On the Hecke algebra of a noncongruence subgroup

A. J. Scholl

1.

Let $\Gamma \subset SL_2(\mathbb{Z})$ be a subgroup of finite index, and let $\mathcal{H}$ denote the Hecke algebra of Γ. The aim of this note is to give some information about the action of $\mathcal{H}$ on spaces of modular forms for certain noncongruence subgroups Γ, which can be deduced from the geometric results of **[9]**.

We begin by recalling standard facts concerning Hecke algebras and modular forms, for details of which the reader is referred to Chapter 3 of Shimura's book **[11]**. By definition $\mathcal{H}$ (in Shimura's notation, $R(\Gamma, GL_2(\mathbb{Q})^+) \otimes \mathbb{Q}$) is the $\mathbb{Q}$-algebra spanned by double cosets $[\Gamma\gamma\Gamma]$, for $\gamma \in GL_2(\mathbb{Q})$ with $\det\gamma > 0$. Write as usual $M_k(\Gamma)$ for the complex vector space of holomorphic modular forms of weight $k \geq 0$, and $S_k(\Gamma)$ for the subspace of cusp forms. There is a natural action of $\mathcal{H}$ on $M_k(\Gamma)$, which preserves $S_k(\Gamma)$. (In fact there is more than one way to normalise this action; the choice is irrelevant for this paper.)

The Petersson inner product

$$<f, g> = \int_{\Gamma\backslash\mathfrak{H}} f(z)\overline{g(z)}y^{k-2}\,dx\,dy$$

is defined for $f \in M_k(\Gamma)$, $g \in S_k(\Gamma)$. The Hecke algebra is $*$-closed with respect to the Petersson inner product and so acts semisimply on $S_k(\Gamma)$, and leaves invariant the orthogonal complement $S_k(\Gamma)^\perp \subset M_k(\Gamma)$.

When Γ is a congruence subgroup the action of $\mathcal{H}$ on $S_k(\Gamma)$ and $M_k(\Gamma)$ is quite well understood, by the work of Hecke **[5]** and Atkin-Lehner **[1]**. Firstly, the structure of $\mathcal{H}$ itself is relatively simple; in particular, it contains a large commutative subalgebra $\mathcal{T}$, generated by the operators denoted here

$$T_p = \Gamma\begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}\Gamma, \quad S_p = \Gamma\begin{pmatrix} p & 0 \\ 0 & p \end{pmatrix}\Gamma$$

for primes p not dividing the level of Γ. (This is not the usual definition of T_p, but agrees with it if $\Gamma = \Gamma_0(N)$.)

Secondly, one knows that $S_k(\Gamma)$ is a direct sum of simple $\mathcal{H}$-modules V_j, which are pairwise non-isomorphic. The complement $S_k(\Gamma)^\perp$ is not always a semisimple $\mathcal{H}$-module, but it is spanned by Eisenstein series and its structure can be explicitly described. Moreover, no subquotient of $S_k(\Gamma)^\perp$ is isomorphic to any V_j.

Finally, if $\Gamma_1(N) \subset \Gamma \subset \Gamma_0(N)$ for some N, one has even more precise information, by Atkin and Lehner's theory of newforms; each V_j contains a unique distinguished element f_j, which is an eigenvector for $\mathcal{T}$, and V_j is determined by the character of $\mathcal{T}$ on f_j (in fact, even by the eigenvalues of the T_p for almost all p).

If Γ is not a congruence subgroup very little is known. Firstly, the structure of $\mathcal{H}$ seems complex and difficult to compute. For example, if Γ is a normal subgroup of $\Gamma(1) = SL_2(\mathbb{Z})$ containing -1 then $\mathcal{H}$ contains the group algebra of the finite group $\Gamma(1)/\Gamma$, which can be any finite group generated by an element of order 2 and one of order 3.

Furthermore, in contrast to the congruence case the operators T_p often act in an essentially trivial way. To be more precise, let $\Gamma' \supset \Gamma$ be the smallest congruence group containing Γ. Write Γ' as a union of double cosets $\cup \Gamma\gamma_i\Gamma$. Then the operator $\mathrm{tr}_{\Gamma,\Gamma'} = \sum[\Gamma\gamma_i\Gamma] \in \mathcal{H}$ clearly maps $M_k(\Gamma)$ into $M_k(\Gamma')$, and $\mathrm{pr}_{\Gamma,\Gamma'} = (\Gamma' : \Gamma)^{-1}\mathrm{tr}_{\Gamma,\Gamma'}$ is a projector onto that subspace. Moreover the restriction of $\mathrm{pr}_{\Gamma,\Gamma'}$ to $S_k(\Gamma)$ is the orthogonal projection, with respect to the Petersson inner product, onto $S_k(\Gamma')$.

Define the spaces of primitive forms on Γ by

$$M_k(\Gamma)^{\mathrm{prim}} = \ker\bigl(\mathrm{tr}_{\Gamma,\Gamma'} : M_k(\Gamma) \to M_k(\Gamma')\bigr); \quad S_k(\Gamma)^{\mathrm{prim}} = S_k(\Gamma) \cap M_k(\Gamma)^{\mathrm{prim}}.$$

Thus $S_k(\Gamma)^{\mathrm{prim}}$ is the orthogonal complement of the subspace of modular forms for congruence groups in $S_k(\Gamma)$.

Conjecture (Atkin). *In $\mathcal{H}$ the identity $T_p = T_p \circ \mathrm{tr}_{\Gamma,\Gamma'}$ holds; in particular, $T_p f = 0$ for every $f \in M_k(\Gamma)^{\mathrm{prim}}$.*

Serre [**10**] proved this conjecture assuming $\Gamma \subset \Gamma' = SL_2(\mathbb{Z})$ is a normal subgroup. In [**12**] Thompson explains how this result may be extended to certain non-normal subgroups. The present note concerns the action of the entire Hecke algebra $\mathcal{H}$ on $M_k(\Gamma)^{\mathrm{prim}}$. We will exhibit some cases when it is rather trivial. We consider the following subgroups of $SL_2(\mathbb{Z})$. Write Γ_{43} for the subgroup generated by the matrices

$$\begin{pmatrix} 1 & 4 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & -1 \\ 2 & -1 \end{pmatrix}$$

and Γ_{52} for the subgroup generated by

$$\begin{pmatrix} 1 & 5 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix}.$$

Γ_{52} and Γ_{43} both have index 7 and two cusps, of widths 4 and 3 (5 and 2, respectively).

Theorem 1. *Let $\Gamma = \Gamma_{43}$ or Γ_{52}, and let k be a positive even integer. Then there is a homomorphism $\chi\colon \mathcal{H} \to \mathbb{C}$ such that*

$$Tf = \chi(T)f \quad \text{for all } T \in \mathcal{H} \text{ and } f \in M_k(\Gamma)^{\mathrm{prim}}.$$

Remarks. (i) In particular, the Hecke algebra cannot distinguish between cusp forms and Eisenstein series on Γ—in contrast to the case of a congruence group. This is of course related to the failure of the Manin-Drinfeld theorem [**4**] for arbitrary Γ, in the case of weight 2 forms. In [**7**] it is explained how this gives rise to Eisenstein series whose Fourier coefficients are transcendental.

(ii) It is possible to write down the character χ of the Theorem in purely group-theoretic terms, by taking f to be a suitable Eisenstein series. But we will not go into this here.

(iii) Let Γ_{711} be the subgroup of index 9 generated by

$$\begin{pmatrix} 1 & 7 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 3 & -4 \\ 1 & -1 \end{pmatrix}, \quad \begin{pmatrix} -1 & -4 \\ 1 & 3 \end{pmatrix}$$

which has a cusp of width 7 and two cusps of width 1. This group was studied in **[2]** and extensively in **[8]**. The same method of proof as of Theorem 1 shows that there is a codimension 1 subspace of $M_k(\Gamma_{711})^{\mathrm{prim}}$ containing $S_k(\Gamma_{711})^{\mathrm{prim}}$, on which $\mathcal{H}$ acts through a character.

2.

For the moment we allow Γ to be arbitrary, and assume that $k \geq 2$. In this situation there are then defined the Eichler-Shimura cohomology groups associated to Γ (**[11]**, Chapter 8). There is a $\mathbb{Q}$-vector space $\mathcal{W}_k(\Gamma)$, together with an inclusion $\iota\colon S_k(\Gamma) \hookrightarrow \mathcal{W}_k(\Gamma) \otimes \mathbb{C}$ such that the map $(f,g) \mapsto \iota f + \overline{\iota g}$ is an isomorphism

$$S_k(\Gamma) \oplus \overline{S_k(\Gamma)} \xrightarrow{\sim} \mathcal{W}_k(\Gamma) \otimes \mathbb{C}. \tag{2.1}$$

To recover this from the results of **[11]**, we take X to be the $(k-2)$-fold symmetric power of the standard representation $\Gamma \hookrightarrow GL_2(\mathbb{Q})$. The space $\mathcal{W}_k(\Gamma)$ is, in the notations of **[11]** §8.1, the cohomology group $H^1_P(\Gamma, X)$. The isomorphism (2.1) is the complexification of Theorem 8.2 of **[11]**.

Moreover if one uses ordinary rather than parabolic cohomology, one obtains a larger space $\mathcal{W}^*_k(\Gamma)$ (which is $H^1(\Gamma, X)$ in the notations of **[11]**) together with an inclusion $M_k(\Gamma) \hookrightarrow \mathcal{W}^*_k(\Gamma) \otimes \mathbb{C}$ giving an isomorphism

$$M_k(\Gamma) \oplus \overline{S_k(\Gamma)} \xrightarrow{\sim} \mathcal{W}^*_k(\Gamma) \otimes \mathbb{C}. \tag{2.2}$$

(This isomorphism is well known but does not figure in **[11]**. The essential ingredients of a proof in exactly the context needed here can be found in §2 of **[6]**, especially 2.13(iii).) The Hecke algebra acts on $\mathcal{W}^*_k(\Gamma)$, leaving invariant $\mathcal{W}_k(\Gamma)$. With respect to this action the isomorphisms (2.1) and (2.2) are $\mathcal{H}$-equivariant. Write

$$\mathcal{W}^*_k(\Gamma)^{\mathrm{prim}} = \ker\bigl(\mathrm{tr}_{\Gamma,\Gamma'}\colon \mathcal{W}^*_k(\Gamma) \to \mathcal{W}^*_k(\Gamma')\bigr).$$

To prove Theorem 1 it is enough to prove that $\mathcal{H}$ acts on $\mathcal{W}^*_k(\Gamma)^{\mathrm{prim}}$ through a character.

Let ℓ be a prime number. Then associated to Γ there is a certain algebraic number field K, and a continuous homomorphism

$$\rho_\ell\colon \mathrm{Gal}(\overline{\mathbb{Q}}/K) \to \mathrm{Aut}(\mathcal{W}^*_k(\Gamma) \otimes \mathbb{Q}_\ell)$$

whose image commutes with that of $\mathcal{H}$. The representation ρ_ℓ leaves the parabolic subspace $\mathcal{W}_k(\Gamma) \otimes \mathbb{Q}_\ell$ invariant. These representations were introduced by Deligne **[3]** in the case of congruence subgroups, in which case one may take $K = \mathbb{Q}$. For general Γ the representations have been studied in **[6]**, **[8]** and **[9]**. We refer to §5.3 and 5.10(ii) of **[6]** for the precise definitions. In the cases considered here, one has in fact $K = \mathbb{Q}$.

Using methods from algebraic geometry, and in particular the theory of vanishing cycles, we obtained in **[9]** a criterion for the image of ρ_ℓ to contain a "long" unipotent element. In particular, in §4 of **[9]** the following result is proved:

Theorem 2. *Let* Γ *be one of* Γ_{52}, Γ_{43}, Γ_{711}. *Let* $p = 7$, 7 *or* 2 *respectively, and let* $\ell \neq p$. *Let* $k > 2$ *be even. Then the image under* ρ_ℓ *of an inertia subgroup at* p *contains a unipotent element* $U \in \mathrm{Aut}(\mathcal{W}_k^*(\Gamma)^{\mathrm{prim}} \otimes \mathbb{Q}_\ell)$ *such that* $(U-1)^{k-2} \neq 0$.

More generally, the conclusion of Theorem 2 holds for any Γ which satisfies the following condition about the reduction of the modular curve attached to Γ; there is a smooth point in the reduction mod p of the modular curve which is the reduction modulo p of exactly two cusps.

From standard formulae for dimensions of spaces of modular forms, one computes without difficulty that in the cases Γ_{43}, Γ_{52} one has $\dim M_k(\Gamma)^{\mathrm{prim}} = \dim S_k(\Gamma)^{\mathrm{prim}} + 1 = k/2$ for even $k \geq 2$. Therefore $\dim \mathcal{W}_k^*(\Gamma)^{\mathrm{prim}} = k-1$, and so the Jordan normal form for U has exactly one block. Also since $S_k(\Gamma)^{\mathrm{prim}}$ has codimension 1 in $M_k(\Gamma)^{\mathrm{prim}}$, the Hecke algebra acts semisimply on $M_k(\Gamma)^{\mathrm{prim}}$, hence also on $\mathcal{W}_k^*(\Gamma)^{\mathrm{prim}}$. Its image in $\mathrm{End}\, \mathcal{W}_k^*(\Gamma)^{\mathrm{prim}}$ commutes with U, and is therefore contained in the scalars. This proves Theorem 1.

For the case of $\Gamma = \Gamma_{711}$ (remark (iii) following Theorem 1) one has $\dim M_k(\Gamma)^{\mathrm{prim}} = \dim S_k(\Gamma)^{\mathrm{prim}} + 2 = k/2 + 1$, but one knows that the Galois representation $\mathcal{W}_k^*(\Gamma)/\mathcal{W}_k(\Gamma) \otimes \mathbb{Q}_\ell$ is the Tate twist of the permutation representation of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ on the cusps of the modular curve. In particular, the inertia group at p acts through a finite quotient, hence the Jordan normal form for U acting of $\mathcal{W}_k(\Gamma)^{\mathrm{prim}} \otimes \mathbb{Q}_\ell$ still has exactly one block. The claimed result follows easily by the same method.

Remark. We chose the three subgroups Γ_{43}, Γ_{52}, Γ_{711} for ease of computation, rather than for any special properties they have. In particular, we expect many noncongruence groups to satisfy the requirements of Theorem 2.

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Department of Mathematical Sciences
Science Laboratories
University of Durham
Durham DH1 3LE
England

e-mail: a.j.scholl@durham.ac.uk