison isomorphisms. In some cases this does not enlarge the category of realisations considered:—

1.4. Proposition. *Let U be quasiprojective over k, and $j: Z \hookrightarrow U$ a closed subset. Then there is for each i a mixed motive $h^i(U \,\mathrm{rel}\, Z)$ over k whose realisations are isomorphic to the relative cohomology groups of (U, Z).*

Proof. Let $Z \hookrightarrow W$ be a closed immersion of Z into some quasi-projective variety which factors through the inclusion $Z \hookrightarrow U$. (For example, take $W = U$.) Let $X \subset W \times \mathbf{A}^1$ denote the union of $U \times \{0\}$, $\widetilde{Z} = Z \times \mathbf{A}^1$ and $W \times \{1\}$. We have a commutative diagram (in, say, singular cohomology):

$$\begin{array}{ccccccc}
 & & H^{i-1}(\widetilde{Z} \cup W) & \xrightarrow{\theta} & H^i(X \,\mathrm{rel}\, \widetilde{Z} \cup W) & \longrightarrow & H^i(X) \\
 & & \downarrow \psi & & \downarrow \wr & & \downarrow \\
H^{i-1}(U) & \longrightarrow & H^{i-1}(Z) & \longrightarrow & H^i(U \,\mathrm{rel}\, Z) & \longrightarrow & H^i(U).
\end{array}$$

Now the inclusion of W in $\widetilde{Z} \cup W$ induces an isomorphism on cohomology, and the restriction $H^{i-1}(W) \to H^{i-1}(Z)$ factors through $H^{i-1}(U)$. Therefore the map ψ factors through $H^{i-1}(U)$ as indicated by the dotted arrow and so θ is zero. Hence $H^i(U \operatorname{rel} Z) \xrightarrow{\sim} \ker\left(H^i(X) \to H^i(\widetilde{Z} \cup W)\right)$, defining the motive $h^i(U \operatorname{rel} Z)$. ■

1.5. *Remarks.* (i) In particular the compatibility with the comparison isomorphisms shows that there is a long exact sequence of mixed motives

$$\longrightarrow h^{i-1}(Z) \longrightarrow h^j(U \operatorname{rel} Z) \longrightarrow h^j(U) \longrightarrow h^j(Z) \longrightarrow \dots$$

(ii) Jannsen's definition in §4 of [**16**] of $\mathcal{MM}_k$ uses only smooth quasiprojective varieties. This is probably inadequate for our purposes. He has suggested the possiblility of enlarging $\mathcal{MM}_k$ to include $h^i(X)$ for simplicial varieties X; then $\mathcal{MM}_k$ would automatically contain motives attached to mapping cones of arbitrary proper morphisms. See [**16**], Appendix C2–3 for further discussion.

1.6. We may define the L-function of a mixed motive E over k to be the Euler product over finite places v of k

$$L(E,s) = \prod_v L_{(v)}(E,s)$$

where

$$L_{(v)}(E,s) = \det\left(1 - q_v^{-s}\mathrm{Frob}_v | E_l^{\mathcal{I}_v}\right)^{-1}, \quad v \not| l.$$

Here it is tacitly assumed that the Euler factors are independent of l (which in this generality is not even known for the good factors). Since the graded pieces $\mathrm{Gr}_j^W E$ of E are pure motives, $L(E,s)$ will in general differ from the product $\prod L(\mathrm{Gr}_j^W E, s)$ by a finite number of Euler factors (as the passage to invariants under inertia is not an exact functor). There is one obvious case in which we have equality.

1.7. Definition. *E is a* mixed motive over $\mathcal{O}$ *if the weight filtration on E_l splits over $\mathcal{I}_v$, for every l, v with $v \not| l$.*

1.8. The mixed motives over $\mathcal{O}$ form a full Tannakian subcategory $\mathcal{MM}_{\mathcal{O}}$ of $\mathcal{MM}_k$, containing $\mathcal{M}_k$. We denote the Yoneda extension groups in $\mathcal{MM}_k$, $\mathcal{MM}_{\mathcal{O}}$ by $\mathrm{Ext}^i_k(\cdot,\cdot)$, $\mathrm{Ext}^i_{\mathcal{O}}(\cdot,\cdot)$. If E, E' are mixed motives over $\mathcal{O}$, then it is clear that $\mathrm{Ext}^0_{\mathcal{O}} = \mathrm{Ext}^0_k = \mathrm{Hom}$, and $\mathrm{Ext}^1_{\mathcal{O}}(E,E')$ is the subgroup of $\mathrm{Ext}^1_k(E,E')$ comprising the classes of extensions $E' \to E'' \to E$ whose l-adic realisation splits over $\mathcal{I}_v$, for all l and all $v \not| l$.

1.9. *Remark.* The definition 1.7 is not the only one possible. One could insist that the weight filtration splits as a representation of the whole decomposition group. However if the pure parts $\mathrm{Gr}_j^W E_l$ satisfy Deligne's conjecture on the purity of the monodromy filtration (see 3.6 below) one can show that these definitions are equivalent. We prefer to use the inertia group, although at times will need to know that the splitting is in fact invariant under $\mathcal{G}_v$. (I am grateful to Jan Nekovář for drawing this last point to my attention.)

2. Kummer theory

2.0. In this section we recall certain properties of extensions of $\mathbf{Q}(0)$ by $\mathbf{Q}(1)$ and local analogues.

2.1. We first review some facts about extensions of Hodge structures. A basic reference for most of this is [**7**]. For $A=\mathbf{Q}$ or $\mathbf{R}$, let $\mathcal{H}_A$ denote the category of mixed A-Hodge structures, and $\mathcal{H}_A^+$ the category of mixed A-Hodge structures over $\mathbf{R}$ (that is, with a Frobenius at infinity Φ_∞). For $H\in\mathcal{H}_\mathbf{R}^+$ one has the "de Rham" $\mathbf{R}$-structure $H_{dR}\subset H_\mathbf{C}$, namely the invariants of the semilinear extension of Φ_∞ to $H_\mathbf{C}=H_\mathbf{R}\otimes\mathbf{C}$. As a matter of notation, for any mixed motive M over $\mathbf{Q}$ we write M_A for the associated mixed A-Hodge structure (with Frobenius at infinity where appropriate).

2.2. Proposition. *(i) Let $H\in\mathcal{H}_\mathbf{R}$ be a pure Hodge structure. Then*

$$\mathrm{Ext}^1_{\mathcal{H}_\mathbf{R}}(\mathbf{R}(0),H)=H_\mathbf{R}\backslash H_\mathbf{C}/F^0(H_\mathbf{C}).$$

(ii) Let $H\in\mathcal{H}_\mathbf{R}^+$ be a pure Hodge structure over $\mathbf{R}$*. Then*

$$\mathrm{Ext}^1_{\mathcal{H}_\mathbf{R}^+}(\mathbf{R}(0),H)=H_\mathbf{R}^+\backslash H_{dR}/F^0(H_{dR}).\qquad\blacksquare$$

2.3. Corollary. *(i) If $w(H)=-1$ then* $\mathrm{Ext}^1_{\mathcal{H}_\mathbf{R}}(\mathbf{R}(0),H)=\mathrm{Ext}^1_{\mathcal{H}_\mathbf{R}^+}(\mathbf{R}(0),H)=0$.

(ii) $\mathrm{Ext}^1_{\mathcal{H}_\mathbf{R}}(\mathbf{R}(0),\mathbf{R}(1))=\mathrm{Ext}^1_{\mathcal{H}_\mathbf{R}^+}(\mathbf{R}(0),\mathbf{R}(1))=\mathbf{R}$. ■

2.4. In the second part of the corollary, we normalise the isomorphism so that $t\in\mathbf{R}$ corresponds to the following extension H_t of $\mathbf{R}(0)$ by $\mathbf{R}(1)$:

— $H_t=\mathbf{C}$, as real vector space, with complex conjugation for Φ_∞; there is an obvious exact sequence $\mathbf{R}(1)\to H_t\to\mathbf{R}$;

— $F^0(H_t\otimes\mathbf{C})$ is generated by

$$1\otimes 1-2\pi i\otimes\frac{t}{2\pi i}\in H_t\otimes\mathbf{C}=\mathbf{C}\otimes_\mathbf{R}\mathbf{C}.$$

For the proof, see [**7**] or [**16**].

2.5. Now let K be a non-archimedean local field, with $\mathrm{Gal}(\bar K/K)=\mathcal{G}$, and inertia group $\mathcal{I}$. Let V be a continuous finite-dimensional l-adic representation of $\mathcal{G}$, where l is a prime number different from the residue characteristic of K. Then we have a canonical isomorphism:

$$H^1(\mathcal{I},V)\xrightarrow{\sim}V(-1)_\mathcal{I}$$

compatible with the action of $\mathcal{G}/\mathcal{I}$. In particular, if 1 is not an eigenvalue of Frobenius on $V(-1)_\mathcal{I}$, any extension $V\to E\to\mathbf{Q}_l$ of $\mathcal{G}$-modules splits over $\mathcal{I}$. Dually, if 1 is not an eigenvalue of Frobenius on $V^\mathcal{I}$, then any extension $\mathbf{Q}_l(1)\to E\to V$ of $\mathcal{G}$-modules splits over $\mathcal{I}$.

2.6. When $V = \mathbf{Q}_l(1)$ we recover the isomorphism:

$$\mathrm{Ext}^1_{\mathcal{I}}(\mathbf{Q}_l, \mathbf{Q}_l(1)) \xrightarrow{\sim} H^1(\mathcal{I}, \mathbf{Q}_l(1)) \xrightarrow{\sim} \mathbf{Q}_l, \tag{2.6.1}$$

the second isomorphism being given by Kummer theory. We fix the normalisation of this isomorphism so that $1 \in \mathbf{Q}_l$ corresponds to the extension

$$0 \to \mathbf{Q}_l(1) \to T_l(\bar{K}^*/\pi^{\mathbf{Z}}) \otimes \mathbf{Q}_l \to \mathbf{Q}_l \to 0$$

with π a uniformiser in K.

2.7. The 1-motives defined in [**8**] can be regarded as mixed motives. In particular, for every $x \in k^*$ there is a 1-motive $K\langle x\rangle = [\mathbf{Z} \xrightarrow{x} \mathbf{G}_m]$ which is an extension of $\mathbf{Q}(0)$ by $\mathbf{Q}(1)$, trivial if and only if x is a root of unity. Recall from [**8**] how $K\langle x\rangle$ may be constructed geometrically:

Let C be the singular curve obtained from $\mathbf{G}_m/k$ by identifying the points 1 and x. Then $h^1(C)(1) = h^1(\mathbf{G}_m \,\mathrm{rel}\,\{1,x\})$ sits in an exact sequence:

$$0 \longrightarrow A \longrightarrow h^1(C)(1) \longrightarrow B \longrightarrow 0$$

where

$$A = \mathrm{coker}\left(h^0(\mathbf{P}^1)(1) \longrightarrow h^0(\{1,x\})(1)\right)$$
$$B = \ker\left(h^2_{\{0,\infty\}}(\mathbf{P}^1)(1) \longrightarrow h^2(\mathbf{P}^1)(1)\right)$$

We fix an isomorphism $A \xrightarrow{\sim} \mathbf{Q}(1)$ by evaluation at x, and an isomorphism $\mathbf{Q}(0) \xrightarrow{\sim} B$ by the difference $[\infty] - [0]$ of the cohomology classes of 0, ∞. In terms of these isomorphisms $h^1(C)(1)$ is isomorphic as an extension to $K\langle x\rangle$. We now recall from §10.3 of *loc. cit.* some of its realisations:—

(l-adic)—the class of the extension $K\langle x\rangle_l$ is

$$x \otimes 1 \in \hat{k}^* \otimes \mathbf{Q}_l = H^1(\bar{k}/k, \mathbf{Q}_l(1)) = \mathrm{Ext}^1_{\mathrm{Gal}(\bar{k}/k)}(\mathbf{Q}_l(0), \mathbf{Q}_l(1)).$$

(Hodge)—the $\mathbf{R}$-Hodge structure $K\langle x\rangle_{\sigma/\mathbf{R}}$ is isomorphic to H_t with $t = \log|\sigma(x)|$.

2.8. It is hoped (cf. for example [**10**], §2.4) that these 1-motives generate $\mathrm{Ext}^1_k(\mathbf{Q}(0), \mathbf{Q}(1))$, *i.e.* that $\mathrm{Ext}^1_k(\mathbf{Q}(0), \mathbf{Q}(1)) = k^* \otimes \mathbf{Q}$. One consequence would be that *any* extension E of $\mathbf{Q}(0)$ by $\mathbf{Q}(1)$ is uniquely determined by the following local data:

– for finite v and each l prime to q_v, the action of $\mathcal{I}_v$ on E_l, given by the invariant

$$t_{v,l} : \mathrm{Ext}^1_k(\mathbf{Q}(0), \mathbf{Q}(1)) \to \mathrm{Ext}^1_{\mathrm{Gal}(\bar{k}/k)}(\mathbf{Q}_l(0), \mathbf{Q}_l(1)) \xrightarrow{\sim} k^* \hat{\otimes} \mathbf{Q}_l \xrightarrow{v} \mathbf{Q}_l;$$

— the class of the $\mathbf{R}$-Hodge structures $E_{\sigma/\mathbf{R}}$, given by the invariant

$$t_\sigma : \mathrm{Ext}^1_k(\mathbf{Q}(0), \mathbf{Q}(1)) \to \mathrm{Ext}^1_{\mathcal{H}_\mathbf{R}}(\mathbf{R}(0), \mathbf{R}(1)) \to \mathbf{R}.$$

Moreover $t_v = t_{v,l}$ would be $\mathbf{Q}$-valued and independent of l, and there would be a "product" formula:

$$\sum_{v \text{ finite}} -\log q_v \cdot t_v + \sum_{\sigma: k \hookrightarrow \mathbf{C}} t_\sigma = 0. \qquad (2.8.1)$$

For motives over $\mathcal{O}$ we would have $\mathrm{Ext}^1_{\mathcal{O}}(\mathbf{Q}(0), \mathbf{Q}(1)) = \mathcal{O}^* \otimes \mathbf{Q}$, and in particular $\mathrm{Ext}^1_{\mathbf{Z}}(\mathbf{Q}(0), \mathbf{Q}(1)) = 0$.

2.9. *Remark.* It is in keeping with the formalism of Beilinson's conjectures to hope that something much more general is true: namely, that if M is any pure motive of weight > -1, then an extension E of M by $\mathbf{Q}(1)$ is determined by the extensions of $\mathbf{R}$-Hodge structures $E_{\sigma/\mathbf{R}}$ together with the extensions of $\mathcal{I}_v$-modules E_l (for every l and every finite $v \nmid l$). For motives E over $\mathcal{O}$ this would essentially amount to the injectivity of Beilinson's regulator.

3. Pairings attached to certain mixed motives

3.0. Let G be a finite-dimensional representation of $\mathrm{Gal}(\bar{k}/k)$ over $\mathbf{Q}$. There is associated to G in the usual way an Artin motive, which we denote $G(0)$. Thus the σ-realisation of $G(0)$ is simply G itself, for any σ. Write $G(n)$ for the Tate twist $G(0) \otimes \mathbf{Q}(n)$.

3.1. Now let E be a mixed motive over k with

$$\mathrm{Gr}^W_{-1}(E) = M, \quad \mathrm{Gr}^W_0(E) = G_1(0), \quad \mathrm{Gr}^W_{-2}(E) = G_2(1)$$

and $\mathrm{Gr}^W_i(E) = 0$ for $i < -2$ and $i > 0$, for Galois representations G_1, G_2 as above. We will construct in this section local pairings

$$b_v = b_{v,E}\colon\ G_1 \times G_2^\vee \longrightarrow \begin{cases} \mathbf{R} & \text{for } v \text{ an infinite place of } k \\ \mathbf{Q}_l & \text{for } v \text{ a finite place, } v \nmid l \end{cases}$$

under certain hypotheses.

3.2. The pairings will transform as follows under finite field extensions k'/k. Let v' be any place of k' over k, of ramification degree $e(v'/v)$. Write E' for the basechange of E to k'. Then

$$b_{v',E'} = e(v'/v) \cdot b_{v,E}.$$

3.3. The pairing at an archimedean place can be constructed unconditionally. By 2.3 there is a canonical splitting in $\mathcal{H}_{\mathbf{R}}$:

$$E_{\sigma/\mathbf{R}} = M_{\sigma/\mathbf{R}} \oplus V_{\sigma/\mathbf{R}}$$

where $V_{\sigma/\mathbf{R}}$ is an extension:

$$0 \to G_2(1)_{\mathbf{R}} \to V_{\sigma/\mathbf{R}} \to G_1(0)_{\mathbf{R}} \to 0.$$

This extension is classified by an element of

$$\mathrm{Ext}^1_{\mathcal{H}_{\mathbf{R}}}(G_1(0)_{\mathbf{R}}, G_2(1)_{\mathbf{R}}) = \mathrm{Hom}(G_1, G_2) \otimes \mathrm{Ext}^1_{\mathcal{H}_{\mathbf{R}}}(\mathbf{R}(0), \mathbf{R}(1)) = \mathrm{Hom}(G_1, G_2) \otimes \mathbf{R}$$

(where the isomorphism is normalised as in 2.4 above) and thus determines a pairing

$$G_1 \times G_2^\vee \to \mathbf{R}$$

which we define to be b_v if v is a real place corresponding to the embedding σ, and $\frac{1}{2}b_v$ if v is complex. It is obvious that this satisfies the compatibility of 3.2.

3.4. To define the pairings $b_{v,E}$ at finite places we need a hypothesis. Denote by M_1, M_2 the intermediate layers of the extension E:

$$M_1 = E/W_{-2}(E), \quad M_2 = W_{-1}(E).$$

3.5. Hypothesis. *The motives M_1, M_2 are motives over $\mathcal{O}$.*

By 2.5, this would follow automatically from the following hypothesis involving only M:

3.6 Hypothesis. *For every l, v with $v\nmid l$, no eigenvalue of* Frob_v *on $M_l^{\mathcal{I}_v}$ or $M_l(-1)_{\mathcal{I}_v}$ is a root of unity.*

(Recall that for $M = h^{2n-1}(X)(n)$, X smooth and proper over k, this would in turn be a consequence of Deligne's conjecture ([**20**], 3.8) on the purity of the monodromy filtration.)

3.7. First assume that $\mathrm{Gal}(\bar{k}/k)$ acts trivially on G_i. Then there is a splitting of the extension of $\mathcal{I}_v$-modules, unique up to isomorphism

$$E_l = M_l \oplus V_{v,l}$$

for every l with $v\nmid l$, where $V_{v,l}$ is an extension

$$0 \to G_2 \otimes \mathbf{Q}_l(1) \to V_{v,l} \to G_1 \otimes \mathbf{Q}_l \to 0,$$

unique up to isomorphism (as an extension). In fact, by hypothesis both the extensions of $\mathcal{I}_v$-modules $W_{-1}(E_l)$, $E_l/W_{-2}(E_l)$ are split. So there is a short exact sequence

$$\begin{aligned} 0 \longrightarrow \mathrm{Ext}^1_{\mathcal{I}_v}(G_1 \otimes \mathbf{Q}_l, G_2 \otimes \mathbf{Q}_l(1)) &\longrightarrow \mathrm{Ext}^1_{\mathcal{I}_v}(G_1 \otimes \mathbf{Q}_l, M_{2,l}) \\ &\longrightarrow \mathrm{Ext}^1_{\mathcal{I}_v}(G_1 \otimes \mathbf{Q}_l, M_l) \longrightarrow 0 \end{aligned}$$

in which the class $[E_l] \in \mathrm{Ext}^1_{\mathcal{I}_v}(G_1 \otimes \mathbf{Q}_l, M_{2,l})$ maps to zero in $\mathrm{Ext}^1_{\mathcal{I}_v}(G_1 \otimes \mathbf{Q}_l, M_l)$. The class of the extension $V_{v,l}$ is then the inverse image of $[E_l]$ in $\mathrm{Ext}^1_{\mathcal{I}_v}(G_1 \otimes \mathbf{Q}_l, G_2 \otimes \mathbf{Q}_l(1))$. The isomorphism (2.6.1) gives

$$\mathrm{Ext}^1_{\mathcal{I}_v}(G_1 \otimes \mathbf{Q}_l, G_2 \otimes \mathbf{Q}_l(1)) = \mathrm{Hom}(G_1, G_2) \otimes \mathbf{Q}_l$$

and the class $[V_{v,l}]$ therefore defines a pairing b_v.

3.8. *(The general case.)* It is simple to check (using 2.6 above) that the pairings just defined satisfy the basechange property 3.2. In order to define them in general, choose an extension k'/k such that $\mathrm{Gal}(\bar{k}/k')$ acts trivially on G_1 and G_2, and set

$$b_{v,E} = \frac{1}{e(v'/v)} b_{v',E'}.$$

3.9. Conjecture. *The pairings $b_{v,E}$ for finite v are $\mathbf{Q}$-valued and independent of l.*

3.10. We need some formal properties of these pairings. The proofs are straightforward consequences of the definitions. In each case the hypothesis 3.5 is assumed to hold.

3.11. Proposition. *Let $\phi_1\colon G_1\to H_1$, $\phi_2\colon H_2\to G_2$ be $\mathrm{Gal}(\bar{k}/k)$-homomorphisms, and let E' be a mixed motive over k with graded pieces $H_2(1)$, M and $H_1(0)$. Write E for the motive with graded pieces $G_2(1)$, M and $G_1(0)$ which is obtained from E' by pullback and pushout via ϕ_1 and ϕ_2. Then the pairings attached to E, E' satisfy*

$$b_{v,E}(x_1,x_2)=b_{v,E'}(\phi_1(x_1),x_2\circ\phi_2). \qquad \blacksquare$$

3.12. Proposition. *Let E be as above, and suppose there is a commutative diagram*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M' & \longrightarrow & M_1' & \longrightarrow & G_1(0) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow{\scriptstyle\omega} & & \| & & \\ 0 & \longrightarrow & M & \longrightarrow & E/W_{-2}(E) & \longrightarrow & G_1(0) & \longrightarrow & 0 \end{array}$$

in which M' is pure of weight -1. Let E' be the mixed motive with graded pieces $G_2(1)$, M', $G_1(0)$ obtained from E by pullback with ω. Then $b_{v,E'}=b_{v,E}$. $\blacksquare$

3.13. Proposition. *Let E, E' both be extensions of $G_1(0)$ by M_2, and let E'' be their Baer sum. Then $b_{v,E''}=b_{v,E}+b_{v,E'}$.* $\blacksquare$

3.14. Proposition. *Let $M=0$ and let E be the extension $K<x>$ of the previous section, with $G_1=G_2=\mathbf{Q}$ (with trivial Galois action), and $x\in k^*$. Then*

$$b_v(1,1)=\begin{cases}\log|x|_v & \text{for } v \text{ infinite}\\ \mathrm{ord}_v(x) & \text{for } v \text{ finite.}\end{cases} \qquad \blacksquare$$

3.15. Proposition. *Assume that G_i are trivial Galois modules and that $k=\mathbf{Q}$. Then the pairing $b_{\infty,E}$ is a perfect pairing if and only if E is a* critical *mixed motive, and if this is the case then*

$$c^+(E)=c^+(M)\cdot\det b_{\infty,E}. \qquad \blacksquare$$

Recall [**21**] that E is *critical* if the period mapping $I^+_\infty(E)$, defined by the commutative diagram

$$\begin{array}{ccccc} E_B\otimes\mathbf{C} & \supset & E_B^+\otimes\mathbf{R} & & \\ \downarrow{\scriptstyle \wr I_\infty(E)} & & \downarrow & \quad I^+_\infty(E) & \\ E_{dR}\otimes\mathbf{C} & \supset & E_{dR}\otimes\mathbf{R} & \longrightarrow & (E_{dR}/F^0)\otimes\mathbf{R} \end{array}$$

is an isomorphism; and that if this is the case the period $c^+(E)\in\mathbf{R}^*/\mathbf{Q}^*$ is the determinant of $I^+_\infty(E)$, calculated with respect to the $\mathbf{Q}$-structures E_B^+, E_{dR}/F^0.

3.16. *Remark.* We tacitly assume, in the construction of the height pairing for extensions of motives, that the local pairings $b_{v,E}$ vanish for all but a finite number of v. Working in the category of mixed motives proposed by Jannsen this is automatic; for the l-adic realisation E_l of any mixed motive is obtained by tensor operations from the l-adic cohomology of some varieties over k. By the theorems on contructibility and generic basechange in l-adic cohomology, E_l therefore extends to a smooth $\mathbf{Q}_l$-sheaf of an open subset $U\subset\mathrm{Spec}\,\mathcal{O}$, and for a finite prime $v\in U$ we then will have $b_{v,E}=0$.

4. The local geometric pairings

4.0. For this section let X be a smooth and projective scheme over a number field k, equidimensional of dimension N. We will relate the pairings of the preceding section to the local height pairings (or link indices) as described in [**2**] §2, whose definition we now recall.

4.1. Assume that X extends to a regular scheme $\mathcal{X}$ which is flat and proper over $\mathcal{O}$. Then there is an intersection pairing (see for example [**13**] §6)

$$<,>_{\mathcal{X}} : CH^a(\mathcal{X})^0 \otimes CH^b(\mathcal{X})^0 \to \mathbf{R}$$

where a and b satisfy $a+b=N+1$, $CH^n(\mathcal{X})$ is the Chow group of codimension n cycles on $\mathcal{X}$ modulo rational equivalence, and $CH^n(\mathcal{X})^0 = \ker\{CH^n(\mathcal{X}) \to H^{2n}(X \otimes \bar{k}, \mathbf{Q}_l(n))\}$.

4.2. Under some restrictions the pairing can be defined at the level of cycles on X rather than $\mathcal{X}$. For this, write $CH^n(X)^{00}_{\mathbf{Q}}$ for the image in $CH^n(X)_{\mathbf{Q}}$ of

$$\bigcap_{\substack{v,l \\ v\nmid l}} \ker\{\mathcal{Z}^n(\mathcal{X})_{\mathbf{Q}} \to H^{2n}(\mathcal{X} \otimes \overline{k(v)}, \mathbf{Q}_l(n))\}.$$

If ξ, η are elements of $CH^*(X)^{00}_{\mathbf{Q}}$, they can be lifted to cycles ξ', η' on $\mathcal{X}$ (with $\mathbf{Q}$-coefficients) whose classes in $H^{2*}(\mathcal{X} \otimes \overline{k(v)}, \mathbf{Q}_l(*))$ are zero for every v, l with $v\nmid l$, and one can then define

$$<\xi,\eta>_X = <\xi',\eta'>_{\mathcal{X}}$$

which depends only on ξ, η.

4.3. Conjecture 2.2.5 of [**2**] asserts that

$$CH^n(X)^{00}_{\mathbf{Q}} = CH^n(X)^0_{\mathbf{Q}} \overset{\text{def}}{=} \ker\{CH^n(X)_{\mathbf{Q}} \to H^{2n}(X \otimes \bar{k}, \mathbf{Q}_l(n))\}.$$

Note that since $H^{2n}(\mathcal{X} \otimes \overline{k(v)}, \mathbf{Q}_l(n)) = H^{2n}(\mathcal{X} \otimes \mathcal{O}_v^{\mathrm{nr}}, \mathbf{Q}_l(n))$ by the proper basechange theorem, the "absolute" cycle map

$$CH^n(X)_{\mathbf{Q}} \to H^{2n}(X \otimes k_v^{\mathrm{nr}}, \mathbf{Q}_l(n)) \tag{4.3.1}$$

is zero on $CH^n(X)^{00}_{\mathbf{Q}}$ for every l and every $v\nmid l$.

4.4. The pairing $<,>_{\mathcal{X}}$ is defined as a sum of local terms. In [**2**] Beilinson expresses the pairing $<,>_X$ as a sum of local terms, each defined cohomologically. The terms for the finite and infinite primes are completely analogous. To describe them in a unified way, introduce a rigid abelian tensor category $\mathcal{T}$, with coefficient ring $A = \mathrm{End}_{\mathcal{T}}(\mathbf{1})$, and objects $\underline{R\Gamma}_c(X)$, $\underline{R\Gamma}_Y(X)$ in the derived category $D^b(\mathcal{T})$ for schemes of finite type X/F and closed subsets $Y \subset X$. Write $\underline{R\Gamma}(X) = \underline{R\Gamma}_X(X)$. The cases we need to consider are:

(i) F is either a number field or a finite extension of $\mathbf{Q}_p^{\mathrm{nr}}$, $\mathcal{T}$ is the category of continuous finite-dimensional representations of $\mathrm{Gal}(\bar{F}/F)$ over $A = \mathbf{Q}_l$, and $\underline{R\Gamma}_\cdot(X) = R\Gamma_\cdot(\overline{X}_{\text{ét}}, \mathbf{Q}_l)$;

(ii) $F=\mathbf{R}$ or $\mathbf{C}$, $\mathcal{T}$ is the category of mixed $\mathbf{R}$-Hodge structures over F, and $\underline{R\Gamma}_{\cdot}(X)$ is the Hodge complex constructed in [**1**].

4.5. In both cases there is a "Tate object" $A(1)$ of $\mathcal{T}$ ($A=\mathbf{Q}_l$ or $\mathbf{R}$), and we write $\underline{R\Gamma}_{\cdot}(X,n)=\underline{R\Gamma}_{\cdot}(X)\otimes A(n)$. The corresponding cohomology objects in $\mathcal{T}$ will be denoted $\underline{H}^i_{\cdot}(X,n)$. We then obtain "absolute" cohomology complexes and groups:

$$R\Gamma_{\mathcal{T}\cdot}(X,n)=R\operatorname{Hom}(\mathbf{1}_{\mathcal{T}},\underline{R\Gamma}_{\cdot}(X,n))\in D(A);$$
$$H^i_{\mathcal{T},\cdot}(X,n)=H^i(R\Gamma_{\mathcal{T}\cdot}(X,n))$$

and the "Hochschild-Serre" spectral sequence:

$$E_2^{ij}=H^i(\mathcal{T},\underline{H}^j_{\cdot}(X,n))=\operatorname{Ext}^i(\mathbf{1}_{\mathcal{T}},\underline{H}^j_{\cdot}(X,n))\Rightarrow H^{i+j}_{\mathcal{T},\cdot}(X,n)).$$

In the case (i) $H_{\mathcal{T}}(-,n)$ is the continuous étale cohomology [**15**] with coefficients $\mathbf{Q}_l(n)$; in (ii) it is the absolute Hodge (or Deligne-Beilinson) cohomology [**1**] $H_{\mathcal{H}}(-,A(n))$.

4.6. The functors $\underline{R\Gamma}_{\cdot}$ enjoy the usual properties of cohomology with supports. For example, there are triangles:

$$\underline{R\Gamma}_Y(X)\to\underline{R\Gamma}(X)\to\underline{R\Gamma}(X-Y)\to\underline{R\Gamma}_Y(X)[1]$$
$$\underline{R\Gamma}_c(X-Y)\to\underline{R\Gamma}_c(X)\to\underline{R\Gamma}_c(Y)\to\underline{R\Gamma}_c(X-Y)[1],$$

duality pairings

$$\underline{R\Gamma}_Y(X)\otimes\underline{R\Gamma}(Y)\to\underline{R\Gamma}_Y(X),\quad \underline{R\Gamma}_c(X)\otimes\underline{R\Gamma}(X)\to\underline{R\Gamma}_c(X)$$

and a trace map

$$\operatorname{Tr}:\underline{R\Gamma}_c(X)\to A(-N)[-2N]$$

if X is smooth of dimension N.

4.7. For X smooth and $Y\subset X$ of codimension d one has the purity

$$\underline{H}^i_Y(X)=0\quad\text{for } i<2d$$

and the cycle class map

$$cl_Y:A(-d)\to\underline{H}^{2d}_Y(X)$$

which is an isomorphism if Y is absolutely irreducible. This induces an absolute cycle map

$$cl_{\mathcal{T},Y}:\mathcal{Z}^d_Y(X)\to H^{2d}_{\mathcal{T},Y}(X,d)$$

which becomes an isomorphism when tensored with A.

4.8. For l-adic cohomology these are all standard facts, simply because $\operatorname{Gal}(\bar F/F)$ acts by transport of structure. In the case of Hodge cohomology the fact that the various arrows are compatible with the Hodge structures is not always obvious, but follows from the results of [**1**] and [**8**].

4.9. We will refer to case (i) with F a finite extension of $\mathbf{Q}_p^{\mathrm{nr}}$ and case (ii) as the *local cases*. In the local cases there is a canonical isomorphism

$$H^1(\mathcal{T}, A(1)) = \mathrm{Ext}^1_{\mathcal{T}}(A(0), A(1)) \to A \tag{4.9.1}$$

given by 2.6 in case (i), and by 2.4 in case (ii).

4.10. In all the cases we are considering, the functor $\underline{H}^{\bullet}$ factors through $\mathcal{MM}_k$ when $k \subset F$, so we can speak of the $\mathcal{T}$-realisation of a mixed motive over k.

Ideally one would like to be able to take $\mathcal{T} = \mathcal{MM}_k$ itself, but at present this is unknown. In this case the groups $H_{\mathcal{T}}$ would hopefully be the motivic cohomology groups $H_{\mathcal{M}}$. This would be part of the formalism of a "derived category of motivic sheaves", as explained in [**2**] §5.10—see also [**9**] (both these references are discussed in [**17**]).

4.11. Although Beilinson's local pairings can be defined in a purely local setting, we will assume that we are in the setting of 4.1, and that one of the local cases for $\mathcal{T}$, with $F \supset k$, has been fixed. To simplify the notation, in the cohomology groups we will write X in place of $X \otimes F$, etc. Let ξ, η be cycles on X of codimensions a, b respectively, with disjoint supports Y, Z. Assume that their global absolute cohomology classes in $H^{2*}_{\mathcal{T}}(X, *)$ vanish; this is true if their rational equivalence classes lie in $CH^*(X)^{00}_{\mathbf{Q}}$, cf. (4.3.1) above. Write $V = X - Z$ and let $\widetilde{cl}_{\mathcal{T}}(\eta) \in H^{2b-1}_{\mathcal{T}}(V, b)$ be any cohomology class whose image in $H^{2b}_{\mathcal{T},Z}(X_k, b)$ is $cl_{\mathcal{T},Z}(\eta)$. The local pairing $<\xi,\eta>_{X,\mathcal{T}}$ is by definition the image of $-cl_{\mathcal{T},Y}(\xi) \otimes \widetilde{cl}_{\mathcal{T}}(\eta)$ under the composite map

$$\begin{array}{ccccc} H^{2a}_{\mathcal{T},Y}(V,a) \otimes H^{2b-1}_{\mathcal{T}}(V,b) & \xrightarrow{\cup} & H^{2N+1}_{\mathcal{T},Y}(V,N+1) & \xrightarrow{\mathrm{Tr}} & H^1_{\mathcal{T}}(\mathcal{T},1) \\ \downarrow \wr & & & & \downarrow \wr (4.9.1) \\ H^{2a}_{\mathcal{T},Y}(X,a) \otimes H^{2b-1}_{\mathcal{T}}(V,b) & & \dashrightarrow & & A \end{array}$$

4.12. The cases of interest here are the non-archimedean case 4.4(i) with $F = k_v^{nr}$, and the archimedean case 4.4(ii) with $F = k_v$ (v infinite). We then write $<,>_{X,v}$ instead of $<,>_{X,\mathcal{T}}$.

If ξ, η are cycles with disjoint support whose rational equivalence classes belong to $CH^*(X)^{00}_{\mathbf{Q}}$, then for v finite $<\xi,\eta>_{X,v}$ is in $\mathbf{Q}$ and (as the notation suggests) is independent of l. The global pairing is given by

$$<\xi,\eta>_X = \sum_{v|\infty} <\xi,\eta>_{X,v} - \sum_{v} \log q_v <\xi,\eta>_{X,v}.$$

4.13. Since we want to be precise about the signs we give some details of the proof of the above compatibility in the case of a finite place. Write R for the ring of integers of $F = k_v^{\mathrm{nr}}$, and use $\bullet$ to denote supports in the closed fibre $\overline{\mathcal{X}}_v$. Let $\pi : \mathcal{X} \to \mathrm{Spec}\,\mathcal{O}$ be the structural morphism, and Y, Z the supports of ξ, η in X. We then have the following diagram:

$$
\begin{array}{ccccc}
H^{2a-2N}_{Y\cup\bullet}(\mathcal{X}_R,\pi^!\mathbf{Q}_l(a-N))\otimes H^{2b-1}(V_k,\mathbf{Q}_l(b)) & \longrightarrow & H^{2a-2N}_{Y}(X_k,\pi^!\mathbf{Q}_l(a-N))\otimes H^{2b-1}(V_k,\mathbf{Q}_l(b)) & = & H^{2a}_{Y}(X_k,\mathbf{Q}_l(a))\otimes H^{2b-1}(V_k,\mathbf{Q}_l(b)) \\
\Big\downarrow{\scriptstyle \mathrm{id}\otimes\partial} & & \Big\downarrow{\scriptstyle \cup} & \textcircled{2} & \Big\downarrow{\scriptstyle \cup} \\
H^{2a-2N}_{Y\cup\bullet}(\mathcal{X}_R,\pi^!\mathbf{Q}_l(a-N))\otimes H^{2b}_{Z\cup\bullet}(\mathcal{X}_R,\mathbf{Q}_l(b)) & \textcircled{1} & H^{1}_{Y}(V_k,\pi^!\mathbf{Q}_l(1)) & = & H^{2N+1}_{Y}(V_k,\mathbf{Q}_l(N+1)) \\
\Big\downarrow{\scriptstyle \cup} & & \Big\downarrow & \textcircled{3} & \\
H^{2}_{\bullet}(\mathcal{X}_R,\pi^!\mathbf{Q}_l(1)) & \xleftarrow{\ \partial\ } & H^{1}(X_k,\pi^!\mathbf{Q}_l(1)) & \mathrm{Tr} & \\
\Big\downarrow{\scriptstyle \mathrm{Tr}} & \textcircled{4} & \Big\downarrow{\scriptstyle \mathrm{Tr}} & & \\
H^{2}_{\bullet}(\mathrm{Spec}\,R,\mathbf{Q}_l(1)) & \xleftarrow{\ \partial\ } & H^{1}(k,\mathbf{Q}_l(1)) & & \\
\Big\uparrow{\scriptstyle cl(\bullet)} & \textcircled{5} & \Big\downarrow{\scriptstyle (2.6.1)} & & \\
\mathbf{Q}_l & = & \mathbf{Q}_l & &
\end{array}
$$

Let $\hat{\xi} \in H^{2a-2N}_{Y\otimes F\cup\bullet}(\mathcal{X}_R, \pi^!\mathbf{Q}_l(a-N))$ be the cohomology class of the extension of ξ to $\mathcal{X}$, whose existence is assured by the definition of $CH(X)^{00}_{\mathbf{Q}}$. The image of $\hat{\xi} \otimes \widetilde{cl}_v(\eta)$ in the first column is the intersection pairing on $\mathcal{X}_R$; the image of $-\hat{\xi} \otimes \widetilde{cl}_v(\eta)$ round the extreme right-hand edge of the diagram is the cohomological definition from 4.11. The parts of the diagram labelled **2, 3** clearly commute; **4** commutes because the trace map $\pi_!\pi^! \to \mathrm{id}$ is compatible with the boundary in the long exact cohomology sequence; and **5** is anti-commutative (see SGA$4\frac{1}{2}$, "Cycle" 2.1.3). The desired compatibility follows from the commutativity of **1**, which is a consequence of 0.1 above.

4.14. Using our sign conventions, the sign in Beilinson's definition [**2**] 2.1.1(i) should be reversed; (iii) is correct as it stands; and the sign in (ii) depends on the normalisation of the signs in the Mayer-Vietoris sequence.

5. Comparison of the local pairings

5.0. Let X be as in 4.0 above, and let a, $b \geq 1$ be integers with $a+b=N+1$. To make the comparison between the motivic and geometric pairings, we assume that X admits a regular model over $\mathcal{O}$ as 4.1, and that the hypothesis 3.6 holds for the motive $M=h^{2a-1}(a)$.

5.1. Let Y, Z, be disjoint closed subschemes of X of codimensions a, b respectively. Let $U = X - Y$, $V = X - Z$. We introduce some further notation. Write $\mathcal{Z}^a_Y(X)$ for the group of cycles of codimension a with coefficients in $\mathbf{Q}$ which are supported on Y, and likewise for Z. Write $\mathcal{Z}^a_Y(X)^0$ for the subgroup of cycles homologically equivalent to zero. Analogously, define

$$H^{2a}_Y(\overline{X}, \mathbf{Q}_l(a))^0 = \ker\big(H^{2a}_{\overline{Y}}(\overline{X}, \mathbf{Q}_l(a)) \to H^{2a}(\overline{X}, \mathbf{Q}_l(a))\big).$$

Write $H = \mathcal{Z}^b_{\overline{Z}}(\overline{X})^0$ and $H' = \mathcal{Z}^a_{\overline{Y}}(\overline{X})^0$. These spaces are equipped with an action of $\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$, and may be viewed as Artin motives. The cycle class map gives isomorphisms

$$H \otimes \mathbf{Q}_l \simeq H^{2b}_Z(\overline{X}, \mathbf{Q}_l(b))^0, \quad H' \otimes \mathbf{Q}_l \simeq H^{2a}_Y(\overline{X}, \mathbf{Q}_l(a))^0$$

which we use without further comment.

5.2. Consider the motive $E = h^{2a-1}(U \operatorname{rel} Z)(a)$ (as in 1.4 above). There are exact sequences

$$h^{2a-2}(U)(a) \longrightarrow h^{2a-2}(Z)(a) \longrightarrow E \longrightarrow h^{2a-1}(U)(a) \longrightarrow 0$$

and

$$0 \longrightarrow M \longrightarrow h^{2a-1}(U)(a) \longrightarrow H'(0) \longrightarrow 0$$

(since $\dim Z = a-1$ and $\operatorname{codim} Y = a$). The trace map defines an isomorphism

$$\frac{h^{2a-2}(Z)(a)}{\mathrm{Im}(h^{2a-2}(U)(a))} \xrightarrow{\sim} H^\vee(1).$$

Therefore E is a motive with $\mathrm{Gr}^W_\bullet E = H^\vee(1) \oplus M \oplus H'(0)$. By 2.5 and the hypotheses of 5.0 the motives $W_{-1}E$ and $E/W_{-2}E$ are motives over $\mathcal{O}$, whence there are the local pairings of §3:

$$b_{v,E} : H \times H' \longrightarrow \begin{cases} \mathbf{Q}_l & \text{for finite } v, \\ \mathbf{R} & \text{for infinite } v. \end{cases}$$

Theorem 5.3. *If $\xi \in H'$, $\zeta \in H$ and their rational equivalence classes belong to $CH^*(X)^{00}_{\mathbf{Q}}$ then $b_{v,E}(\xi,\zeta) = <\xi,\zeta>_{X,v}$ for every place v of k.*

Proof. We first recall that the local geometric pairing enjoys a basechange property analogous to 3.2—this is almost automatic from the definition. Therefore we may assume (enlarging k if necessary) that the action of $\mathrm{Gal}(\bar{k}/k)$ on H and H' is trivial. Fix v and take the corresponding $\mathcal{T}$-cohomology as in 4.4 above. By abuse of notation write $H(n)$ for the objects $H \otimes A(n)$ of $\mathcal{T}$. Define a map θ by the commutativity of the diagram:

$$\begin{array}{ccccccc}
 & H' & \xrightarrow{cl_{\mathcal{T},Y}} & H^{2a}_{\mathcal{T},Y}(X,a) & \longrightarrow & H^0(\mathcal{T},\underline{H}^{2a}(X,a)) \\
 & \downarrow & & \downarrow & & \downarrow \wr \\
\theta & \mathrm{Fil}^1 H^{2a}_{\mathcal{T},c}(V,a) & \longrightarrow & H^{2a}_{\mathcal{T},c}(V,a) & \xrightarrow{e^0} & H^0(\mathcal{T},\underline{H}^{2a}_c(V,a)) \\
 & \downarrow e^1 & & & & \\
 & H^1(\mathcal{T},\underline{H}^{2a-1}_c(V,a)) & & & &
\end{array}$$

(Here $\mathrm{Fil}^{\bullet}$ and e^i are the filtration and edge homomorphisms coming from the "Hochschild-Serre" spectral sequence

$$E_2^{ij} = H^i(\mathcal{T},\underline{H}^j_c(V,a)) \Rightarrow H^*_{\mathcal{T},c}(V,a);$$

c.f. 0.3 above.)

Proposition 5.4. *θ is the classifying map for the extension:*

$$\begin{array}{ccccccc}
0 & \longrightarrow & \underline{H}^{2a-1}(X\,\mathrm{rel}\,Z,a) & \longrightarrow & \underline{H}^{2a-1}(U\,\mathrm{rel}\,Z,a) & \longrightarrow & H'(0) \longrightarrow 0 \\
 & & \| & & & & \\
 & & \underline{H}^{2a-1}_c(V,a) & & & &
\end{array}$$

This is a very mild generalisation of [**16**], Lemma 9.4 (l-adic) and Lemma 9.2 and Remark 9.7(c) (Hodge). ■

We now have split short exact sequences

$$\begin{array}{ccccccccc}
0 & \longrightarrow & H^{\vee}(1) & \xrightarrow{-\partial_{\varsigma}} & \underline{H}^{2a-1}_c(V,a) & \longrightarrow & \underline{H}^{2a-1}(X,a) & \longrightarrow & 0 \\
0 & \longrightarrow & \underline{H}^{2b-1}(X,b) & \longrightarrow & \underline{H}^{2b-1}(V,b) & \xrightarrow{\partial} & H(0) & \longrightarrow & 0
\end{array}$$

which are dual, by lemma 0.2. Choose splittings in $\mathcal{T}$

$$\sigma: H(0) \longrightarrow \underline{H}^{2b-1}(V,b)$$
$$\tau: \underline{H}^{2a-1}_c(V,a) \rightarrow H^{\vee}(1)$$

which are adjoint with respect to this duality; in other words, so that the diagram

$$\begin{array}{ccc} \underline{H}_c^{2a-1}(V,a-1)\otimes H & \stackrel{1\otimes\sigma}{\longrightarrow} & \underline{H}_c^{2a-1}(V,a-1)\otimes \underline{H}^{2b-1}(V,b) \\ \Big\downarrow{\scriptstyle \tau\otimes 1} & & \Big\downarrow{\scriptstyle \text{P.D.}} \\ H^\vee \otimes H(0) & \stackrel{\text{contract}}{\longrightarrow} & A(0) \end{array} \qquad (5.4.1)$$

is commutative. Without loss of generality we may also assume (by choice of $\widetilde{cl}_{\mathcal{T}}$ in 4.8) that the diagram

$$\begin{array}{ccc} H\otimes A & = & H^0(\mathcal{T}, H(0)) \\ \Big\downarrow{\scriptstyle \widetilde{cl}_{\mathcal{T}}} & & \Big\downarrow{\scriptstyle H^0(\sigma)} \\ H^{2b-1}_{\mathcal{T}}(V,b) & \stackrel{e^0}{\longrightarrow} & H^0(\mathcal{T}, \underline{H}^{2b-1}(V,b)) \end{array}$$

commutes. We now have a large diagram:

$$
\begin{array}{ccccccccc}
 & & H^{2a}_{\mathcal{T},Y}(X,a)^0 \otimes H^{2b-1}_{\mathcal{T}}(V,b) & & & & & & \\
 & & \big\downarrow {\scriptstyle \text{obvious}\otimes\text{id}} & \cup & & & & & \\
 & & \operatorname{Fil}^1 H^{2a}_{\mathcal{T},c}(V,a) \otimes H^{2b-1}_{\mathcal{T}}(V,b) & \xrightarrow{\cup} & H^{2N+1}_{\mathcal{T},c}(V,N+1) & & & & \\
 & cl_{\mathcal{T},Y}\otimes\widetilde{cl}_{\mathcal{T}} & \big\downarrow {\scriptstyle e^1\otimes e^0} & & \text{Ⓐ} & \mathrm{Tr}_V & & & \\
 & & H^1(\mathcal{T},\underline{H}^{2a-1}_c(V,a)) \otimes H^0(\mathcal{T},\underline{H}^{2b-1}(V,b)) & \xrightarrow{\cup} & H^1(\mathcal{T},\underline{H}^{2N+1}_c(V,N+1)) & \xrightarrow{\mathrm{Tr}_V} & H^1(\mathcal{T},A(1)) & & \\
 & \nearrow \theta\otimes\sigma & \big\uparrow {\scriptstyle \text{id}\otimes\sigma} & & \text{Ⓑ} & & \big\uparrow {\scriptstyle \text{contract}} & & \\
H'\otimes H & \xrightarrow{\theta\otimes\text{id}} & H^1(\mathcal{T},\underline{H}^{2a-1}_c(V,a)) \otimes H & \xrightarrow{\tau_*\otimes\text{id}} & H^1(\mathcal{T},H^\vee\otimes A(1))\otimes H & = & H^\vee\otimes H\otimes H^1(\mathcal{T},A(1)) & &
\end{array}
$$

of the Hochschild-Serre spectral sequence. ■

7.3. By passage to the limit over Y we obtain the desired homomorphism α. Now by Poincaré duality there is an isomorphism $h^{2b-1}(X)(b) \simeq M^\vee(1)$, which in accordance with the conventions for signs (cf. Lemma 0.2) is normalised by taking the cup-product in the order

$$h^{2a-1}(X)(a) \times h^{2b-1}(X)(b) \to h^{2N}(X)(N+1) \xrightarrow{Tr} \mathbf{Q}(1).$$

We thus obtain a homomorphism

$$CH^b(X)^0 \longrightarrow \mathrm{Ext}^1_{\mathbf{Q}}(\mathbf{Q}(0), h^{2b-1}(X)(b)) = \mathrm{Ext}^1_{\mathbf{Q}}(\mathbf{Q}(0), M^\vee(1)) = \mathrm{Ext}^1_{\mathbf{Q}}(M, \mathbf{Q}(1))$$

which we take to be $-\beta$.

7.4. In a moment we will need an alternative description of β. Consider the exact sequence

$$h^{2a-2}(X)(a) \longrightarrow h^{2a-2}(Z)(a) \longrightarrow h^{2a-1}(X \,\mathrm{rel}\, Z)(a) \longrightarrow h^{2a-1}(X)(a) \longrightarrow 0. \qquad (7.4.1)$$

A cycle $\zeta \in \mathcal{Z}^b_Z(X)$ gives a trace map $h^{2a-2}(Z) \to \mathbf{Q}(-a+1)$, which vanishes on the image of $h^{2a-2}(X)$ if ζ is homologically equivalent to zero. By pushout we obtain directly an extension of M by $\mathbf{Q}(1)$, whose class we denote $\beta'(\zeta)$.

Proposition 7.5. *The extension classes $\beta(\zeta)$ and $\beta'(\zeta)$ are equal.*

Proof. By the compatibility 0.2, the l-adic realisation of the exact sequence (7.4.1) is dual to the local cohomology sequence

$$0 \longrightarrow H^{2b-1}(X, b-1) \longrightarrow H^{2b-1}(U, b-1) \xrightarrow{-\partial} H^{2b}_Z(X, b-1) \longrightarrow H^{2b}(X, b-1).$$

By the comparison theorems the same is true in the other realisations (in a way compatible with the comparison isomorphisms) and the trace map is dual to the cycle class map $\mathcal{Z}^b_Z(X) \to H^{2b}_Z(X, b)$. The extension classes therefore agree. ■

7.6. One hopes that $\alpha \otimes \mathbf{Q}$, $\beta \otimes \mathbf{Q}$ are isomorphisms, and one might dream that $<\alpha(x), \beta(y)>_{\mathcal{M}} = <x, y>_X$. In the absence of 6.3 we will be content with the following motivic interpretation of $<-,->_X$.

Theorem 7.7. *Let $G \subset CH^b(X)^{00}_{\mathbf{Q}}$, $G' \subset CH^a(X)^{00}_{\mathbf{Q}}$ be any finite-dimensional subspaces. Assume the hypotheses of 5.0 hold. Then there is a motive $\widetilde{M}$ over $\mathbf{Z}$ with the following properties:*

(i) $\mathrm{Gr}^W_0 \widetilde{M} = G'(0)$, $\mathrm{Gr}^W_{-1} \widetilde{M} = M$ *and* $\mathrm{Gr}^W_{-2} \widetilde{M} = G^\vee(1)$; $\mathrm{Gr}^W_i \widetilde{M} = 0$ *for all other i.*

(ii) The classes of the intermediate extensions

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M & \longrightarrow & \widetilde{M}/W_{-2}\widetilde{M} & \longrightarrow & G'(0) & \longrightarrow & 0 \\ 0 & \longrightarrow & G^\vee(1) & \longrightarrow & W_{-1}\widetilde{M} & \longrightarrow & M & \longrightarrow & 0 \end{array}$$

are given by α, β respectively.

(iii) If $x \in G'$, $y \in G$ then $b_{\infty,\widetilde{M}}(x,y) = <x,y>_X$.

Proof. We first construct a motive E' over $\mathbf{Q}$ satisfying (i) and (ii). By the moving lemma, there are disjoint closed subschemes Y, Z of X, of codimensions a, b respectively, such that any element of G' (resp. G) is rationally equivalent to a cycle supported in Y (resp. Z). Using the same notations as in 5.1 above, we define $E'' = h^{2a-1}(U \operatorname{rel} Z)(a)$.

Choose splittings over G and G'

$$spl_Y : G' \longrightarrow \mathcal{Z}^a_Y(X)^0 \subset H', \quad spl_Z : G \longrightarrow \mathcal{Z}^b_Z(X)^0 \subset H$$

of the cycle class maps. Applying to E'' pullback by spl_Y and pushout by the transpose of spl_Z, we obtain a motive E' with $\mathrm{Gr}^W_\bullet E' = G^\vee(1) \oplus M \oplus G'(0)$. From the construction and 7.2 above it is clear that the extensions $E'/W_{-2}E'$ and $W_{-1}E'$ are classified by the homomorphisms α, β, and in particular do not depend on the choice of splittings. (Of course E' itself does in general depend on this choice.)

By 5.3 above and 3.11 it follows that, for any $y \in G'$ and $z \in G$,

$$b_{v,E'}(y,z) = < spl_Y(y), spl_Z(z) >_{\mathcal{G},v} \quad \text{for } v = \infty \text{ or } p.$$

This and the hypotheses imply that the pairings $b_{p,E'}$ are $\mathbf{Q}$-valued, independent of l. By 6.1 above we may therefore construct $\widetilde{M} = E$, a motive over $\mathbf{Z}$, satisfying all the requirements of the theorem. ∎

7.8. To explain the connection with special values of L-functions, we first briefly review the reformulation of the conjectures of Beilinson and others in terms of periods of mixed motives [**21**]. If M is a pure motive of weight $w \in \mathbf{Z}$, then under some general hypotheses it is possible to construct a certain associated mixed motive, which in the present context we have denoted $\widetilde{M}$, in four steps:

(i) First remove any submotive isomorphic to $\mathbf{Q}(0)$. This means replace M by the quotient motive M_1 in the exact sequence

$$0 \to \mathrm{Hom}(\mathbf{Q}(0), M) \otimes \mathbf{Q}(0) \to M \to M_1 \to 0.$$

(ii) Next remove any quotient motive isomorphic to $\mathbf{Q}(1)$. This replaces M_1 by M_2, which is the kernel:

$$0 \to M_2 \to M_1 \to \mathrm{Hom}(M_1, \mathbf{Q}(1))^\vee \otimes \mathbf{Q}(1) \to 0$$

(iii) Now take the universal extension of M_2 by $\mathbf{Q}(1)$ on the left and by $\mathbf{Q}(0)$ on the right. This comes from the two exact sequences

$$0 \to \mathrm{Ext}^1_{\mathbf{Z}}(M_2, \mathbf{Q}(1))^\vee \otimes \mathbf{Q}(1) \to M_3 \to M_2 \to 0$$
$$0 \to M_3 \to \widetilde{M} \to \mathrm{Ext}^1_{\mathbf{Z}}(\mathbf{Q}(0), M_3) \otimes \mathbf{Q}(0) \to 0$$

Using the hypothesis $\mathrm{Ext}^i_{\mathbf{Z}}(\mathbf{Q}(0), \mathbf{Q}(1)) = 0$ it is easily seen that in step (iii) the order in which the extensions are made is immaterial, and that $\widetilde{M}$ has a three-step filtration, with associated graded pieces $\mathrm{Ext}^1_{\mathbf{Z}}(\mathbf{Q}(0), M) \otimes \mathbf{Q}(0)$, M_2 and $\mathrm{Ext}^1_{\mathbf{Z}}(M, \mathbf{Q}(1))^\vee \otimes \mathbf{Q}(1)$.

7.9. In [**21**] it was explained that the conjectures of Beilinson and Bloch are equivalent to the conjecture:

The mixed motive $\widetilde{M}$ *is* critical *(see 3.15 above) and* $L(\widetilde{M},0)/c^+(\widetilde{M}) \in \mathbf{Q}^*$.

7.10. This is a special case of the conjectures A–C of [**21**]. In fact it was shown in section VI of that paper that this is in some sense the essential case of those conjectures. We now explain this in greater detail in the present case when $M = h^{2a-1}(a)$ has weight -1. This is the only case in which both of the groups $\mathrm{Ext}^1_{\mathbf{Z}}(M,\mathbf{Q}(1))$ and $\mathrm{Ext}^1_{\mathbf{Z}}(\mathbf{Q}(0),M)$ can be non-zero. We write ρ, ρ' for their dimensions (assumed to be finite in the entire discussion). To make the link between the motivic and the geometric setting we need to assume that the maps α, β are isomorphisms.

The L-function of $\widetilde{M}$ is

$$L(\widetilde{M},s) = L(M,s)\cdot\zeta(s+1)^{\rho}\cdot\zeta(s)^{\rho'}.$$

(Note that we have exact equality here as $\widetilde{M}$ is a motive over $\mathbf{Z}$; were this not the case we would have to remove one or more Euler factors of the form $(1-p^{-s})^{-1}$, which would change the order of $L(\widetilde{M},s)$ at $s=0$.) On the other hand, combining 7.7(iii) with 3.15 above, we see that $\widetilde{M}$ is critical if and only if the height pairing $<\cdot,\cdot>_{\mathcal{G}}$ is non-singular; and if this is the case, then $c^+(\widetilde{M}) = c^+(M)\cdot\det<\cdot,\cdot>_{\mathcal{G}}$. Thus the "motivic" conjecture 7.9 is in this case equivalent to the conjunction of the statements:

- The geometric height pairing $<\cdot,\cdot>_{\mathcal{G}}$ is non-singular;
- $\mathrm{ord}_{s=0} L(M,s) = \rho$, and the leading coefficient in the Taylor series of $L(M,s)$ about $s=0$ is a rational multiple of $c^+(M)\cdot\det<\cdot,\cdot>_{\mathcal{G}}$.

This is precisely the generalisation by Beilinson and Bloch of the Birch-Swinnerton-Dyer conjectures to arbitrary Chow groups.

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