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A trace formula for F-crystals

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Introduction

Let X be a smooth and proper curve over the finite field $k=\mathbf{F}_q$. If $\mathbb{E}$ is an F-crystal [K] on X, then the q-power Frobenius acts on the crystalline cohomology $H^*(X/W, \mathbb{E})$, where W is the ring of Witt vectors of k. In this note we show how the method of Monsky ([M1], [M2]) may be used to prove a Lefchetz-type formula for the alternating sum of the traces of Frobenius, provided that q is odd.

For the application we have in mind [S] we need to consider a slightly wider class of systems of coefficients; the underlying differential equation of $\mathbb{E}$ is permitted to have regular singular points. The formalism of such "F-crystals with logarithmic singularities" (which were first considered by Dwork [Dw], from a somewhat different point of view) is described in the first part of the paper; the treatment largely parallels the exposition of Katz [K], with which we assume some familiarity. To avoid later difficulties (cf. § 4), we have assumed that the divided powers which arise are topologically *nilpotent* - which accounts for the restriction on the characteristic of k. The idea of using a chain homotopy to define the action of Frobenius on the cohomology, as in § 2, was suggested by Deligne.

The principle of the proof of the trace formula is, roughly speaking, as follows: one considers the rigid analytic space associated to a lifting of X, and removes p-adic discs of radius $1-\varepsilon$ around each point of $X(k)$. As ε tends to zero, one is left with a "dagger space", which has no points over W, and therefore by a general result of Monsky ([M2], § 3) the alternating sum of the traces of Frobenius on its cohomology vanishes. In order to express the trace over the whole space as a sum of local terms, it therefore remains to calculate the contribution from the excised discs. This can be done since $\mathbb{E}$, being an F-crystal (and not merely a crystal), can be locally trivialised over an open disc of radius one.

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It is clear that some of the theory presented here could be developed without the hypothesis that X be one-dimensional. We have not done this for two reasons. Firstly, the construction of the chain homotopies of § 2 becomes more involved in the multi-dimensional case; secondly, the present approach would require unattractive liftability hypotheses. A more intrinsic approach would obviate both these difficulties.

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§ 1. *F*-crystals

1.1. We fix the following notation:

R	a complete discrete valuation ring
K	the field of quotients of R, of characteristic 0
π	a uniformising parameter in R
k	the residue field $R/(\pi)$, of characteristic $p \neq 0$
e	the absolute ramification degree of R
ord_p	the normalised ordinal function with $\mathrm{ord}_p(p)=1$
σ	a lifting of the p-power Frobenius endomorphism of k to R.

We always assume $e<p-1$ (so that in particular, $p \neq 2$).
We often abbreviate the divided powers $x^n/n!$ by $x^{[n]}$.

1.2 (X, Y) denotes one of the following:

i) X a smooth separated R-scheme of relative dimension one, and Y a closed subscheme of X which is finite and étale over R;

ii) $X=\operatorname{Spec} R'[[t]]$, and Y either empty or the closed subscheme $t=0$, where R' is a finite étale R-algebra.

In either case, t denotes a local parameter on X which, if Y is nonempty, is also a local equation for Y.

$\Omega^1_X(\log Y)=\Omega^1_{X/R}(\log Y)$ denotes the module of relative differentials with at most simple poles along Y.

${}^\infty$ (resp. †) denotes p-adic completion (resp. weak p-adic completion, in the sense of [M – W], [Me]).

1.3 We say that a lifting ϕ: $X^\infty \to X^\infty$ (or $X^\dagger \to X^\dagger$) of the absolute Frobenius endomorphism of $X \otimes k$ is *admissible* if

a) ϕ^* is σ-linear; and

b) $\phi^*(\mathscr{I}_Y)=\mathscr{I}_Y^p$, where $\mathscr{I}_Y$ is the ideal sheaf of Y.

In local coordinates, b) is equivalent to:

b′) $\phi^*(t)=t^p \cdot u$, where $u \in 1+\pi\mathcal{O}_X^\infty$ (or $1+\pi\mathcal{O}_X^\dagger$).

1.4 Let $\mathscr{E}$ be an $\mathcal{O}_X^\infty$-module, and

$$\nabla\colon \mathscr{E} \to \mathscr{E} \otimes \Omega^1_X(\log Y)^\infty$$

a connection with logarithmic singularities. If ϕ, ϕ' are two σ-linear liftings of Frobenius to X^∞, and Y is empty, then there is an isomorphism

$$\chi(\phi', \phi):\ \phi'^* \mathscr{E} \xrightarrow{\sim} \phi^* \mathscr{E}$$

given locally by the well-known formula

$$\chi(\phi', \phi)(\phi'^* e) = \sum_{n \geqq 0} \phi^* \left(\nabla \left(\frac{d}{dt}\right)^n (e)\right) (\phi'^*(t) - \phi^*(t))^{[n]}. \tag{1.4.1}$$

If Y is nonempty, $\chi(\phi', \phi)$ is still defined provided that ϕ, ϕ' are admissable, as we may rewrite (1.4.1) as

$$\sum_{n \geqq 0} \phi^* \left(\nabla \left(t \frac{d}{dt}\right)^n (e)\right) \cdot \left(\log \frac{\phi'^*(t)}{\phi^*(t)}\right)^{[n]}. \tag{1.4.2}$$

Let us indicate a proof of the equality between (1.4.1) and (1.4.2). First note that, from b′) above,

$$\frac{\phi'^*(t)}{\phi^*(t)} - 1 \in \pi \mathcal{O}_X^\infty$$

whence we may define

$$\eta = \log \frac{\phi'^*(t)}{\phi^*(t)} \in \pi \mathcal{O}_X^\infty. \tag{1.4.3}$$

The infinite sum (1.4.2) is therefore convergent. Substituting (1.4.3) into (1.4.1), we are reduced to verifying the formal identity of differential operators

$$\sum_{n \geqq 0} \frac{1}{n!} X^n \frac{\partial^n}{\partial X^n} (e^Y - 1)^n = \sum_{n \geqq 0} \frac{1}{n!} \left(X \frac{\partial}{\partial X}\right)^n Y^n$$

which is elementary (calculate the action of each side in turn on the monomials X^k, for $k = 0, 1, \ldots$).

1.5. We define an F-crystal with logarithmic singularities on (X^∞, Y^∞) to be a triple $(\mathscr{E}, \nabla, F)$, where $(\mathscr{E}, \nabla)$ is as above, and F is a rule which associates, to each admissable lifting ϕ of Frobenius to an open $U \subseteq X^\infty$, a horizontal endomorphism F_ϕ of the restriction $\mathscr{E}_{|U}$ of $\mathscr{E}$ to U, satisfying:

i) F_ϕ is ϕ^*-linear, and the assignment $(U, \phi) \to F_\phi$ is compatible with restriction to open subsets; and

ii) if $\bar{F}_\phi$: $\phi^* \mathscr{E} \to \mathscr{E}$ denotes the linearisation of F_ϕ, then if ϕ' is another admissable lifting

$$\bar{F}_{\phi'} = \bar{F}_\phi \circ \chi(\phi', \phi).$$

1.6 An F-crystal in the usual sense [K] on X^∞ gives an F-crystal in our sense, with Y empty. (In fact the usual notion is somewhat stronger, as ∇ is assumed to satisfy a condition of nilpotence, which we have not needed as $e < p - 1$.)

Because of condition ii), in order to give $(\mathscr{E}, \nabla)$ the structure of an F-crystal with logarithmic singularities, it suffices to specify F_ϕ locally for just one choice of lifting ϕ. In particular, if $(\mathscr{E}, \nabla)$ already defines an F-crystal (without singularities) on the complement $X^\infty - Y^\infty$, it suffices to find an open neighbourhood U of Y^∞ and an admissible ϕ on U such that the endomorphism F_ϕ over $U - Y^\infty$ extends to the whole of U.

1.7. Let $\operatorname{res}_Y\colon \Omega_X^1(\log Y) \to \mathcal{O}_Y$ denote the residue map

$$\operatorname{res}_Y\left(a \frac{dt}{t}\right) = a$$

and $\mathscr{R}_Y$ the residue map of ∇ along Y, which is the $\mathcal{O}_Y$-linear endomorphism of $\mathscr{E} \otimes \mathcal{O}_Y$ which makes the diagram

$$\begin{array}{ccc} \mathscr{E} & \xrightarrow{\nabla} & \mathscr{E} \otimes \Omega_X^1(\log Y)^\infty \\ \downarrow & & \downarrow{\scriptstyle \mathrm{id} \otimes \operatorname{res}_Y} \\ \mathscr{E} \otimes \mathcal{O}_Y & \xrightarrow[\mathscr{R}_Y]{} & \mathscr{E} \otimes \mathcal{O}_Y \end{array} \tag{1.7.1}$$

commute (where the left-hand vertical arrow is the natural map).

We restrict from now on to $(\mathscr{E}, \nabla, F)$ which satisfy the additional conditions

i) $\mathscr{E}$ is locally free of finite rank;

ii) $\mathscr{R}_Y$ is nilpotent; and

iii) $\bar{F}_\phi \otimes \mathbf{Q}$ is an isomorphism.

For the rest of the paper, by "F-crystal" *we shall mean* "F-crystal with logarithmic singularities, satisfying i), ii) and iii)".

§2. Cohomology

2.1. Let $(\mathscr{E}, \nabla, F)$ be an F-crystal on (X^∞, Y^∞), and ϕ, ϕ' admissable liftings of Frobenius on X^∞. Define a σ-linear mapping

$$L(\phi', \phi)\colon \mathscr{E} \otimes \Omega_X^1(\log Y)^\infty \to \mathscr{E}$$

by

$$L(\phi', \phi)(e\, dt) = \sum_{n \geq 0} F_\phi \left(\nabla \left(\frac{d}{dt}\right)^n (e)\right) \cdot (\phi'^*(t) - \phi^*(t))^{[n+1]} \tag{2.1.1}$$

if t is a local parameter away from Y, and

$$L(\phi', \phi)\left(e \frac{dt}{t}\right) = \sum_{n \geq 0} F_\phi \left(\nabla \left(t \frac{d}{dt}\right)^n (e)\right) \cdot \left(\log \frac{\phi'^*(t)}{\phi^*(t)}\right)^{[n+1]} \tag{2.1.2}$$

if t is a local equation for Y. The proof that these two formulae are compatible parallels the argument of 1.4 above.

2.2. **Proposition.** i) *$L(\phi', \phi)$ does not depend on the choice of parameter t (and is therefore globally well-defined).*

ii) *On the complex* $\mathscr{E}\otimes\Omega^{\bullet}_X(\log Y)^{\infty}$

$$\nabla\circ L(\phi',\phi)+L(\phi',\phi)\circ\nabla=F_{\phi'}-F_{\phi}.$$

iii) *If* ϕ'' *is a third admissable lifting of Frobenius, then*

$$L(\phi',\phi)+L(\phi'',\phi')=L(\phi'',\phi)$$

and $L(\phi,\phi')=-L(\phi',\phi)$.

iv) $L(\phi',\phi)\equiv 0 \pmod{\pi}$.

Proof. i) Suppose that u and t are two different local parameters. Since we need only check the invariance of the definition of $L(\phi',\phi)$ in a formal neighbourhood of a closed point of X (the sheaves in question being locally free), and since it is clearly invariant under a translation $t\mapsto t+a$ (where a is a constant) we may assume that $X=\operatorname{Spec} R'[[t]]$ as in 1.2.ii), and $u=u(t)\in R'[[t]]$, with $du/dt\in R'[[t]]^*$. We then need to check

$$\sum_{n\geq 0} F_{\phi}\left(\nabla\left(\frac{d}{dt}\right)^n\left(\frac{du}{dt}e\right)\right)\cdot(\phi'^*(t)-\phi^*(t))^{[n+1]}$$
$$=\sum_{n\geq 0} F_{\phi}\left(\nabla\left(\frac{d}{du}\right)^n(e)\right)\cdot(\phi'^*(u)-\phi^*(u))^{[n+1]}.$$

This would follow from the formal identity of differential operators

$$\sum_{n\geq 0}(t'-t)^{[n+1]}\cdot\frac{d^n}{dt^n}\cdot\frac{du}{dt}=\sum_{n\geq 0}(u(t')-u(t))^{[n+1]}\cdot\frac{d^n}{du^n}. \tag{2.2.1}$$

If $f\in R'[[t]]$, choose $g\in R'\otimes\mathbf{Q}[[t]]$ such that $dg/dt=f\,du/dt$. Then (2.2.1) applied to f yields

$$\sum_{n\geq 1}(t'-t)^{[n]}\cdot\frac{d^n g}{dt^n}=\sum_{n\geq 1}(u(t')-u(t))^{[n]}\cdot\frac{d^n g}{du^n}$$

which is indeed a valid identity – it is the Taylor expansion for $g(u(t'))-g(u(t))$ expressed in two different ways.

ii) We have

$$L(\phi',\phi)(\nabla e)=L(\phi',\phi)\left(\nabla\left(\frac{d}{dt}\right)(e)\,dt\right)$$
$$=\sum_{n\geq 1}F_{\phi}\left(\nabla\left(\frac{d}{dt}\right)^n(e)\right)\cdot(\phi'^*(t)-\phi^*(t))^{[n]}$$
$$=F_{\phi}(\chi(\phi',\phi)(e))-F_{\phi}(e)$$
$$=F_{\phi'}(e)-F_{\phi}(e)$$

and similarly for $\nabla\circ L(\phi',\phi)$.

iii) Write ∇_t for $\nabla(d/dt)$, and s, s', s'' for $\phi^*(t)$, etc.

Then

$$\begin{aligned}(L(\phi'',\phi')+L(\phi',\phi))(e\,dt) &= \sum_{n\ge 0} F_{\phi'}(\nabla_t^n(e))\cdot(s''-s')^{[n+1]}+\sum_{n\ge 0} F_\phi(\nabla_t^n(e))\cdot(s'-s)^{[n+1]}\\ &= \sum_{\substack{n\ge 0\\ r\ge 0}} F_\phi(\nabla_t^{n+r}(e))\cdot(s'-s)^{[r]}(s''-s')^{[n+1]}+\sum_{n\ge 0} F_\phi(\nabla_t^n(e))\cdot(s'-s)^{[n+1]}\\ &= \sum_{n\ge 0} F_\phi(\nabla_t^n(e))\sum_{r=0}^{n+1}(s'-s)^{[r]}(s''-s')^{[n+1-r]}\\ &= \sum_{n\ge 0} F_\phi(\nabla_t^n(e))\cdot(s''-s)^{[n+1]}=L(\phi'',\phi)(e\,dt).\end{aligned}$$

The first part follows since clearly $L(\phi,\phi)=0$.

iv) Since $\phi'\equiv\phi\,(\operatorname{mod}\pi)$, this is immediate.

2.3 As in [D] §7.4, the above allows us to define a canonical σ-linear endomorphism F of the cohomology $\mathbf{H}^*(X^\infty, \mathscr{E}\otimes\Omega^{\cdot}_X(\log Y)^\infty)$. We recall the construction: let $\mathfrak{U}=\{U_i\}$ be an open covering of X^∞, and let ϕ_i be an admissable lifting of Frobenius on U_i, for each i. Then an endomorphism F of the Čech complex

$$\check{C}^{\cdot}(\mathfrak{U}, \mathscr{E}\otimes\Omega^{\cdot}_X(\log Y)^\infty)$$

is defined as the sum $u+v+w$, where

$$u\colon \bigoplus_i \Gamma(U_i, \mathscr{E}\otimes\Omega^{\cdot}_X(\log Y)^\infty)\to\bigoplus_i \Gamma(U_i, \mathscr{E}\otimes\Omega^{\cdot}_X(\log Y)^\infty)$$
$$\{e_i\}\mapsto\{F_{\phi_i}(e_i)\}$$

$$v\colon \bigoplus_{i<j} \Gamma(U_i\cap U_j, \mathscr{E}\otimes\Omega^{\cdot}_X(\log Y)^\infty)\to\bigoplus_{i<j} \Gamma(U_i\cap U_j, \mathscr{E}\otimes\Omega^{\cdot}_X(\log Y)^\infty)$$
$$\{e_{ij}\}\mapsto\{F_{\phi_i}(e_{ij})\}$$

$$w\colon \bigoplus_i \Gamma(U_i, \mathscr{E}\otimes\Omega^1_X(\log Y)^\infty)\to\bigoplus_{i<j} \Gamma(U_i\cap U_j, \mathscr{E})$$
$$\{e_i\}\mapsto\{L(\phi_j,\phi_i)(e_{i|U_i\cap U_j})\}.$$

By 2.2. iii) and §7.5 of [D], F does not depend, up to homotopy, on the choice of liftings $\{\phi_i\}$.

2.4. If (X,Y) is as in 1.2.ii), then of course we need only take the trivial covering $\mathfrak{U}=\{X\}$ above, and choose any admissable lifting ϕ on X, to define F. If (X,Y) is as in 1.2.i), and if $\hat{X}$ is the formal completion of X about a closed point, then there is a natural map

$$\mathbf{H}^*(X^\infty,\mathscr{E}\otimes\Omega^{\cdot}_X(\log Y)^\infty)\to\mathbf{H}^*(\hat{X},\mathscr{E}\otimes\Omega^{\cdot}_X(\log Y)\hat{\ })$$

with respect to which F is evidently functorial.

2.5 Let $\Omega^{\cdot}(\mathscr{E})$ denote the subcomplex

$$[\mathscr{E}\xrightarrow{\nabla}\nabla(\mathscr{E})+\mathscr{E}\otimes\Omega^1_{X^\infty}]$$

of $\mathscr{E}\otimes\Omega_X^{\cdot}(\log Y)^{\infty}$. It may be characterised by

$$\Omega^1(\mathscr{E})=(\mathrm{id}\otimes\mathrm{res}_Y)^{-1}(\mathrm{Im}\,\mathscr{R}_Y)$$

(cf. the diagram (1.7.1)). The procedure of 2.3 gives a canonical endomorphism F of $\mathbf{H}^*(X^{\infty},\Omega^{\cdot}(\mathscr{E}))$, which is functorial with respect to completion about a closed point of X, as in the previous paragraph.

§ 3. Local structure

3.1. For this section we assume that (X, Y) is as in 1.2.ii) above, with $R'=R$. Denote by $K\{t\}$ the ring of power series in $K[[t]]$ which converge on the p-adic disc $\{z\colon \mathrm{ord}_p(z)>0\}$. Let $(\mathscr{E}, \nabla, F)$ be an F-crystal on (X, Y), and write $\mathscr{E}_0$ for the fibre $\mathscr{E}\underset{t=0}{\otimes} R$. Let F_0 denote the endomorphism of $\mathscr{E}_0$ deduced from F_ϕ by passage to the quotient (it is independent of ϕ, cf. (1.4.1)), and write $\mathscr{R}_0$ for the residue map $\mathscr{R}_Y$.

3.2. **Proposition**

i) $p\mathscr{R}_0^{\sigma}\circ F_0=F_0\circ\mathscr{R}_0$.

ii) *There is a unique isomorphism of* $K\{t\}$*-modules*

$$\mathscr{E}\underset{R[[t]]}{\otimes}K\{t\}\xrightarrow{\sim}\mathscr{E}_0\underset{R}{\otimes}K\{t\}$$

which reduces to the identity mod(t), *and for which the actions of* ∇, F_ϕ *on* $\mathscr{E}_0\otimes K\{t\}$ *satisfy*

$$\begin{aligned}\nabla(e\otimes 1)&=\mathscr{R}_0(e)\otimes\frac{dt}{t}\\ F_\phi(e\otimes 1)&=F_0(e)\otimes 1.\end{aligned}\tag{3.2.1}$$

Proof ([Dw], [K]). Let $\phi^*(t)=t^p$, and let $\{e_i\colon 1\leqq i\leqq d\}$ be a basis for $\mathscr{E}$, with

$$\nabla(e_i)=\sum_j (B_{ij}+t^{-1}g_{ij})e_j\cdot dt,$$

$$F_\phi(e_i)=\sum_j A_{ij}e_j,$$

where

$$A_{ij}, B_{ij}\in R[[t]],\qquad g_{ij}\in R.$$

We require a basis $\{e_i'\}$ for $\mathscr{E}\otimes K\{t\}$ such that

$$\nabla(e_i')=\sum_j g_{ij}e_j'\frac{dt}{t}.$$

If $e_i'=\sum D_{ij}e_j$, then the matrix $\mathbf{D}=(D_{ij})$ is to satisfy

$$t\frac{d}{dt}(\mathbf{D})=[\mathbf{g},\mathbf{D}]-t\,\mathbf{D}\cdot\mathbf{B}.\tag{3.2.2}$$

By hypothesis, $\mathbf{g}$ is nilpotent, whence there is a unique power series solution $\mathbf{D} \in M_d(K[[t]])$ of (3.2.2) for which $\mathbf{D}(0)=\mathbf{I}$, and the entries of $\mathbf{D}$ have a non-zero radius of convergence (cf. [C]).

If we write $F_\phi(e'_i)=\sum a_{ij} e'_j$, then

$$t \frac{d}{dt}(\mathbf{a}) = p\,\mathbf{g}^\sigma \cdot \mathbf{a} - \mathbf{a} \cdot \mathbf{g} \tag{3.2.3}$$

whence $\mathbf{a}$ is a constant matrix. The relation $\mathbf{D}^\phi \cdot \mathbf{A} = \mathbf{a} \cdot \mathbf{D}$ then shows firstly that $a_{ij} = A_{ij}(0)$, whence by (3.2.3) we have i), and secondly that $\mathbf{D}$ converges and is invertible on the whole open disc of radius 1 (compare [K], 3.1.2 and [Dw], Theorem 6; note that $\mathbf{a}$ and $\mathbf{A}$ are invertible matrices, by 1.7.iii) above).

§ 4. Overconvergence

4.1. Assume that we are in case 1.2.i), with X *proper* over R. If $(\mathscr{E}, \nabla, F)$ is an F-crystal on (X^∞, Y^∞), then the differential equation $(\mathscr{E}, \nabla)$ descends uniquely to $(X^\dagger, Y^\dagger)$ (indeed, even to (X, Y), by EGA III 5.1.4). Let $U \subseteq X$ be an open subscheme, and ϕ, ϕ': $U^\dagger \to U^\dagger$ admissable liftings of Frobenius. Then there are defined mappings

$$F_\phi \colon \mathscr{E}_{|U^\infty} \to \mathscr{E}_{|U^\infty}$$
$$L(\phi', \phi) \colon \mathscr{E} \otimes \Omega^1_X(\log Y)_{|U^\infty} \to \mathscr{E}_{|U^\infty}.$$

4.2. *F_ϕ and $L(\phi', \phi)$ extend to mappings over $U^\dagger$.*

We reduce this to a local statement as follows. Let Z be a closed subscheme of X, finite and étale over R, such that $U^\infty = X^\infty - Z^\infty$. Without loss of generality we may assume that X is integral, $U = X - Z$, and $Z \neq \emptyset$. Let $\Gamma(Z, \mathcal{O}_Z) = R'$; then the formal completion of X along Z is isomorphic to $\operatorname{Spf} R'[[z]]$ for a local equation z for Z. Consider the commutative square

$$\begin{array}{ccc} \Gamma(U^\dagger, \mathcal{O}_X^\dagger) & \hookrightarrow & R'((z))^\dagger \\ \downarrow & & \downarrow \\ \Gamma(U^\infty, \mathcal{O}_X^\infty) & \hookrightarrow & R'((z))^\infty. \end{array}$$

(Recall that $R'((z))^\infty$ is the ring of formal Laurent series

$$P(z) = \sum_{n \in \mathbf{Z}} a_n z^n$$

where $a_n \in R'$ and $\operatorname{ord}_p a_{-n} \to \infty$ as $n \to \infty$; and that $R'((z))^\dagger$ is the subring comprised of series $P(z)$ such that for some $\alpha > 0$, depending on P,

$$\operatorname{ord}_p a_{-n} \geqq \alpha n - 1$$

for every $n \geqq 0$.)

Viewing the modules in this diagram as submodules of $R'((z))^{\infty}$, we claim

$$\Gamma(U^{\dagger}, \mathcal{O}_X^{\dagger}) = \Gamma(U^{\infty}, \mathcal{O}_X^{\infty}) \cap R'((z))^{\dagger}.$$

To prove this, choose a finite flat morphism

$$f: X \to \mathbf{P}^1_R$$

such that $f^{-1}(\infty)_{\mathrm{red}} = Z$. If x denotes the coordinate on $\mathbf{P}^1$, then

$$\Gamma(U^?, \mathcal{O}_X^?) = \Gamma(U, \mathcal{O}_X) \bigotimes_{R[x]} R[x]^?,$$
$$R'((z))^? = \Gamma(U, \mathcal{O}_X) \bigotimes_{R[x]} R((x^{-1}))^?$$

where ? denotes either † or ∞. By faithful flatness we are reduced to showing

$$R[x]^{\dagger} = R[x]^{\infty} \cap R((x^{-1}))^{\dagger}$$

which is obvious from the definitions.

To prove 4.2, it now suffices to prove the following:

Let $(\mathscr{E}, \nabla, F)$ be an F-crystal on $(\mathrm{Spf}\, R'[[t]], t=0)$, and let ϕ, ϕ' be σ-linear liftings of Frobenius to $R'((t))^{\dagger}$. Then the mappings F_{ϕ}, $L(\phi', \phi)$ (which are *a priori* defined over $R'((t))^{\infty}$) extend to mappings over $R'((t))^{\dagger}$.

Let θ be the σ-linear lifting of Frobenius to $R'[[t]]$ with $\theta^*(t) = t^p$ (extending σ to R' in the unique way). By the transitivity properties 1.5.ii) and 2.2.iii), it suffices to show that $L(\phi, \theta)$ and $\chi(\phi, \theta)$ are defined over $R'((t))^{\dagger}$.

Since $\phi^*(t) \in R'((t))^{\dagger}$ and $\phi^*(t) \equiv t^p \pmod{\pi}$,

$$\log \frac{\phi^*(t)}{t^p} \in R'((t))^{\dagger}$$

whence we may write

$$\log \left(\frac{\phi^*(t)}{\theta^*(t)}\right)^n = \pi^n \sum_{l \in \mathbf{Z}} b_n(l)\, t^l \tag{4.2.1}$$

where $b_n(l) \in R'$, and for some $\alpha > 0$

$$\mathrm{ord}_p\, b_n(l) \geqq \max(0, -\alpha l - n). \tag{4.2.2}$$

Then if $e \in \mathscr{E}$ and $k \geqq 0$, the formula (2.1.2) gives

$$L(\phi, \theta)\left(e \frac{dt}{t^{k+1}}\right) = \sum_{\substack{n \geqq 0 \\ l \in \mathbf{Z}}} F_{\theta}\left(\nabla\left(t \frac{d}{dt}\right)^n \left(\frac{e}{t^k}\right)\right) \pi^{[n+1]} b_{n+1}(l)\, t^l. \tag{4.2.3}$$

Now write, for $n \geqq 1$,

$$F_{\theta}\left(\nabla\left(t \frac{d}{dt}\right)^{n-1} \left(\frac{e}{t^k}\right)\right) = t^{-pk} \sum_i \sum_{s \geqq 0} a_n(i, k, s)\, t^s e_i$$

where $\{e_i\}$ is a basis for $\mathscr{E}$ and $a_n(i,k,s)\in R'$. Then (4.2.3) becomes

$$\sum_{n\geqq 1}\frac{\pi^n}{n!}\sum_{l\in\mathbf{Z}} b_n(l)\,t^l \sum_{s\geqq 0}\sum_i t^{s-pk}\,a_n(i,s,k)\,e_i$$
$$=\sum_i e_i \sum_{m\in\mathbf{Z}} c(i,k,m)\,t^m$$

where

$$c(i,k,m)=\sum_{\substack{n\geqq 1\\ s\geqq 0}}\frac{\pi^n}{n!}\,a_n(i,k,s)\,b_n(m-s+pk).$$

From (4.2.2) and the well-known formula

$$\operatorname{ord}_p\left(\frac{\pi^n}{n!}\right)\geqq n\left(\frac{1}{e}-\frac{1}{p-1}\right)$$

we have

(4.2.4) $$\operatorname{ord}_p c(i,k,m)\geqq \min_{\substack{s\geqq 0\\ n\geqq 1}}\left(\lambda n+\max(0,-\alpha(m-s+pk)-n)\right)$$

writing $\lambda=\frac{1}{e}-\frac{1}{p-1}$, so that $0<\lambda<1$.

If $n\geqq\alpha(s-pk-m)$, the expression in brackets in (4.2.4) is $\lambda n\geqq\alpha\lambda(-pk-m)$, and if $n\leqq\alpha(s-pk-m)$, it is

$$\begin{aligned}\lambda n+\alpha(s-pk-m)-n&=\alpha(s-pk-m)-(1-\lambda)\,n\\ &\geqq\alpha(s-pk-m)-(1-\lambda)\,\alpha(s-pk-m)\\ &\geqq\alpha\lambda(-pk-m)\end{aligned}$$

and thus (4.2.4) gives

$$\operatorname{ord}_p c(i,k,m)\geqq\max(0,\alpha\lambda(-pk-m)).$$

Now suppose that for each $k\geqq 0$ we have $d_k\in R'$, satisfying

$$\operatorname{ord}_p d_k\geqq\beta k-1,\qquad \beta>0.$$

Then

(4.2.5) $$L(\phi,\theta)\left(\sum_{k\geqq 0} d_k\,e\,\frac{dt}{t^{k+1}}\right)=\sum_i e_i\sum_{\substack{k\geqq 0\\ m\in\mathbf{Z}}} d_k\,c(i,k,m)\,t^m$$

and

(4.2.6) $$\begin{aligned}\operatorname{ord}_p\Big(\sum_k d_k\,c(i,k,m)\Big)&\geqq\min_{k\geqq 0}\left(\operatorname{ord}_p d_k+\operatorname{ord}_p c(i,k,m)\right)\\ &\geqq\min_{k\geqq 0}\left(\max(0,\beta k-1)+\max(0,\alpha\lambda(-m-pk))\right).\end{aligned}$$

Evaluating the bracketed expression at its changes of slope, $k=-m/p$ and $k=1/\beta$, we find that (4.2.6) is greater than or equal to

$$\max\left\{0, \min\left(\beta\left(-\frac{m}{p}\right)-1, \alpha\lambda\left(-m-\frac{p}{\beta}\right)\right)\right\}$$
$$=\max\left\{0, \left(-\frac{\beta m}{p}-1\right)\min\left(1, \frac{\alpha\lambda p}{\beta}\right)\right\}.$$

Hence the right hand expression of (4.2.5) is an element of $\mathscr{E}\otimes R'((t))^\dagger$, and since every element of $\mathscr{E}\otimes R'((t))^\dagger$ is a finite linear combination of elements of the form

$$\sum_{k\geqq 0} d_k t^{-k}\cdot e$$

with $e\in\mathscr{E}$, $d_k\in R'$, and $\operatorname{ord}_p d_k\geqq\beta k-1$, for some $\beta>0$, we have the desired result for $L(\phi,\theta)$. An expression for $\chi(\phi,\theta)$, similar to (4.2.3), may be obtained from (1.4.2) above, and the rest of the reasoning above then applies.

§ 5. Trace formula

5.1. In this section we assume that $k=\mathbf{F}_q$ is finite, where $q=p^r$. For $s\geqq 1$, write $k_s=\mathbf{F}_{q^s}$, and let R_s denote the (unique) unramified extension of R with residue field k_s. Let $(\mathscr{E},\nabla,F)$ be an F-crystal on (X^∞, Y^∞). To each $x\in X(k_s)$ we associate an R_s-module $\mathscr{E}_x$ and R_s-linear endomorphisms F_x^{rs}, $\mathscr{R}_x$, as follows.

If $x\notin Y(k_s)$, define $\mathscr{E}_x$ to be $\tilde{x}^*\mathscr{E}$, for any lifting $\tilde{x}\in X(R_s)$ of x. We take F_x^{rs} to be the restriction of $(F_\phi)^{rs}$ for a lifting ϕ of Frobenius for which $\phi^{rs}(\tilde{x})=\tilde{x}$, and write $\mathscr{R}_x=0$. This assignment is independent of the choices of $\tilde{x}$ and ϕ (cf. [K] 1.4).

If $x\in Y(k_s)$, we let $\mathscr{E}_x=\tilde{x}^*\mathscr{E}$, where $\tilde{x}\in Y(R_s)$ is the unique lifting of x to Y (Hensel's lemma). F_x^{rs} is the restriction of $(F_\phi)^{rs}$ for an admissable lifting ϕ of Frobenius to a neighbourhood of x in X^∞, and $\mathscr{R}_x$ is the restriction of the residue map $\mathscr{R}_Y$.

In either case, the relation

$$q^s\mathscr{R}_x\circ F_x^{rs}=F_x^{rs}\circ\mathscr{R}_x \tag{5.1.1}$$

holds (cf. 3.2 above), and F_x^{rs} is injective (by 1.7.iii)).

Assume now that (X, Y) is as in 1.2.i), and write $U=X-Y$. For a $\mathbf{Z}$-module M, abbreviate $M\otimes\mathbf{Q}$ by $M_\mathbf{Q}$.

5.2. **Theorem.** *Let X be* proper *over R.*

i) *F is bijective on $\mathbf{H}^i(X^\infty,\mathscr{E}\otimes\Omega_X^\cdot(\log Y)^\infty)_\mathbf{Q}$, and*

$$\sum_i(-1)^i\operatorname{Tr}(F^{-rs}\colon \mathbf{H}^i(X,\mathscr{E}\otimes\Omega_X^\cdot(\log Y)^\infty)_\mathbf{Q})$$
$$=q^{-s}\sum_{x\in U(k_s)}\operatorname{Tr}(F_x^{-rs}\colon \mathscr{E}_{x,\mathbf{Q}}).$$

ii) *F is bijective on $\mathbf{H}^i(X^\infty,\Omega^\cdot(\mathscr{E}))_\mathbf{Q}$, and*

$$\sum_i(-1)^i\operatorname{Tr}(F^{-rs}\colon \mathbf{H}^i(X^\infty,\Omega^\cdot(\mathscr{E}))_\mathbf{Q})=q^{-s}\sum_{x\in X(k_s)}\operatorname{Tr}(F_x^{-rs}\colon \operatorname{coker}(\mathscr{R}_x)_\mathbf{Q}).$$

5.3. *Remark.* Note that the complex $\Omega^\cdot(\mathscr{E})$ does not change if Y is enlarged, whereas $\mathscr{E}\otimes\Omega^\cdot_X(\log Y)$ does. If $\mathscr{E}$ is viewed as the analogue of a local system $\mathscr{F}$ on U, then the cohomology of the first complex corresponds to $H^*(X, j_*\mathscr{F})$, and the second to $H^*(U,\mathscr{F})$, for $j\colon U\hookrightarrow X$ the inclusion.

5.4. *Proof of* 5.2. By Meredith's comparison theorem ([Me], 5, Theorem 4) and EGA III 4.1.5, the natural map

$$\mathbf{H}^i(X^\dagger, \mathscr{E}\otimes\Omega^\cdot_X(\log Y)^\dagger) \to \mathbf{H}^i(X^\infty, \mathscr{E}\otimes\Omega^\cdot_X(\log Y)^\infty)$$

is an isomorphism of R-modules of finite type. In § 6 we shall prove:

The natural map

$$\mathbf{H}^i(X^\dagger, \mathscr{E}\otimes\Omega^\cdot_X(\log Y)^\dagger)_{\mathbf{Q}} \to \mathbf{H}^i(U^\dagger, \mathscr{E}\otimes\Omega^\cdot_{U^\dagger})_{\mathbf{Q}} \tag{5.4.1}$$

is an isomorphism.

Granted this, we deduce 5.2 in a series of steps, following Monsky [M2]. By extension of scalars to R_s, we may assume $s=1$.

5.5. *Suppose that $U(k)$ is empty, and that there is a σ-linear lifting ϕ of Frobenius on $U^\dagger$. Then* 5.2.i) *holds.*

By 4.2, F_ϕ extends to a σ-linear endomorphism of $\mathscr{E}\otimes\Omega^\cdot_{U^\dagger}$. Denoting by $\bar{F}_\phi$ the linearisation of F_ϕ, as in 1.5ii), there is a factorisation

$$\begin{array}{ccccc} \mathscr{E}\otimes\Omega^\cdot_{U^\dagger} & \xrightarrow{\phi^*} & \mathscr{E}\otimes\phi^*\Omega^\cdot_{U^\dagger} & \xrightarrow[\sim]{\alpha} & \phi^*(\mathscr{E}\otimes\Omega^\cdot_{U^\dagger}) \\ & \searrow^{F_\phi} & & & \downarrow{\scriptstyle \bar{F}_\phi} \\ & & & & \mathscr{E}\otimes\Omega^\cdot_{U^\dagger}. \end{array}$$

By Theorem 8.5 of [M–W] there is a trace map

$$\Psi\colon\ \phi^*\Omega^\cdot_{U^\dagger} \to \Omega^\cdot_{U^\dagger}$$

such that

$$\Psi(\phi^*\omega) = p\omega$$

for $\omega\in\Omega^\cdot_{U^\dagger}$. Since $\mathcal{O}^\infty_U$ is faithfully flat over $\mathcal{O}^\dagger_U$ (cf. 5.8 below), the map $\bar{F}_\phi\otimes\mathbf{Q}$ is an isomorphism (by 1.7.iii)), and using the diagram above we may define an endomorphism

$$\Upsilon = (\mathrm{id}_{\mathscr{E}}\otimes\Psi)\circ\alpha^{-1}\circ(\bar{F}_\phi\otimes\mathbf{Q})^{-1}$$

of $\mathscr{E}\otimes\Omega^\cdot_{U^\dagger}\otimes\mathbf{Q}$, which satisfies

$$\Upsilon\circ F_\phi = p.$$

Thus $p^{-1}\Upsilon$ induces a left inverse to F on $\mathbf{H}^i(U^\dagger, \mathscr{E}\otimes\Omega^\cdot_{U^\dagger})_{\mathbf{Q}}$. By (5.4.1) these spaces have finite dimension over K, and hence F is bijective on them.

Since the $\mathcal{O}_U^\dagger$-modules $\mathcal{E}\otimes\Omega^\bullet_{U^\dagger}$ are finitely generated, for some power p^k of p we have

$$p^k\,\Upsilon(\mathcal{E}\otimes\Omega^\bullet_{U^\dagger})\subseteq\mathcal{E}\otimes\Omega^\bullet_{U^\dagger};$$

it then remains to prove that

$$\sum_i(-1)^i\,\mathrm{Tr}(p^{kr}\,\Upsilon^r\colon\ \mathbf{H}^i(U^\dagger,\mathcal{E}\otimes\Omega^\bullet_{U^\dagger})_{\mathbf{Q}})=0$$

if $U(k)$ is empty. But the endomorphism $p^{kr}\Upsilon^r$ of $\mathcal{E}\otimes\Omega^\bullet_{U^\dagger}$is a "Dwork operator", in the terminology of [M2], and so by Theorems 3.3 and 3.5 of *loc. cit.*, the alternating sum of the traces is zero.

5.6. *The conclusions* i) *and* ii) *of Theorem* 5.2 *are equivalent.*

By 2.4 above, there is an exact sequence

$$0\to\Omega^1(\mathcal{E})\to\mathcal{E}\otimes\Omega^1_X(\log Y)^\infty\to\operatorname{coker}\mathcal{R}_Y\to 0$$

whence a long exact sequence of cohomology

$$\begin{aligned}(5.6.1)\quad &0\to\mathbf{H}^0(X^\infty,\Omega^\bullet(\mathcal{E}))\to\mathbf{H}^0(X^\infty,\mathcal{E}\otimes\Omega^\bullet_X(\log Y)^\infty)\to 0\\ &0\to\mathbf{H}^1(X^\infty,\Omega^\bullet(\mathcal{E}))\to\mathbf{H}^1(X^\infty,\mathcal{E}\otimes\Omega^\bullet_X(\log Y)^\infty)\\ &\to\operatorname{coker}\mathcal{R}_Y(-1)\to\mathbf{H}^2(X^\infty,\Omega^\bullet(\mathcal{E}))\to\mathbf{H}^2(X^\infty,\mathcal{E}\otimes\Omega^\bullet_X(\log Y)^\infty)\to 0\end{aligned}$$

which we view as an exact sequence of R-modules with σ-linear operators F. Here $\operatorname{coker}\mathcal{R}_Y(-1)$ is the Tate twist of $\operatorname{coker}\mathcal{R}_Y$; it has the same underlying R-module, but the operator F (deduced from F_ϕ by passage to the quotient) is replaced by pF. The compatibility of (5.6.1) with F is a consequence of 3.2.i), or an obvious variant of it. Since F is bijective on $(\operatorname{coker}\mathcal{R}_Y)_{\mathbf{Q}}$ (by the hypothesis 1.7.iii)), the first assertions of 5.2.i) and 5.2.ii) are equivalent; for the rest, it suffices to prove:

$$\mathrm{Tr}(F^{-r}\colon\ (\operatorname{coker}\mathcal{R}_Y)_{\mathbf{Q}})=\sum_{x\in Y(k)}\mathrm{Tr}(F_x^{-r}\colon\ (\operatorname{coker}\mathcal{R}_x)_{\mathbf{Q}}).$$

Since both sides of this equation are additive in Y, it suffices to show that if Z is a component of Y with $Z(k)$ empty, then

$$\mathrm{Tr}(F^{-r}\colon\ (\operatorname{coker}\mathcal{R}_Z)_{\mathbf{Q}})=0.$$

But replacing F^{-r} by p^kF^{-r} for some $k\geqq 0$, we obtain a Dwork operator on $(\operatorname{coker}\mathcal{R}_Z)/(\pi\text{-torsion})$, and again the trace vanishes.

5.7. *End of proof*

By 5.6, it suffices to prove 5.2.ii), and the latter statement does not depend on the subscheme Y (cf. 5.3 above). Enlarge Y so that the conditions of 5.5 are satisfied. Applying 5.6 again, it suffices to prove 5.2.i) for this choice of Y, which we have done in 5.5.

5.8. We finally fill the gap in 5.5. By Theorem 1.4 of [M−W], $\mathcal{O}_U^\infty$ is the π-adic completion of $\mathcal{O}_U^\dagger$; hence by Theorem 1.6 of *loc. cit.* and Theorem 56, page 172, [Ma], it is faithfully flat over $\mathcal{O}_U^\dagger$.

§ 6. The isomorphism (5.4.1)

6.1. We retain the notations of the previous section. Let i, j denote the inclusions

$$U^\dagger \underset{j}{\hookrightarrow} X^\dagger \underset{i}{\hookleftarrow} Y^\dagger.$$

Write

$$A^\bullet = \mathscr{E} \otimes \Omega^\bullet_X(\log Y)^\dagger.$$

The exact sequence

$$0 \to A^\bullet \to j_* j^* A^\bullet \to (j_* j^* A^\bullet)/A^\bullet \to 0$$

gives (since j is affine) a long exact sequence of cohomology

$$\ldots \to \mathbf{H}^i(X^\dagger, A^\bullet) \to \mathbf{H}^i(U^\dagger, j^* A^\bullet) \to \mathbf{H}^i(X^\dagger, (j_* j^* A^\bullet)/A^\bullet) \to \ldots$$

By Lemma 1.4 and Theorem 2.3 of [M1], we have an isomorphism

$$(6.1.1) \qquad \frac{j_* \mathcal{O}_U^\dagger}{\mathcal{O}_X^\dagger} \xrightarrow{\sim} i_* \left(\frac{\mathcal{O}_Y((t))^\dagger}{\mathcal{O}_Y[[t]]} \right).$$

(Note that, in the notation of [M1], we have $\mathcal{O}_Y\langle T \rangle = \mathcal{O}_Y[[T]]$ since Y has dimension zero over R.)

Extending the differential of $A^\bullet$ in the obvious way, (6.1.1) gives

$$\mathbf{H}^*(X^\dagger, (j_* j^* A^\bullet)/A^\bullet) \xrightarrow{\sim} H^* \left(A^\bullet \otimes \frac{B((t))^\dagger}{B[[t]]} \right)$$

where $B = \Gamma(Y, \mathcal{O}_Y)$. To prove that (5.4.1) is an isomorphism, it therefore suffices to prove that the complex

$$(6.1.2) \qquad A^\bullet \otimes \frac{B((t))^\dagger}{B[[t]]} \otimes \mathbf{Q}$$

is acyclic.

6.2. Let $K\{t\}$ be the ring of convergent power series in t, as in 3.1, and write $K\{\{t\}\}$ for the set of all formal Laurent series

$$P(t) = \sum_{n=-\infty}^{\infty} a_n t^n, \qquad a_n \in K,$$

such that, for all $\varepsilon > 0$,

$$\operatorname{ord}_p(a_n) + \varepsilon n \to +\infty \qquad \text{as } n \to +\infty$$

and such that for some constant $\alpha > 0$ (depending on P)

$$\operatorname{ord}_p(a_{-n}) - \alpha n \to +\infty \qquad \text{as } n \to +\infty.$$

Thus $K\{\{t\}\}$ is the ring of formal Laurent series converging on some annulus $0 < \operatorname{ord}_p(z) < \delta$. From the definitions, we have

$$\frac{K\{\{t\}\}}{K\{t\}} \xrightarrow{\sim} \frac{R((t))^\dagger}{R[[t]]} \otimes_R K.$$

6.3. Write B as a direct sum of domains B_i, so that $B\bigotimes_R K=\bigoplus_i K_i$, where K_i is the field of fractions of B_i. The complex (6.1.2) is, by the above, isomorphic to

$$\bigoplus_i A^{\bullet}\otimes\frac{K_i\{\{t\}\}}{K_i\{t\}}.$$

Write $\mathscr{E}_i$ for $\mathscr{E}\otimes B_i$, and $\mathscr{R}_i$ for the restriction of $\mathscr{R}_Y$ to $\mathscr{E}_i$. The complex $A^{\bullet}\otimes K_i\{\{t\}\}/K_i\{t\}$ is then by Proposition 3.2 isomorphic to

$$\mathscr{E}_i\otimes\frac{K_i\{\{t\}\}}{K_i\{t\}}\xrightarrow{\nabla_i}\mathscr{E}_i\otimes\frac{K_i\{\{t\}\}\cdot dt}{K_i\{t\}\cdot t^{-1}dt}$$

where the differential ∇_i is given by

$$\nabla_i(e\otimes y)=e\otimes dy+\mathscr{R}_i(e)\otimes y\cdot t^{-1}\,dt.$$

This complex is acyclic; indeed, an explicit inverse to ∇_i is the mapping

$$e\otimes\sum_{n\in\mathbf{Z}}a_n t^{n-1}\,dt\mapsto\sum_{k\geqq 0}(-1)^k\mathscr{R}_i^k(e)\otimes\sum_{n<0}a_n\frac{t^n}{n^{k+1}}$$

which is well-defined since $\mathscr{R}_i$ is nilpotent.

§ 7. Complements

7.1. Let (X, Y) and (X', Y') be as in 1.2.i), and let $f\colon X'\to X$ be a (non-constant) R-morphism, with $f^{-1}(Y)=Y'$. If $(\mathscr{E},\nabla,F)$ is an F-crystal without singularities on X^{∞}, then $f^*\mathscr{E}$ defines an F-crystal on X'^{∞} (by functoriality of Frobenius). If $\mathscr{E}$ has singularities, we can at any rate give $f^*\mathscr{E}$ the structure of an F-crystal with singularities under the hypothesis:

f is étale away from a closed subscheme $Z\subseteq X$, whose intersection with Y is finite and étale over R.

Indeed, since $f^*\,\Omega^1_{X/R}(\log Y)\subseteq\Omega^1_{X'/R}(\log Y')$, there is a natural extension of ∇ to $f^*\mathscr{E}$. By the above, we have an F-crystal structure on $f^*\mathscr{E}$ away from Y. It therefore suffices to define the map $F_{\phi'}$ for some admissable ϕ' in a neighbourhood of Y'; we may assume then that $Z=Y$.

If there are admissable liftings ϕ, ϕ' of Frobenius to X, X' with

$$f\circ\phi'=\phi\circ f$$

we can take $F_{\phi'}=f^*(F_{\phi})$. So it suffices to find a pair (ϕ,ϕ'). Choose ϕ arbitrarily. Since f is étale away from Y, and X' is separated, such a ϕ', if it exists, is uniquely determined; and the same is true if X' is replaced by any X'' étale over X'. It therefore suffices to construct ϕ' locally for the étale topology. But by Abhyankar's lemma X' is locally isomorphic to

$$X[s]/(s^d-t),$$

where t is a local equation for Y, and $(d,p)=1$. Since ϕ is admissable,

$$\phi^* t = t^p \cdot u$$

for some $u \in 1+\pi \cdot \mathcal{O}_X^\infty$. Since $p \nmid d$, we can solve

$$v^d = u, \qquad v \equiv 1 \pmod{\pi}$$

for some $v \in \mathcal{O}_X^\infty$, and then

$$\phi'^*\colon\ s \mapsto s^p \cdot v$$

is the required lifting.

7.2 Let H be a finite group of automorphisms of X which preserve Y. If ϕ is an admissable lifting of Frobenius to X^∞, then so is $h^{-1} \circ \phi \circ h$ for any $h \in H$. We may therefore define the notion of an action of H on an F-crystal $(\mathscr{E}, \nabla, F)$ on (X^∞, Y^∞) as the data: for each $h \in H$, a horizontal isomorphism

$$\tilde{h}\colon\ h^* \mathscr{E} \xrightarrow{\ \sim\ } \mathscr{E},$$

satisfying the usual compatibilities for a group action, and such that if ϕ is an admissable lifting of Frobenius to an open $U \subseteq X^\infty$, and $h \in H$ such that $h^{-1}(U) = U$, then the diagram of sheaves on U

$$\begin{array}{ccc}
h^* \phi^* \mathscr{E} & \xrightarrow{h^*(F_\phi)} & h^* \mathscr{E} \\
\Vert & & \\
\phi'^* h^* \mathscr{E} & & \simeq \Big\downarrow \tilde{h} \\
\simeq \Big\downarrow \phi'^*(\tilde{h}) & & \\
\phi'^* \mathscr{E} & \xrightarrow[F_{\phi'}]{} & \mathscr{E}
\end{array}$$

commutes, with $\phi' = h^{-1} \circ \phi \circ h$.

Let $\mathfrak{U} = \{U_i\}$ be an open covering of X, where $h^{-1}(U_i) = U_i$ for each i and for a fixed $h \in H$. Choose admissable ϕ_i on each U_i, and write F for the endomorphism of the Čech complex $\check{C}^\cdot(\mathfrak{U}, \Omega^\cdot(\mathscr{E}))$ constructed in 2.3 from $\{\phi_i\}$; write F' for the homotopic endomorphism constructed from $\{\phi'_i\}$, where $\phi'_i = h^{-1} \circ \phi_i \circ h$. The pair $(h^*, \tilde{h})$ defines an endomorphism h of the Čech complex, and by the diagram above, $h \circ F = F' \circ h$, whence the endomorphisms h and F of cohomology commute.

7.3 In [S] a filtered form of the complex $\Omega^\cdot(\mathscr{E})$ is employed. Write, for $i \geqq 0$,

$$\langle i \rangle = \min \left\{ \operatorname{ord}_\pi \left(\frac{\pi^j}{j!} \right) : j \geqq i \right\}.$$

We consider F-crystals $(\mathscr{E}, \nabla, F)$ equipped with a decreasing filtration

$$\mathscr{E} = \mathrm{Fil}^0 \mathscr{E} \supseteq \mathrm{Fil}^1 \mathscr{E} \supseteq \ldots \supseteq \mathrm{Fil}^{k+1} \mathscr{E} = 0$$

whose successive quotients are locally free, satisfying the conditions

$$\nabla(\mathrm{Fil}^{i+1}) \subseteq \mathrm{Fil}^i \otimes \Omega^1_{X/R}(\log Y)^\infty$$

and

$$F_\phi(\mathrm{Fil}^i) \subseteq \pi^{\langle i \rangle} \mathscr{E}$$

Write $\mathrm{Fil}^i = 0$ for $i > k$ and $Fil^i = \mathscr{E}$ for $i \leqq 0$. We may then define a filtration on $\mathscr{E} \otimes \Omega^{\cdot}_{X/R}(\log Y)^\infty$ by

$$\mathrm{Fil}^i(\mathscr{E} \otimes \Omega^j) = \mathrm{Fil}^{i-j}(\mathscr{E}) \otimes \Omega^j$$

and, by restriction, a filtration on $\Omega^{\cdot}(\mathscr{E})$. Define

$$^{(\pi)}\Omega^{\cdot}(\mathscr{E}) = \sum_k \pi^{\langle k+1-i \rangle} \mathrm{Fil}^i(\Omega^{\cdot}(\mathscr{E})).$$

One checks, using (2.1.1) and the formula $\langle i \rangle + \langle j \rangle \geqq \langle i+j \rangle$, that $^{(\pi)}\Omega^{\cdot}(\mathscr{E})$ is a complex, and that it is stable under the action of F_ϕ and $L(\phi, \phi')$. Thus its cohomology has a canonical σ-linear endomorphism F, which is compatible with that defined in 2.3, and functorial with respect to formal completion at a closed point.

7.4. As an example of the situation of the preceding section, let $(\mathscr{E}, \nabla, F)$ be an F-crystal on (X^∞, Y^∞), and assume that we have a filtration

$$\mathscr{E} \supseteq \mathscr{F} \supseteq 0 \tag{7.4.1}$$

with $\mathscr{F}$, $\mathscr{E}/\mathscr{F}$ locally free, such that $F_\phi(\mathscr{F}) \subseteq \pi \cdot \mathscr{E}$. Let $\Gamma^k(\mathscr{E}, \mathscr{F})$ denote the $\mathcal{O}^\infty_X$-submodule of $\mathrm{Sym}^k \mathscr{E} \otimes \mathbf{Q}$ generated locally by sections of the form

$$\frac{e_1^{r_1} \dots e_m^{r_m} \cdot f_1^{s_1} \dots f_n^{s_n}}{s_1! \dots s_n!}$$

where $e_i \in \mathscr{E}$, $f_j \in \mathscr{F}$, and r_i, $s_j \geqq 0$, $\sum r_i + \sum s_j = k$. Then $\Gamma^k(\mathscr{E}, \mathscr{F})$ has a natural filtration

$$\Gamma^k(\mathscr{E}, \mathscr{F}) = \mathrm{Fil}^0 \supseteq \mathrm{Fil}^1 \dots \supseteq \mathrm{Fil}^{k+1} = 0 \tag{7.4.2}$$

satisfying the conditions of § 7.3, with successive quotients

$$\mathrm{Fil}^i/\mathrm{Fil}^{i+1} \simeq \Gamma^i(\mathscr{F}) \otimes \mathrm{Sym}^{k-i}(\mathscr{E}/\mathscr{F}).$$

Observe that if there is a pairing on $\mathscr{E}$ with respect to which (7.4.1) is autodual, then it induces a natural pairing on $\Gamma^k(\mathscr{E}, \mathscr{F})$ for which (7.4.2) is autodual. (Of course this is not true for either $\mathrm{Sym}^k \mathscr{E}$ or $\Gamma^k(\mathscr{E})$, in general.)

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