Classical motives

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Introduction

This paper is based on a talk given on the first day of the conference, entitled "Examples". Its purpose was to give an elementary survey of some examples of classical motives from a purely geometric standpoint (that is, without recourse to any cohomological methods). This report has the same aims and consequent shortcomings. There are four main parts:

(i) An account of the definitions and basic properties of motives. This is included for completeness; it can all be found, in a somewhat different form but in greater detail, in the definitive accounts of Kleiman [**12**] and Manin [**19**].

(ii) The relation between the motive of a curve and its Jacobian variety, due in essence to Weil.

(iii) The motives $h^1(X)$ and $h^{2d-1}(X)$ for a variety X of dimension d. Here we follow Murre [**23**], with minor modifications.

(iv) An elementary proof of the canonical decomposition of the motive of an abelian variety, inspired by the work of Deninger and Murre [**6**] and Künnemann [**15, 16**].

There is therefore little which is original contained in these pages. I have given a more or less complete proof of Murre's result in §4, so as to make the comparison between the different decompositions in 5.3. Also included is a proof (3.5) of the unsurprising fact that the category of motives constructed using rational equivalence of cycles is in general not an abelian category. Otherwise proofs have been sketched or omitted.

A word about the notion of motive used in this paper is appropriate. Grothendieck's definition of a motive involves replacing the category of varieties by a category with the same objects, but whose morphisms are correspondences, modulo a suitable equivalence relation. Depending on the equivalence relation chosen, one gets rather different theories (see 6.2 below for a discussion).

It is usual to take numerical or homological equivalence, obtaining motive categories $\mathcal{M}_k^{\mathrm{num}}$ and $\mathcal{M}_k^{\mathrm{hom}}$, but in this report we concentrate more on $\mathcal{M}_k^{\mathrm{rat}}$, the category of motives for rational equivalence (sometimes called Chow motives). One reason to do this is that rational equivalence is the finest adequate equivalence on cycles, so that $\mathcal{M}_k^{\mathrm{rat}}$ is in some sense universal.

Another reason is that many interesting cohomology-like functors factor through $\mathcal{M}_k^{\mathrm{rat}}$ but not in general through $\mathcal{M}_k^{\mathrm{num}}$ or $\mathcal{M}_k^{\mathrm{hom}}$. Examples of such functors include absolute ℓ-adic cohomology $H^*(X_{/k}, \mathbf{Q}_\ell(n))$, Deligne-Beilinson (absolute Hodge) cohomology, and motivic cohomology (see Nekovář's paper in these Proceedings for details of both of these). In particular, to formulate Beilinson's conjectures for motives it is at present necessary to work in $\mathcal{M}_k^{\mathrm{rat}}$.

The obvious disadvantage of using rational equivalence is that of not being in an abelian (or even conjecturally abelian) category. Arguments that are trivial when one uses numerical or homological equivalence can become cumbersome. Philosophically, one has to study $\mathcal{M}_k^{\mathrm{rat}}$ as well as $\mathcal{M}_k^{\mathrm{hom}}$ for the same reasons that in cohomology it is often necessary to work with complexes (i.e., in the derived category) rather than simply with cohomology groups. Unfortunately we do not yet have the whole derived category of motives to play with.

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1. Formal properties of motives

1.1. We fix a base field k. Let $\mathcal{V}_k$ denote the category of smooth and projective k-schemes. We refer to the objects of $\mathcal{V}_k$ as simply *varieties*—note that they are not necessarily irreducible or even equidimensional. However we will only consider connected varieties from §3 onwards. If $\phi: Y \to X$ is a morphism in $\mathcal{V}_k$, we denote by $\Gamma_\phi \subset X \times Y$ its graph.

1.2. For a variety X and an integer d, the cycle group $\mathcal{Z}^d(X)$ is the free abelian group generated by irreducible subvarieties of X of codimension d. Central to the definition of motives is the choice of an *adequate equivalence relation* $\sim$ on cycles. For a precise definition of what that entails we refer to [**12**]; there are three important examples of adequate equivalence relations:

(i) rational equivalence;

(ii) homological equivalence, with respect to a (fixed) Weil cohomology theory H^*;

(iii) numerical equivalence.

We write (with apologies for the notation) $A^d(X) = \mathcal{Z}^d(X) \otimes \mathbf{Q}/\sim$, where $\sim$ is a fixed adequate equivalence relation. If Z is a cycle on X we write $[Z]$ for its class in $A^d(X)$. The definition of adequate implies that the groups $A^d(X)$ enjoy a number of functorial properties:—

- For a morphism $\phi: X \to Y$ there are pullback and push-forward maps $\phi^*: A^*(Y) \to A^*(X)$, $\phi_*: A^*(X) \to A^{*+\dim Y - \dim X}(Y)$.
- There is a product structure $A^d(X) \otimes A^e(X) \to A^{d+e}(X)$ given by intersection theory.

In this report we will be mostly concerned with the case when $\sim$ is rational equivalence, in which case $A^d(X)$ is the usual codimension d Chow group tensored with $\mathbf{Q}$. The motives arising from this choice are sometimes called *Chow motives*. For further comments on the effect of choosing a different equivalence relation, see §§3.5, 6.2 below.

1.3. Let X, Y be in $\mathcal{V}_k$. Define $\mathrm{Corr}^r(X,Y)$, the group of correspondences of degree r from X to Y, as follows. If X is purely d-dimensional, then

$$\mathrm{Corr}^r(X,Y) = A^{d+r}(X \times Y).$$

In general, let $X = \coprod X_i$ where each X_i is a connected variety, and set

$$\mathrm{Corr}^r(X,Y) = \bigoplus \mathrm{Corr}^r(X_i, Y) \subset A^*(X \times Y).$$

For varieties X, Y and Z there is a composition of correspondences

$$\mathrm{Corr}^r(X,Y)\otimes\mathrm{Corr}^s(Y,Z)\to\mathrm{Corr}^{r+s}(X,Z)$$

defined by

$$f\otimes g\mapsto g\circ f=p_{13*}(p_{13}^*f\cdot p_{23}^*g)$$

where p_{ij} are the projections of $X\times Y\times Z$ onto products of factors.

1.4. The category $\mathcal{M}_k$ of k-motives is now defined as follows: an object of $\mathcal{M}_k$ is a triple (X,p,m) where X is a k-variety, m is an integer and $p=p^2\in\mathrm{Corr}^0(X,X)$ is an idempotent. If (X,p,m) and (Y,q,n) are motives, then

$$\mathrm{Hom}_{\mathcal{M}_k}((X,p,m),(Y,q,n))=p\,\mathrm{Corr}^{n-m}(X,Y)q\subset\mathrm{Corr}^*(X,Y)$$

and composition is given by composition of correspondences.

This is the definition of the category of motives as given, for example, in [**9**]. It is equivalent to the definition found in other places (for example [**19**] and [**12**]) because of the following elementary fact:

Lemma 1.5. *Let p, q be commuting endomorphisms of an abelian group B, which are idempotents. Then the map $x\mapsto pqx$ gives an isomorphism*

$$\frac{\ker(p-q)}{\ker p\cap\ker q}\xrightarrow{\sim}pqB.$$

Theorem 1.6. *$\mathcal{M}_k$ is an additive, $\mathbf{Q}$-linear category, which is pseudoabelian.*

If (X,p,m) and (Y,q,n) are motives with $m=n$ then their direct sum is defined to be

$$(X,p,m)\oplus(Y,q,m)\overset{\mathrm{def}}{=}(X\textstyle\coprod Y,p\oplus q,m).$$

It is immediate that this satisfies the necessary properties. The general construction of the direct sum is given in §1.14 below; the reader can check that the proof is not circular.

Now recall that an additive category such as $\mathcal{M}_k$ is pseudoabelian if every projector $f\in\mathrm{End}\,M$ has an image, and the canonical map $\mathrm{Im}(f)\oplus\mathrm{Im}(1-f)\to M$ is an isomorphism. In this case it follows formally from the definition that there is a decomposition

$$M=(X,pfp,m)\oplus(X,p-pfp,m)$$

if $M=(X,p,m)$ is a motive and $f=pfp\in\mathrm{End}\,M$ is a projector.

1.7. *Remark.* One should bear in mind that the category $\mathcal{M}_k$ is in general *not* abelian (see 3.5 below), and some caution must therefore be exercised when discussing kernels of arbitrary morphisms. In view of this it is worth making the following trivial observation:

> If $f\colon M\to N$ is a morphism in $\mathcal{M}_k$ (or any pseudoabelian category) which has a left inverse, then it has an image, which is (non-canonically!) a direct factor of N, and $M\to\mathrm{Im}\,f$ is an isomorphism.

In fact if g is any such retract of f, then fg is an idempotent, so $fgN \subset N$ exists, and $f\colon M \xrightarrow{\sim} fgN$. Dually, if f has a section, then N is canonically a quotient object of M, and non-canonically a direct factor of M.

1.8. There is a functor

$$h\colon \mathcal{V}_k^{\mathrm{opp}} \to \mathcal{M}_k$$

which on objects is given by $h(X) = (X, \mathrm{id}, 0)$, and on morphisms $\phi\colon Y \to X$ by

$$h(\phi) = [\Gamma_\phi] \in \mathrm{Corr}^0(X,Y) = \mathrm{Hom}(h(X), h(Y))$$

(usually one writes ϕ^* for $h(\phi)$).

1.9. There is a tensor product on $\mathcal{M}_k$, defined on objects by

$$(X,p,m) \otimes (Y,q,n) = (X \times Y, p \otimes q, m+n)$$

and on morphisms by

$$p_1 f_1 q_1 \otimes p_2 f_2 q_2 = (p_1 \otimes p_2)(f_1 \otimes f_2)(q_1 \otimes q_2) \in \mathrm{Corr}^*(X_1 \times X_2, Y_1 \times Y_2)$$
$$\text{if } p_i f_i q_i\colon (X_i, p_i, m_i) \to (Y_i, q_i, n_i).$$

One writes $\mathbb{1} = (\mathrm{Spec}\, k, \mathrm{id}, 0)$ (the unit motive) and $\mathbb{L} = (\mathrm{Spec}\, k, \mathrm{id}, -1)$ (the Lefschetz motive). Then $\mathbb{1}$ is the identity for the tensor product, and every motive is a direct factor of $h(X) \otimes \mathbb{L}^{\otimes n}$ for suitable X and n; in fact, if $p \in \mathrm{Corr}^0(X,X) = \mathrm{End}\, h(X)$ is a projector, then

$$(X,p,m) = ph(X) \otimes \mathbb{L}^{\otimes -m} \subset h(X) \otimes \mathbb{L}^{\otimes -m}.$$

In future we write simply $\mathbb{L}^n$ for $\mathbb{L}^{\otimes n}$, and for a morphism $f\colon M \to N$ of motives we will write f also for the tensor product $M \otimes \mathbb{L}^{\otimes n} \to N \otimes \mathbb{L}^{\otimes n}$.

The diagonal $\Delta\colon X \to X \times X$ defines a product structure on $h(X)$, given by the composite

$$m_X\colon h(X) \otimes h(X) = h(X \times X) \xrightarrow{\Delta^*} h(X).$$

1.10. If $\phi\colon Y \to X$ and X and Y are purely d- and e-dimensional, respectively, then the transpose $[{}^t\Gamma_\phi] \in A^d(Y \times X)$ is a correspondence of degree $d-e$ from Y to X, and so defines a morphism

$$\phi_*\colon h(Y) \to h(X) \otimes \mathbb{L}^{e-d}.$$

Suppose that $d = e$ and that ϕ is generically finite, of degree r. Then the composite $\phi_* \circ \phi^* \in \mathrm{End}\, h(X)$ is multiplication by r. In fact,

$$\phi_* \circ \phi^* = p_{13*}(p_{13}^*[\Gamma_\phi] \cdot p_{24}^*[{}^t\Gamma_\phi]) = p_{13*}(\phi, \mathrm{id}, \phi)_*[Y] = r[\Delta_X]$$

where $\Delta_X \subset X \times X$ is the diagonal.

If $f \in \mathrm{Corr}^0(X,Y) = \mathrm{Hom}(h(X), h(Y))$ and $\phi\colon X \to X'$, $\psi\colon Y \to Y'$ are morphisms in $\mathcal{V}_k$ then the formula for the composite map of motives is

$$\psi_* \circ f \circ \phi^* = (\phi \times \psi)_* f.$$

Similarly if $\phi: X' \to X$ and $\psi: Y' \to Y$ then

$$\psi^* \circ f \circ \phi_* = (\phi \times \psi)^* f.$$

Because of the inherent confusion in formulae of this type, we will attempt to distinguish between operations on cycles (direct and inverse image and intersection) and on correspondences by frequent use of the usual symbols $\cdot$ and $\circ$. In particular, the notation c^2 will generally denote $c \cdot c$ and not $c \circ c$.

1.11. Suppose that X is irreducible of dimension d, and that there is a k-rational point $x \in X(k)$. Denote by $\alpha: X \to \operatorname{Spec} k$ the structural morphism. Then $x^* \circ \alpha^* = \mathrm{id}$, so by the remark in 1.7, the map

$$\alpha^*: \mathbb{1} \to h(X)$$

is a subobject of $h(X)$. Similarly, as $\alpha_* \circ x_* = \mathrm{id}$, the map

$$\alpha_*: h(X) \to \mathbb{L}^d$$

is a quotient object.

More generally, let X be irreducible, and write $k' = \Gamma(X, \mathcal{O}_X)$, and $\alpha: X \to \operatorname{Spec} k'$ for the structural morphism. Let k''/k' be a finite separable extension such that there exists $x \in X(k'')$. Write $\gamma: \operatorname{Spec} k'' \to \operatorname{Spec} k'$ for the natural map. Then by 1.10, the composite

$$h(\operatorname{Spec} k') \xrightarrow{\alpha^*} h(X) \xrightarrow{x^*} h(\operatorname{Spec} k'') \xrightarrow{\gamma_*} h(\operatorname{Spec}(k')$$

is multiplication by $[k'':k']$. Therefore

$$\alpha^*: h(\operatorname{Spec} k') \to h(X)$$

defines a subobject of $h(X)$, denoted $h^0(X)$. It is well-defined as a subobject up to unique isomorphism.

We denote the quotient of $h(X)$ by $h^0(X)$ as $h^{\geq 1}(X)$. The quotient exists because $h^0(X)$ is (non-canonically) a direct factor of $h(X)$. The choice of a point x determines a splitting $h(X) = h^0(X) \oplus h^{\geq 1}(X)$.

Remark: When $\sim$ is homological or numerical equivalence, the class of Γ_x in $A^*(X \otimes_{k'} k'')$ is independent of x, and so the splitting is in this case canonical.

In particular, if X is absolutely irreducible, then $h^0(X) = \mathbb{1}$ is a direct summand of $h(X)$. We can use this to eliminate the nuisance of dealing with varieties with components of different dimension as follows:

Proposition 1.12. *Any motive M can be expressed as a direct factor of some $h(X') \otimes \mathbb{L}^n$, with X' equidimensional.*

Proof. It is enough to show this for $M = h(X)$. Let $X = \coprod X_i$ be the decomposition of X into its components. Choose integers $d_i \geq 0$ such that $\dim X_i + d_i$ does not depend on i. We then have

$$h(X) = \bigoplus h(X_i) = \bigoplus \big(h(X_i) \otimes h^0(\mathbf{P}^{d_i})\big)$$

and we have seen that this is a direct factor of $\bigoplus h(X_i \otimes \mathbf{P}^{d_i}) = h(\coprod X_i \times \mathbf{P}^{d_i})$.

1.13. Continuing with the assumptions of 1.11, let $d = \dim X$. Then the composite

$$h(\operatorname{Spec} k') \otimes \mathbb{L}^d \xrightarrow{x_* \gamma^*} h(X) \xrightarrow{\alpha_*} h(\operatorname{Spec} k') \otimes \mathbb{L}^d$$

is multiplication by $[k'' \colon k']$, and so

$$\alpha_* \colon h(X) \to h(\operatorname{Spec} k') \otimes \mathbb{L}^d$$

is a quotient object of $h(X)$, denoted $h^{2d}(X)$.

If X is irreducible and has a rational point $x \in X(k)$ then

$$h^0(X) \simeq (X, \{x\} \times X, 0) \simeq \mathbb{1}$$
$$h^{2d}(X) \simeq (X, X \times \{x\}, 0) \simeq \mathbb{L}^d$$

Assuming only that X is irreducible, let Z be any zero-cycle on X whose degree d is positive. Then $p_0 = (1/d)[Z \times X] \in A^d(X \times X)$ is an idempotent, and gives rise to canonical isomorphisms

$$h^0(X) \simeq (X, p_0, 0), \quad h^{2d}(X) \simeq (X, p_{2d}, 0) \quad \text{where } p_{2d} = {}^t p_0.$$

For example, consider $X = \mathbf{P}^1$. Since the cycles $\mathbf{P}^1 \times \{x\}$, $\{x\} \times \mathbf{P}^1$ on $\mathbf{P}^1 \times \mathbf{P}^1$ do not depend on the choice of a rational point $x \in \mathbf{P}^1(k)$, and since their sum is rationally equivalent to the diagonal, we have canonically

$$h(\mathbf{P}^1) = h^0(\mathbf{P}^1) \oplus h^2(\mathbf{P}^1) = \mathbb{1} \oplus \mathbb{L}.$$

In most treatments this is taken as the definition of $\mathbb{L}$.

1.14. We can now construct arbitrary direct sums in $\mathcal{M}_k$. Let $M = (X, p, m)$ and $N = (Y, q, n)$ be motives. Assume that $m < n$. Then

$$M = (X, p, n) \otimes L^{n-m} = (X, p, n) \otimes h^2(\mathbf{P}^1)^{n-m} = (X \times (\mathbf{P}^1)^{n-m}, p', n)$$

for a suitable projector p', and the direct sum of M and N is then

$$\left(X \times (\mathbf{P}^1)^{n-m} \coprod Y, p' \oplus q, n\right)$$

as in 1.6 above.

1.15. There is an involution ${}^\vee \colon \mathcal{M}_k^{\mathrm{opp}} \to \mathcal{M}_k$, defined on objects by

$$(X, p, m)^\vee = (X, {}^t p, d - m) \quad \text{if } X \text{ is purely } d\text{-dimensional}$$

and on morphisms as the transpose of correspondences. In particular $h(X)^\vee = h(X) \otimes \mathbb{L}^{-d}$ ("Poincaré duality"). Clearly $M^{\vee\vee} = M$ for every M, and the standard formula

$$\operatorname{Hom}(M \otimes N, P) = \operatorname{Hom}(M, N^\vee \otimes P)$$

is trivially seen to hold. Then one can define an internal Hom in $\mathcal{M}_k$ by the formula $\mathbf{Hom}(M, N) = M^\vee \otimes N$. These constructions give $\mathcal{M}_k$ the structure of a rigid additive tensor category, once commutativity and associativity constraints are defined. (This tensor structure gives what in the Tannakian setting is usually called the false category of motives—see [**5**], p.200ff.)

1.16. It is sometimes convenient to construct a decomposition of motives over an extension field (this is needed in §4, for instance). Because of what was said in 1.10, the groups A^* satisfy Galois descent: if k'/k is a finite Galois extension then $A^*(X)$ is the subspace of invariants of $A^*(X \otimes k')$ under $\mathrm{Gal}(k'/k)$. Therefore Galois-invariant decompositions of motives descend to the ground field: more precisely, we have the following easy result.

Lemma 1.17. *Let $X \in \mathcal{V}_k$ be purely d-dimensional, and let k'/k be a finite Galois extension of degree m. Write $X' = X \otimes_k k'$, and denote by β the canonical map $X' \times_{k'} X' \to X \times_k X$. Suppose that $p'_1, \dots, p'_r \in A^d(X' \times_{k'} X')$ is a complete system of orthogonal idempotents, which are invariant under* $\mathrm{Gal}(k'/k)$*. Then the correspondences $p_i = (1/m)\beta_*(p'_i)$ form a complete system of orthogonal idempotents in $A^d(X \times_k X)$.*

2. Cycles and Manin's identity principle

2.1. An immediate consequence of the definition of motives is that the cycle class groups A^* can be interpreted in terms of $\mathcal{M}_k$: in fact

$$A^d(X) = \mathrm{Hom}(\mathbb{L}^d, h(X)).$$

If $\xi \in A^d(X)$, then we write $\xi_* \colon \mathbb{L}^d \to h(X)$ for the corresponding mapping of motives, and $\xi^* \colon h(X) \to \mathbb{L}^{\dim X - d}$ for its transpose. (If ξ is the class of a rational point $x \in X(k)$, this then agrees with the notations x_* and x^*.) There is then defined a morphism $\bar{\xi} \colon h(X) \otimes \mathbb{L}^d \to h(X)$ ("cup-product with ξ") by

$$h(X) \otimes \mathbb{L}^d \xrightarrow{\mathrm{id} \otimes \xi_*} h(X) \otimes h(X) \xrightarrow{m_X} h(X).$$

The morphism $\bar{\xi}$ is represented by the cycle class

$$\Delta_*(\xi) \in \mathrm{Corr}^d(X, X) \subset A^*(X \times X).$$

If η is another cycle class then

$$\bar{\xi} \circ \bar{\eta} = \overline{\xi \cdot \eta} \quad \text{and} \quad \bar{\xi} \circ \eta_* = (\xi \cdot \eta)_*. \tag{2.1.1}$$

In possible conflict with the convention of 1.10 the notation $\bar{\xi}^i$ will mean the correspondence induced by ξ^i, which is the i-th iterate of $\bar{\xi}$ and not the i-fold intersection. We define for any motive M and $d \in \mathbf{Z}$ the cycle groups of M by

$$A^d(M) \stackrel{\mathrm{def}}{=} \mathrm{Hom}(\mathbb{L}^d, M).$$

Then $A^*(-)$ is a $\mathbf{Z}$-graded additive functor from $\mathcal{M}_k$ to $\mathrm{Vect}_{\mathbf{Q}}$, the category of $\mathbf{Q}$-vector spaces. For $M = h(X)$ this therefore agrees with the previous notation.

2.2. If M and $N \in \mathcal{M}_k$ then

$$\mathrm{Hom}(N, M) = \mathrm{Hom}(\mathbb{1}, M \otimes N^\vee) = A^0(M \otimes N^\vee).$$

Therefore by the Yoneda lemma the functor which attaches to $M \in \mathcal{M}_k$ the functor $A^0(M \otimes -): \mathcal{M}_k \to \mathrm{Vect}_{\mathbf{Q}}$ is fully faithful.

Now any $N \in \mathcal{M}_k$ is a direct factor of some $h(Y) \otimes \mathbb{L}^n$ with $Y \in \mathcal{V}_k$ and $n \in \mathbf{Z}$, and $A^0(M \otimes h(Y) \otimes \mathbb{L}^n) = A^{-n}(M \otimes h(Y))$. Therefore if we denote by $\omega_M: \mathcal{V}_k^{\mathrm{opp}} \to \mathrm{Vect}_{\mathbf{Q}}$ the functor

$$\omega_M(Y) = A^*(M \otimes h(Y))$$

then $M \mapsto \omega_M$ is fully faithful. From this one can deduce (using for example [**20**], II.7.1):

2.3 (Manin's identity principle). *(i) Let f, $g: M \to N$ be morphisms of motives. Then f is an isomorphism if and only if the induced map*

$$\omega_f(Y): A^*(M \otimes h(Y)) \to A^*(N \otimes h(Y))$$

is an isomorphism for every $Y \in \mathcal{V}_k$; and $f = g$ if and only if $\omega_f(Y) = \omega_g(Y)$ for every Y.

(ii) A sequence $0 \longrightarrow M' \xrightarrow{f} M \xrightarrow{g} M'' \longrightarrow 0$ in $\mathcal{M}_k$ is exact if and only if, for every $Y \in \mathcal{V}_k$, the sequence

$$0 \longrightarrow A^*(M' \otimes h(Y)) \xrightarrow{\omega_f(Y)} A^*(M \otimes h(Y)) \xrightarrow{\omega_g(Y)} A^*(M'' \otimes h(Y)) \longrightarrow 0$$

is exact.

2.4. As a first example we calculate the motive of a projective bundle following Manin [**19**]. Let $S \in \mathcal{V}_k$ and let $\mathcal{E}$ be a locally free sheaf on S of constant rank $r + 1 \geq 1$. Let $X = \mathbf{P}_S[\mathcal{E}] \xrightarrow{\pi} S$ be the associated projective bundle, and $\xi = c_1(\mathcal{O}_X(1)) \in A^1(X)$ be the divisor class of the tautological line bundle on X.

Recall that $A^*(X)$ is a free module over $A^*(S)$ (via π^*) with basis $1, \xi, \ldots, \xi^r$, and that the multiplication is given by

$$\xi^{r+1} = \sum_{j=0}^{r} (-1)^{r-j} c_{r-j+1}(\mathcal{E}) \xi^j$$

where $c_i(\mathcal{E}) \in A^i(S)$ are the Chern classes of $\mathcal{E}$. For any integer $n \geq 0$, write $\xi^n = \sum_{j=0}^{r} \theta_{n,j} \xi^j$. Then $\theta_{n,j} \in A^{n-j}(S)$ are given by certain universal polynomials in the Chern classes.

Theorem 2.5. *The map*

$$\sum_{i=0}^{r} \bar{\xi}^i \circ \pi^* : \bigoplus_{i=0}^{r} h(S) \otimes \mathbb{L}^i \longrightarrow h(X)$$

is an isomorphism of motives. In terms of this isomorphism, the product structure on $h(X)$ is given by

$$\big(h(S) \otimes \mathbb{L}^i\big) \otimes \big(h(S) \otimes \mathbb{L}^{n-i}\big) \xrightarrow{m_S} h(S) \otimes \mathbb{L}^n \xrightarrow{(\bar{\theta}_{n,j})} \bigoplus_{j=0}^{r} h(S) \otimes \mathbb{L}^j$$

Proof. Manin's identity principle implies that it is equivalent to know that the corresponding mapping at the level of cycle class groups

$$\bigoplus_{i=0}^{r} A^{*-i}(S \times Y) \longrightarrow A^*(X \times Y)$$

$$(z_i) \mapsto \sum_{i=0}^{r} \xi^i \cdot (\pi \times \mathrm{id}_Y)^*(z_i)$$

is a ring isomorphism for every $Y \in \mathcal{V}_k$. But since the Chern classes of $pr_1^*\mathcal{E}$ on $S \times Y$ are simply $pr_1^* c_i(\mathcal{E})$, this follows from the facts recalled in 2.4 (replacing X/S by $X \times Y/S \times Y$).

2.6. Similar results hold for other varieties which admit cellular decompositions relative to a base (grassmanians, etc...). Suppose that $\pi: X \to S$ is a flat morphism of pure relative dimension n, and that X admits a filtration by closed subschemes $X = X_0 \supset X_1 \supset \ldots$ such that $X_i - X_{i+1}$ is S-isomorphic to the affine space $\mathbf{A}_S^{n-d_i}$, for some $d_i \in \mathbf{Z}$. Then there is an isomorphism

$$\bigoplus_i A^{*-d_i}(S) \longrightarrow A^*(X) \tag{2.6.1}$$

which is functorial with respect to cartesian squares

$$\begin{array}{ccc} X' & \longrightarrow & X \\ \big\downarrow{\scriptstyle \pi'} & & \big\downarrow{\scriptstyle \pi} \\ S' & \longrightarrow & S \end{array}$$

and so by Manin's identity principle $h(X) \simeq \bigoplus h(S) \otimes \mathbb{L}^{d_i}$. See [**14**], particularly the appendix, for a proof of (2.6.1) and further examples. Calculations for many of the classical varieties of this type can be found already in SGA6.

2.7. The identity principle can also be used to calculate the motive of a blowup. Let $Y \subset X$ be a non-singular subvariety which is purely of codimension $r+1 > 1$, and let

$$\begin{array}{ccc} Y' & \stackrel{i'}{\hookrightarrow} & X' \\ \big\downarrow{\scriptstyle \pi'} & & \big\downarrow{\scriptstyle \pi} \\ Y & \stackrel{i}{\hookrightarrow} & X \end{array}$$

be the blowup of X along Y. Then $Y' \to Y$ is a projective bundle; let $\xi \in A^1(Y')$ be the class of the tautological line bundle. Consider the sequence of motives:

$$0 \to h(Y) \otimes \mathbb{L}^{r+1} \xrightarrow{(i_*, \bar{\xi}^r \cdot \pi^*)} h(X) \oplus (h(Y) \otimes \mathbb{L}) \xrightarrow{\pi'^* \oplus (-i'_*)} h(X') \to 0. \tag{1}$$

There is a retract of the first arrow given by $0 \oplus \pi_*: h(X) \oplus (h(Y) \otimes \mathbb{L}) \to h(Y) \otimes \mathbb{L}^{r+1}$.

Theorem 2.8 (Manin). *The sequence* (1) *is split exact.*

The proof relies on the identity principle and the behaviour of A^* under blowups. For details we refer the reader to [**19**]. This result was used there by Manin to prove the Weil conjectures for unirational varieties of dimension three.

3. Curves and abelian varieties (I)

3.1. One reason for Grothendieck's introduction of motives was to serve as analogues of the Jacobian of a curve in higher dimensions. Here we explain the precise relationship between the motive of a curve and its Jacobian. *For the rest of the paper all varieties will be assumed connected.*

3.2. Let $X \in \mathcal{V}_k$ be a curve, with field of constants k'. As explained in §1.11–1.13 there is a submotive $h^0(X) \simeq h(\operatorname{Spec} k')$ and a quotient motive $h^2(X) \simeq h(\operatorname{Spec} k') \otimes \mathbb{L}$ of $h(X)$. The choice of a zero-cycle Z on X of positive degree determines projectors p_0, $p_2 \in \operatorname{Corr}^0(X,X) = \operatorname{End} h(X)$ with $h^i(X) \simeq (X, p_i, 0)$ for $i = 0$, 2 as in 1.13.

Let $p_1 = 1 - p_0 - p_2 \in \operatorname{Corr}^0(X,X)$, and write $h^1(X) = (X, p_1, 0) \in \mathcal{M}_k$. Then there is a direct sum decomposition

$$h(X) = h^0(X) \oplus h^1(X) \oplus h^2(X)$$

which in general depends on the choice of Z (more specifically, its class in $A^1(X)$). However $h^1(X)$ is well-defined up to unique isomorphism; in fact, it is the kernel of the composite map $h^{\geq 1}(X) \to h(X) \to h^2(X)$, so is well-defined as a subquotient of $h(X)$. The theory of the motives $h^1(X)$ is essentially that of Jacobian varieties:

Proposition 3.3. *If X, X' are curves with Jacobian varieties J, J', then*

$$\operatorname{Hom}(h^1(X), h^1(X')) = \operatorname{Hom}(J, J') \otimes \mathbf{Q}$$

and

$$\operatorname{Hom}(L, h^1(X)) = \begin{cases} 0 & \text{if } \sim \text{ is numerical (or homological) equivalence} \\ J(k) \otimes \mathbf{Q} & \text{if } \sim \text{ is rational equivalence.} \end{cases}$$

(Some explanation for the dichotomy will be given at the end of the paper.)

Proof. By [**27**] we have $A^1(X \times X') = A^1(X) \oplus A^1(X') \oplus \operatorname{Hom}(J, J') \otimes \mathbf{Q}$ giving the first formula. For the second, we have $\operatorname{Hom}(L, h(X)) = A^1(X)$ and $\operatorname{Hom}(L, h^0(X)) = 0$, while $\operatorname{Hom}(L, h^2(X)) = \mathbf{Q}$ is generated by the class of a closed point in $A^1(X)$. Hence $\operatorname{Hom}(L, h^1(X)) = \ker(\deg: A^1(X) \to \mathbf{Q})$, giving the second formula.

Corollary 3.4. *Let $\mathcal{C}_k$ be the full subcategory of $\mathcal{M}_k$ whose objects are direct summands of motives of the form $h^1(X)$, where X is of dimension one. Then $\mathcal{C}_k$ is equivalent to the category of abelian varieties over k up to isogeny.*

This follows from the fact that every abelian variety is an abelian subvariety of a Jacobian, and Poincaré's complete reducibility theorem. (Note that the result is independent of the particular equivalence relation chosen to define $\mathcal{M}_k$.)

Corollary 3.5. *Assume that k is not contained in the algebraic closure of a finite field, and that $\sim$ is rational equivalence. Then $\mathcal{M}_k^{\mathrm{rat}}$ is not an abelian category.*

Proof. . Under the hypothesis on k, one can find an elliptic curve E/k and a point $P \in E(k)$ of infinite order. Let $\xi \in A^1(E)$ be the class of the divisor $(P)-(0)$. Then the morphism $\xi_*: \mathbb{L} \to h^1(E)$ is non-zero by 3.3. The composite $\xi_* \circ \xi^*: h^1(E) \otimes \mathbb{L} \to h^1(E)$ is represented by the class of the zero-cycle $\eta(P,P)+(0,0)-(P,0)-(0,P)$ on $E \times E$. If $P = 2Q$ for $Q \in E(\bar{k})$, then in $CH^2(E \times E_{/\bar{k}})$ we can write

$$\eta = [(P,P)+(0,0)-2(Q,Q)]+[2(Q,Q)-(P,0)-(0,P)]$$

and this is rationally equivalent to zero. Therefore η is a torsion class in $CH^2(E \times E)$, whence $\xi_* \circ \xi^* = 0$. Thus ξ_* is not a monomorphism. If $\mathcal{M}_k^{\mathrm{rat}}$ were abelian then $\ker \xi_*$ would be a proper subobject of $\mathbb{L}$. Tensoring by $\mathbb{L}^{-1}$ this would imply that $\mathbb{1}$ had a nontrivial subobject. But the unit object in a rigid abelian tensor category is completely decomposable, by [**5**] Prop. 1.17, so as $\operatorname{End} \mathbb{1} = \mathbf{Q}$ this is impossible.

3.6. In the next section we will give a generalisation of the above to higher dimensional varieties. First we recall the bare essentials of the theory of Albanese and Picard varieties (see for example [**17**]).

3.7. Let X be a variety over k. The Albanese variety J_X is an abelian variety over k equipped with a morphism $\nu: X \times X \to J_X$, satisfying the following universal property: any morphism $\phi: X \times X \to A$ to an abelian variety such that $\phi(x,y)+\phi(y,z)=\phi(x,z)$ (or equivalently, $\phi(x,x)=0$) factors uniquely as $\phi = \beta \circ \nu$ for some homomorphism $\beta: J_X \to A$.

If X has a k-rational point x_0 and $\gamma: X \to J_X$ is the morphism $\gamma(x)=\nu(x,x_0)$, then γ satisfies the usual universal property for morphisms from X to abelian varieties. If $\psi: X \to Y$ is a morphism, then the universal property defines a homomorphism $J_\psi: J_X \to J_Y$, so that J is a covariant functor.

3.8. Let $\mathcal{P}_X$ be the functor on varieties given by

$$\mathcal{P}_X(S) = \frac{\left\{\begin{matrix}\text{isomorphism classes of line bundles } \mathcal{L} \text{ on } X \times S \text{ such that} \\ \mathcal{L}\big|_{X \times s} \text{ is algebraically equivalent to 0 for all } s \in S(\bar{k})\end{matrix}\right\}}{\left\{pr_2^* \mathcal{G} \text{ for } \mathcal{G} \in \operatorname{Pic} S\right\}}$$

Recall that the Picard variety P_X is an abelian variety over k such that there are functorial injections

$$P_X(S) \hookrightarrow \mathcal{P}_X(S)$$

for varieties S, which are bijections whenever $X(S)$ is non-empty. (See for example §5(d) in chapter 0 of [**22**].) P_X is contravariant in X, and is (functorially) isomorphic to the dual abelian variety of J_X.

Theorem 3.9. *Let X and $Y \in \mathcal{V}_k$ be varieties of dimensions d and e. Then:*

(i)
$$\operatorname{Hom}(J_X, P_Y) \otimes \mathbf{Q} \simeq \frac{A^1(X \times Y)}{pr_1^* A^1(X) + pr_2^* A^1(Y)}$$

(ii) Let $\xi \in A^d(X)$, $\eta \in A^e(Y)$ be 0-cycles of positive degree. Then there is an isomorphism

$$\Omega : \operatorname{Hom}(J_X, P_Y) \otimes \mathbf{Q} \xrightarrow{\sim} \left\{ c \in A^1(X \times Y) \,\middle|\, c \circ \zeta_* = 0 \text{ and } \eta^* \circ c = 0. \right\}$$

Recall the proof: first assume that $X(k)$ and $Y(k)$ are nonempty, and let $x_0 \in X(k)$. Then by the universal properties we have

$$\begin{aligned} \operatorname{Hom}(J_X, P_Y) &= \{ \phi : X \to P_Y \mid \phi(x_0) = 0 \} \\ &= \frac{\left\{ \begin{array}{c} \text{isomorphism classes of line bundles } \mathcal{L} \text{ on} \\ X \times Y \text{ such that } \mathcal{L}\big|_{\{x_0\} \times Y} \simeq \mathcal{O}_Y \end{array} \right\}}{\left\{ pr_1^* \mathcal{G} \text{ for } \mathcal{G} \in \operatorname{Pic} X \right\}} \\ &= \frac{\operatorname{Pic} X \times Y}{pr_1^* \operatorname{Pic} X + pr_2^* \operatorname{Pic} Y} \end{aligned}$$

by the seesaw theorem. In general this will hold when k is replaced by a finite Galois extension k'/k. Tensoring with $\mathbf{Q}$ and taking invariants under $\operatorname{Gal}(k'/k)$ then yields (i). The isomorphism (ii) is obtained by combining (i) and seesaw.

The following easy proposition gives the functorial behaviour of Ω:

Proposition 3.10. *Let $\phi : X' \to X$, $\psi : Y' \to Y$ be morphisms of varieties. Choose positive zero-cycles ξ', η' on X', Y' with direct images ξ, η on X and Y. If $\beta : J_X \to P_Y$ is a homomorphism, then*

$$\Omega(P_\psi \circ \beta) = \psi^* \circ \Omega(\beta) \quad \text{and} \quad \Omega(\beta \circ J_\phi) = \Omega(\beta) \circ \phi_*$$

(where Ω denotes the isomorphism of 3.9(ii) with respect to the chosen 0-cycles.)

4. The motives $h^1(X)$ and $h^{2d-1}(X)$

4.1. Let X/k be a variety of dimension d, and fix a projective embedding of X of degree m. Let $\xi \in A^1(X)$ be the class of a hyperplane section. Following Murre [**23**] we will define projectors p_1, $p_{2d-1} \in A^d(X \times X)$ such that the corresponding motives $h^i(X) = (X, p_i, 0)$ satisfy the analogue of the hard Lefschetz theorem:

$$\bar{\xi}^{d-1}: h^1(X) \xrightarrow{\sim} h^{2d-1}(X) \otimes \mathbb{L}^{1-d}$$

and are closely related to the Picard and Albanese varieties of X. This generalises work of Grothendieck, Kleiman and Lieberman, who proved this when $\sim$ is homological equivalence (see [**11**], Appendix to §2).

4.2. Assume from now on that there is a 1-dimensional linear section C of X which is a smooth (and connected) curve. This does not involve loss of generality, since one can always find such a section after a finite base extension k'/k. Then the projectors p_i may be constructed over k' and descended to k by Lemma 1.17.

Let Z be a 0-cycle on C which is cut out by a hyperplane, and let $\zeta \in A^d(X)$ be the class of Z. Write $i: C \hookrightarrow X$ for the embedding. Then

$$\bar{\xi}^{d-1} = i_* \circ i^*: h(X) \longrightarrow h(X) \otimes \mathbb{L}^{1-d}.$$

4.3. The functoriality of the Picard and Albanese varieties defines a composite homomorphism

$$\alpha: P_X \xrightarrow{P_i} P_C = J_C \xrightarrow{J_i} J_X.$$

The construction relies upon the theorem of Weil ([**28**], cor. 1 to thm. 7) that α is an isogeny, and does not depend on the choice of section C. Choose $n \geq 1$ and $\beta: J_X \to P_X$ such that $\alpha \circ \beta = [\times n]$. Since duality interchanges the functors P and J, one has $\widehat{\alpha} = \alpha$ and $\widehat{\beta} = \beta$.

By 3.9, β corresponds to a cycle $\widetilde{\beta} := \Omega(\beta) \in A^1(X \times X) = \mathrm{Hom}(h(X) \otimes \mathbb{L}^{1-d}, h(X))$ satisfying $\widetilde{\beta} \circ \zeta_* = 0$ and $\zeta^* \circ \widetilde{\beta} = 0$. Since $\widehat{\beta} = \beta$ we have ${}^t\widetilde{\beta} = \widetilde{\beta}$. Normalise the projectors p_0, p_{2d} of section 1.13 above by $p_0 = \frac{1}{m}[Z \times X] = {}^t p_{2d}$, and define

$$p_1^? = \frac{1}{n} \widetilde{\beta} \circ \bar{\xi}^{d-1} = \frac{1}{n} \widetilde{\beta} \cdot [C \times X] = \frac{1}{n} \widetilde{\beta} \circ i_* \circ i^*, \text{ and}$$
$$p_{2d-1}^? = {}^t p_1^? = \frac{1}{n} i_* \circ i^* \circ \widetilde{\beta}.$$

Define $p_1 = p_1^? \circ (1 - \frac{1}{2} p_{2d-1}^?)$ and $p_{2d-1} = {}^t p_1$. (We will show that $p_1 = p_1^?$ if $d > 2$; see the proof of (i) below.) Let $h^i(X) = (X, p_i, 0) \in \mathcal{M}_k$.

Theorem 4.4. *(i)* p_0, p_1, p_{2d-1}, p_{2d} *are orthogonal idempotents.*

(ii) The composite morphism

$$h^1(X) \hookrightarrow h(X) \xrightarrow{\bar{\xi}^{d-1}} h(X) \otimes \mathbb{L}^{1-d} \twoheadrightarrow h^{2d-1}(X) \otimes \mathbb{L}^{1-d}$$

is an isomorphism.

(iii) Assume $\sim$ is rational equivalence. Let $A^r(X)^0 \subset A^r(X)$ be the subgroup of cycle classes numerically equivalent to zero, and let $\mathrm{Alb}\colon A^d(X)^0 \to J_X(k)\otimes \mathbf{Q}$ *be the Albanese map. The cycle class groups of $h^i(X)$ are given as follows:*

$$A^r(h^1(X)) = \begin{cases} 0 & \text{if } r \neq 1 \\ A^1(X)^0 & \text{if } r = 1; \end{cases}$$
$$A^r(h^{2d-1}(X)) = \begin{cases} 0 & \text{if } r \neq d \\ A^d(X)^0/\ker(\mathrm{Alb}) & \text{if } r = d. \end{cases}$$

Proof. (i) We first consider the $p_i^?$. We have

$$n^2 p_1^? \circ p_1^? = \widetilde{\beta}\circ i_* \circ i^* \circ \widetilde{\beta} \circ i_* \circ i^*.$$

Now if $f = i^* \circ \widetilde{\beta} \circ i_* \in A^1(C\times C)$ and $g = \widetilde{\beta}\circ i_* \in A^1(C\times X)$ then $\Omega^{-1}(f) = P_i \circ \beta \circ J_i$ and $\Omega^{-1}(g) = \beta \circ J_i$, by 3.10. Therefore

$$\Omega^{-1}(g\circ f) = \beta\circ J_i \circ P_i \circ \beta \circ J_i = [\times n]\circ \beta \circ J_i \in \mathrm{Hom}(J_C, P_X).$$

Since $\zeta^* \circ f = \zeta^* \circ g \circ f = 0$ and $f \circ \zeta_* = g\circ f \circ \zeta_* = 0$, from 3.9 we deduce that $g\circ f = ng$. Therefore

$$p_1^? \circ p_1^? = p_1^?. \tag{4.4.1}$$

Next, $\widetilde{\beta}\circ \zeta_* = 0 = \zeta^* \circ \widetilde{\beta}$ implies $pr_{2*}(\widetilde{\beta}\cdot [Z\times X]) = pr_{1*}(\widetilde{\beta}\cdot[X\times Z]) = 0$, and so we get

$$p_1^? \circ p_{2d} = 0 = p_0 \circ p_1^?. \tag{4.4.2}$$

Also $p_1^? \circ p_0 = (1/mn)[Z]\times pr_{2*}(p_1^?) = a p_0$ for some $a\in\mathbf{Q}$. Squaring gives $a^2 p_0 = (p_1^?\circ p_0)^2 = 0$, so $a = 0$, and

$$p_1^? \circ p_0 = 0. \tag{4.4.3}$$

A similar argument shows that

$$p_{2d}\circ p_1^? = 0 \tag{4.4.4}$$

and by transposition

$$p_{2d-1}^? \circ p_i = p_i \circ p_{2d-1}^? = 0 \quad (i = 0,\ 2d) \tag{4.4.5}$$

Now consider

$$n^2 p_{2d-1}^? \circ p_1^? = i_* \circ i^* \circ \widetilde{\beta} \circ \widetilde{\beta} \circ i_* \circ i^*.$$

We have $\widetilde{\beta}\circ\widetilde{\beta} \in A^{2-d}(X\times X)$. So if $d > 2$ then $\widetilde{\beta}\circ\widetilde{\beta}$ vanishes, and if $d = 2$ then $\widetilde{\beta}\circ\widetilde{\beta} = a[X\times X]$ for some $a \in \mathbf{Q}$. Now $\widetilde{\beta}\circ \zeta_* = 0$, whereas $[X\times X]\circ\zeta_* = m[X]$, so in fact $a = 0$. Thus in every case $\widetilde{\beta}\circ\widetilde{\beta} = 0$, and

$$p_{2d-1}^? \circ p_1^? = 0 \tag{4.4.6}$$

Combining (4.4.1–4.4.6) with the definition of p_1 and p_{2d-1} we get all the required orthogonalities after a series of trivial computations.

For completeness we finally calculate $p_1^{?} \circ p_{2d-1}^{?}$. We have

$$n^2 p_1^{?} \circ p_{2d-1}^{?} = \widetilde{\beta} \circ (i_* \circ i^*) \circ (i_* \circ i^*) \circ \widetilde{\beta}.$$

If $d > 2$ then $(i_* \circ i^*) \circ (i_* \circ i^*) = 0$ since $[C]^2 = 0$ in $A^*(X)$. But if $d = 2$ then we have

$$(i_* \circ i^*) \circ (i_* \circ i^*) = [\Delta_*(Z)].$$

If it happens that

$$n[\Delta_*(Z)] = [Z \times Z] \in A^4(X \times X)$$

then we will have $p_1^{?} \circ p_3^{?} = n^{-3} \widetilde{\beta} \circ \zeta_* \circ \zeta^* \circ \widetilde{\beta} = 0$. However in general it appears that this need not hold (the situation is rather similar to Remark (iv) below), and the correcting terms in the definition of p_1 and p_3 are needed.

(ii) Consider the morphisms

$$h^1(X) \underset{p_1 \circ \widetilde{\beta} \circ p_{2d-1}}{\overset{p_{2d-1} \circ \bar{\xi}^{d-1} \circ p_1}{\rightleftarrows}} h^{2d-1}(X) \otimes \mathbb{L}^{1-d}.$$

Now

$$p_1 \circ \widetilde{\beta} \circ p_{2d-1} = p_1^{?} \circ (1 - \frac{1}{2} p_{2d-1}^{?}) \circ \widetilde{\beta} \circ (1 - \frac{1}{2} p_1^{?}) \circ p_{2d-1}^{?}.$$

As $\widetilde{\beta} \circ \widetilde{\beta} = 0$ (cf. the proof of (i) above) we get $p_{2d-1}^{?} \circ \widetilde{\beta} = 0 = \widetilde{\beta} \circ p_1^{?}$ and so

$$p_1 \circ \widetilde{\beta} \circ p_{2d-1} = p_1^{?} \circ \widetilde{\beta} \circ p_{2d-1}^{?} = p_1^{?} \circ \widetilde{\beta} = \widetilde{\beta} \circ p_{2d-1}^{?}.$$

Therefore

$$p_1 \circ \widetilde{\beta} \circ p_{2d-1} \circ \bar{\xi}^{d-1} \circ p_1 = p_1^{?} \circ \widetilde{\beta} \circ \bar{\xi}^{d-1} \circ p_1 = n p_1.$$

Similarly,

$$p_{2d-1} \circ \bar{\xi}^{d-1} \circ p_1 \circ \widetilde{\beta} \circ p_{2d-1} = p_{2d-1} \circ \bar{\xi}^{d-1} \circ \widetilde{\beta} \circ p_{2d-1}^{?} = n p_{2d-1}$$

and thus the arrows are isomorphisms.

(iii) See [**23**] for details of this part.

Proposition 4.5. *Let X, $X' \in \mathcal{V}_k$ with chosen projective embeddings. Then*

$$\mathrm{Hom}(h^1(X), h^1(X')) = \mathrm{Hom}(P_X, P_{X'}) \otimes \mathbf{Q}$$

and

$$\mathrm{Hom}(h^{2d-1}(X), h^{2d-1}(X')) = \mathrm{Hom}(J_{X'}, J_X) \otimes \mathbf{Q}.$$

Proof. Use $'$ to denote the corresponding objects in the construction above when applied to X'. By 4.4(ii) we have

$$\begin{aligned}\mathrm{Hom}(h^1(X),h^1(X')) &\simeq \mathrm{Hom}(h^{2d-1}(X)\otimes\mathbb{L}^{1-d},h^1(X'))\\ &=\{c\in A^1(X\times X')\,|\,p'_1\circ c\circ p_{2d-1}=c\}\end{aligned}$$

and the isogeny β gives an isomorphism

$$\mathrm{Hom}(P_X,P_{X'})\otimes\mathbf{Q}\simeq\mathrm{Hom}(J_X,P_{X'})\otimes\mathbf{Q}\simeq\{c\in A^1(X\times X')\,|\,\zeta'^*\circ c=0=c\circ\zeta_*\}.$$

SInce $\zeta'^*\circ p'_1=0=p_{2d-1}\circ\zeta_*$ by 4.4(i) the first subspace of $A^1(X\times X')$ is contained in the second. To get an inclusion in the other direction, suppose that $\zeta'^*\circ c=0=c\circ\zeta_*$, so that $c=\Omega(\nu)$ for some $\nu\in\mathrm{Hom}(J_X,P_{X'})\otimes\mathbf{Q}$. Then

$$\begin{aligned}c\circ p^?_{2d-1}&=\frac{1}{n}\Omega(\nu)\circ i_*\circ i^*\circ\Omega(\beta)\\ &=\frac{1}{n}\Omega(\nu\circ J_i)\circ\Omega(P_i\circ\beta)\\ &=\frac{1}{n}\Omega(\nu\circ J_i\circ P_i\circ\beta)=\frac{1}{n}\Omega([\times n]\circ\nu)=c\end{aligned}$$

and by transposition $p^{?\prime}_1\circ c=c$ also. This settles the case $d>2$ completely. If $d=2$, then consider $c\circ p^?_1=c\circ\widetilde{\beta}\circ i_*\circ i^*$. We have $c\circ\widetilde{\beta}\circ i_*\in A^0(C\times X')$, so $c\circ\widetilde{\beta}\circ i_*=a[C\times X']$ for some $a\in\mathbf{Q}$. Since $\zeta'^*\circ c=0$ we have $a=0$. Thus $c\circ p^?_1=0$, hence $c\circ p_3=c\circ p^?_3=c$ and likewise $p_1\circ c=c$.

The second equality in the proposition follows from the first by duality.

4.6. By 4.4(i) we can write

$$h(X)=h^0(X)\oplus h^1(X)\oplus M\oplus h^{2d-1}(X)\oplus h^{2d}(X)$$

for some M. Suppose now that X has dimension 2. We can then define $p_2=1-p_0-p_1-p_3-p_4$ and $h^2(X)=(X,p_2,0)=M$, and there is a decomposition

$$h(X)=\bigoplus_{i=0}^{4}h^i(X).$$

When $\sim$ is rational equivalence the cycle groups $A^j(h^i(X))$ are given by the following table:

$M=$	$h^0(X)$	$h^1(X)$	$h^2(X)$	$h^3(X)$	$h^4(X)$
$A^0(M)=$	$A^0(X)$	0	0	0	0
$A^1(M)=$	0	$A^1(X)^0$	$NS(X)\otimes\mathbf{Q}$	0	0
$A^2(M)=$	0	0	$\ker(\mathrm{Alb})$	$A^2(X)^0/\ker(\mathrm{Alb})$	$\mathbf{Q}$

Remarks. (i) Murre calls the motives $h^1(X)$ and $h^{2d-1}(X)$ the *Picard* and *Albanese* motives of X, respectively, in view of the previous result. Observe that p_1 factors through $h(C)$, and in fact $h^1(X)$ is a direct summand of $h^1(C)$.

(ii) The minor differences between the construction we give and that in [**23**] are in slightly different normalisations of the projectors p_i; we use the zero-cycle ζ rather than an auxiliary rational point to normalise p_0 and p_{2d}, and in the case $d=2$ we have made a different choice of p_1 and p_3 to preserve Poincaré duality and the analogue of the hard Lefschetz theorem.

(iii) To complete the picture in 4.5 one would like to know for surfaces X, X' the nature of the group $\mathrm{Hom}(h^2(X), h^2(X'))$, and in particular that it did not depend on the choice of equivalence relation on cycles. This would be answered by enough knowledge about the conjectural filtration on Chow groups; see §6.2 and Jannsen's paper in these proceedings [**10**] for more information.

(iv) The reader is warned against reading too much into 4.4(ii). In particular, in $\mathcal{M}_k^{\mathrm{rat}}$ it will not generally be the case that $h(X)$ has a decomposition into primitive pieces which satisfy a naïve analogue of the hard Lefschetz theorem. The analogous situation in the derived category of $\mathbf{Q}_\ell$-sheaves is considered in [**3**] and especially [**4**]; here we give a simple example.

Consider a curve X of genus g over an algebraically closed field embedded in projective space by a multiple of $P+Q$, where P, Q are points on X. We take $\sim$ to be rational equivalence. Then cup-product with the hyperplane section ξ does of course give an isomorphism

$$h^0(X) \xrightarrow{\sim} h^2(X) \otimes \mathbb{L}^{-1}$$

where the projectors are taken to be

$$p_0 = \frac{1}{2}[(P+Q)\times X] = {}^t p_2, \quad p_1 = 1 - p_0 - p_2.$$

However it also induces a morphism

$$h^1(X) \longrightarrow h^1(X) \otimes \mathbb{L}^{-1}$$

which is represented by a non-zero multiple of the cycle

$$\eta = (P,P) + (Q,Q) - (P,Q) - (Q,P).$$

Using arguments of Bloch, Mumford and Roitman one sees that for $g > 1$ and P, Q sufficiently generic, η does not vanish. (This can be deduced fairly easily from Theorem 3.1(a) of [**2**], for instance.) So $h^1(X)$ is not killed by $\bar{\xi}$ in general.

5. Abelian varieties (II)

5.1. In this section we will consider an abelian variety X over k of dimension g. For an integer n, write $[\times n]: X \to X$ for multiplication by n. Also let $\mu: X \times X \to X$ be the group law, $\varepsilon \in X(k)$ the identity element and $\sigma: X \to X$ multiplication by -1.

Theorem 5.2.

(i) There is a unique decomposition in $\mathcal{M}_k$

$$h(X) = \bigoplus_{i=0}^{2g} h^i(X)^{\mathrm{can}}$$

which is stable under $[\times n]^*$, *and such that* $[\times n]^*\big|_{h^i(X)}$ *is multiplication by the scalar* n^i, *for every* $n \in \mathbf{Z}$.

(ii) The iterated product maps

$$h(X) \otimes \ldots \otimes h(X) = h(X \times \ldots \times X) \xrightarrow{\mathrm{diag}^*} h(X)$$

induce for every $i \geq 0$ *isomorphisms*

$$\bigwedge^i h^1(X)^{\mathrm{can}} \xrightarrow{\sim} h^i(X)^{\mathrm{can}}.$$

(iii) Let $\xi \in A^1(X)$ *be the class of an ample symmetric line bundle on* X. *Then there is a commutative diagram*

$$\begin{array}{ccc} h^i(X) & \hookrightarrow & h(X) \\ \downarrow \wr & & \downarrow \xi^{g-i} \\ h^{2g-i}(X) \otimes \mathbb{L}^{i-g} & \hookrightarrow & h(X) \otimes \mathbb{L}^{i-g} \end{array}$$

in which the horizontal arrows are the obvious inclusions.

(Recall that $\xi \in A^*(X)$ is symmetric if $\sigma^*\xi = \xi$.) There is also a relation between (i) and the decomposition of the previous section. Let $p_i^{\mathrm{can}} \in \operatorname{End} h(X)$ be the projectors for which $h^i(X)^{\mathrm{can}} = (X, p_i^{\mathrm{can}}, 0)$, and for $i = 0,\ 1,\ 2g-1,\ 2g$ let p_i be the projectors defined in §4, using the class ξ of a very ample line bundle on X.

Theorem 5.3. *If* ξ *is symmetric, then* $p_i^{\mathrm{can}} = p_i$ *for* $i = 0,\ 1,\ 2g-1$ *and* $2g$.

5.4. For numerical (or homological) equivalence, Theorem 5.2 was proved by Grothendieck, Kleiman and Lieberman; see the appendix to §2 in [**11**]. For rational equivalence, the existence of a decomposition $h(X) = \oplus h^i(X)$ in $\mathcal{M}_k^{\mathrm{rat}}$ satisfying (ii) was first proved by Manin and Shermenev [**26**], using Jacobians. An elegant proof of (i) was recently found by Deninger and Murre [**6**], who used the *Fourier transform* on Chow groups [**1**]. Künnemann extended their ideas to prove (ii); see his paper [**15**] in these proceedings, where he gives an elegant explicit formula for the projectors p_i^{can} using the Pontryagin product on the Chow groups of X. The work of Deninger-Murre and Künnemann applies more generally to abelian schemes over any smooth variety. Finally, in his 1992 Ph.D. thesis Künnemann not only proves (iii), but also obtains a complete Lefschetz decomposition of $h(X)$, just as one has in cohomology. For details see [**16**].

Rather than reproduce any of these arguments here, we will give an elementary proof of 5.2(i) from which 5.2(iii) and 5.3 will be easy consequences, and which uses Fourier theory at just one point (5.7 below).

5.5. We first introduce some simple notations. If $i\in\mathbf{Z}$ then write $A^*(X)^{(i)}$ for the subspace comprising all $c\in A^*(X)$ such

$$[\times n]^*(c)=n^i c \quad \text{for every } n\in\mathbf{Z}.$$

Likewise if X' is a second abelian variety, of dimension g', write $A^*(X\times X')^{(i,j)}$ for the set of all $c\in A^*(X\times X')$ such that

$$([\times m]\times[\times n])^*(c)=m^i n^j c \quad \text{for all } m,\, n\in\mathbf{Z}.$$

From 1.10 it follows that $c\in A^*(X\times X')^{(i,j)}$ if and only if, for every $n\in\mathbf{Z}$, one has identities of correspondences

$$[\times n]^*_{X'}\circ c=n^j c \quad \text{and} \quad c\circ[\times n]^*_X=n^{2g-i}c.$$

Therefore if $c\in A^*(X\times X')^{(i,j)}$ and $d\in A^*(X'\times X'')^{(r,s)}$ one has $d\circ c=0$ unless $j=2g'-r$.

5.6. Recall ([**21**], §II.6) that if $\mathcal{L}$ is any line bundle on X then

$$[\times n]^*\mathcal{L}\simeq\mathcal{L}^{\otimes n(n+1)/2}\otimes(\sigma^*\mathcal{L})^{\otimes n(n-1)/2}.$$

Therefore $\xi\in A^1(X)$ is symmetric if and only if $\xi\in A^1(X)^{(2)}$, or equivalently $[\times n]^*\circ\bar{\xi}=n^2\bar{\xi}\circ[\times n]^*$. Likewise, ξ is antisymmetric if and only if $\xi\in A^1(X)^{(1)}$ (which holds if and only if ξ is algebraically equivalent to zero).

5.7. If $\xi\in A^1(X)$ is the class of an ample line bundle we have the usual formula

$$\deg(\xi^g)=g!d$$

where d^2 is the degree of the polarisation determined by ξ. Using Fourier theory one can show that if ξ is symmetric then

$$\xi^g=g!d[\varepsilon]\in A^g(X)$$

(see for example [**1**], middle of page 249).

5.8. Now pick a symmetric ξ which is the class of an ample line bundle, and write $\lambda=\mu^*\xi-pr_1^*\xi-pr_2^*\xi\in A^1(X\times X)$. Then

$$\lambda\in A^1(X\times X)^{(1,1)}.$$

Indeed, the restrictions of λ to the fibres of pr_1 and pr_2 are algebraically equivalent to zero, so $([\times m]\times[\times n])^*\lambda - mn\lambda$ has zero restriction to the fibres by 5.6, hence is zero.

Remark. If $\phi_{\mathcal{L}}: X \to \widehat{X}$ is the usual homomorphism $x \to T_x^*\mathcal{L}\otimes\mathcal{L}^\vee$ attached to a line bundle $\mathcal{L}$ and $\xi = c_1(\mathcal{L})$, then λ is the pullback by $\mathrm{id}\times\phi_{\mathcal{L}}$ of the class of the Poincaré bundle on $X\times\widehat{X}$. This is the link between the formulae given below and Fourier theory.

5.9. *Proof of* 5.2(i), (iii). Firstly, it is obvious that the decomposition (i) is unique if it exists. This shows in particular that the choice of ξ is unimportant.

If $0\le i\le 2g$ define

$$f_i = \sum_{\max(0,i-g)\le j\le i/2} \frac{1}{j!(g-i+j)!(i-2j)!} pr_1^*\xi^j \cdot pr_2^*\xi^j \cdot \lambda^{i-2j}$$

$$q_i = \sum_{\max(0,i-g)\le j\le i/2} \frac{1}{j!(g-i+j)!(i-2j)!} pr_1^*\xi^{g-i+j} \cdot pr_2^*\xi^j \cdot \lambda^{i-2j}.$$

If $0\le i\le g$ then

$$q_i = pr_1^*\xi^{g-i}\cdot f_i = f_i\circ\bar{\xi}^{g-i} \quad\text{and}\quad q_{2g-i} = pr_2^*\xi^{g-i}\cdot f_i = \bar{\xi}^{g-i}\circ f_i. \qquad (5.9.1)$$

Also we have

$$f_i\in A^i(X\times X)^{(i,i)} \quad\text{and}\quad q_i\in A^g(X\times X)^{(2g-i,i)}.$$

In particular $q_i\circ q_{i'} = 0$ if $i\ne i'$. Now

$$\begin{aligned}\sum_{i=0}^{2g} q_i &= \sum_{\substack{0\le i\le 2g\\ \max(0,i-g)\le j\le i/2}} \frac{1}{j!(g-i+j)!(i-2j)!} pr_1^*\xi^{g-i+j}\cdot pr_2^*\xi^j\cdot\lambda^{i-2j}\\ &= \frac{1}{g!}(pr_1^*\xi + pr_2^*\xi + \lambda)^g\\ &= \frac{1}{g!}\mu^*\xi^g = d\mu^*[\varepsilon] = d[\Gamma_\sigma].\end{aligned}$$

Define $p_i^{\mathrm{can}} = \dfrac{1}{d}\sigma^*\circ q_i = \dfrac{(-1)^i}{d} q_i \in A^g(X\times X)^{(2g-i,i)}$. Then by the above,

$$\sum p_i^{\mathrm{can}} = 1 \quad\text{and}\quad \sum p_i^{\mathrm{can}}\circ p_i^{\mathrm{can}} = (\sum p_i^{\mathrm{can}})^2 = 1.$$

This forces $p_i^{\mathrm{can}}\circ p_i^{\mathrm{can}} = p_i^{\mathrm{can}}$, and then $h^i(X)^{\mathrm{can}} = (X, p_i^{\mathrm{can}}, 0)$ satisfies (i).

By 5.6 and (i) the cup-product $\bar{\xi}^{g-i}$ maps $h^i(X)^{\mathrm{can}}$ into $h^{2g-i}(X)^{\mathrm{can}}\otimes\mathbb{L}^{i-g}$. Then the formulae (5.9.1) show that it is an isomorphism, giving (iii).

Corollary 5.10. *The natural map* $\mathrm{Hom}(X,X')\otimes\mathbf{Q}\to\mathrm{Hom}(h^1(X')^{\mathrm{can}}, h^1(X)^{\mathrm{can}})$ *is an isomorphism.*

Proof. Composition with $\bar{\xi}^{g-1}$ gives

$$\begin{aligned}\mathrm{Hom}(h^1(X')^{\mathrm{can}}, h^1(X)^{\mathrm{can}}) &\simeq \mathrm{Hom}(h^{2g-1}(X')^{\mathrm{can}} \otimes \mathbb{L}^{1-g}, h^1(X)^{\mathrm{can}}) \\ &= A^1(X' \times X)^{(1,1)} \\ &\simeq A^1(X' \times X)/pr_1^* A^1(X') + pr_2^* A^1(X) \\ &\simeq NS(X' \times X) \otimes \mathbf{Q} \\ &\simeq \mathrm{Hom}(X', X) \otimes \mathbf{Q}\end{aligned}$$

the last isomorphism coming from the polarisation ξ. It is easily checked that this is inverse to the map of the corollary.

5.11. A formal consequence of 5.2(i) is the eigenspace decomposition of the Chow groups: we have

$$A^d(X) = \mathrm{Hom}(\mathbb{L}^d, h(X)) = \bigoplus_{i=0}^{2g} \mathrm{Hom}(\mathbb{L}^d, h^i(X)^{\mathrm{can}}) = \bigoplus_{i=0}^{2g} A^d(X)^{(i)}.$$

Since composition with $\bar{\xi}^{g-i}$ gives an isomorphism $A^d(X)^{(i)} \xrightarrow{\sim} A^{g+d-i}(X)^{(2g-i)}$ one gets by dimensional considerations that $A^d(X)^{(i)} = 0$ unless $d \le i \le g+d$. Similarly, we have

$$A^d(X \times X') = \bigoplus_{(i,j)} A^d(X \times X')^{(i,j)}$$

where the sum is over pairs $(i,j) \in \mathbf{Z}^2$ such that $d - g' \le i \le d + g$, $d - g \le j \le d + g'$ and $d \le i + j \le d + g + g'$. These decompositions are of course direct consequences of Fourier theory [**1, 6**], and it was using them that Deninger and Murre proved 5.2(i).

The interpretation of motives for rational equivalence as complexes (see 6.2 below) suggests that a stronger vanishing result

$$A^d(X)^{(i)} = \mathrm{Hom}(\mathbb{L}^d, h^i(X)^{\mathrm{can}}) = 0 \quad \text{for } i > 2d$$

holds. See [**10**], where the conjectural filtration on Chow groups (of which this vanishing is a part) is explained.

Proof of 5.3. We have $p_{2g-i}^{\mathrm{can}} = {}^t p_i^{\mathrm{can}}$ and $p_{2g-i} = {}^t p_i$, so it is enough to treat p_0 and p_1. We have $p_0 = \dfrac{1}{g!d}(\xi^g \times [X])$, so by 5.7 this is the same as p_0^{can}.

For p_1 we first observe that the difficulties arising in the previous section in dimension 2 disappear here; for by 5.7 we have $\xi^g = d!g[\varepsilon]$ so that (4.4.7) obviously holds. So in every case we have $p_1 = p_1^?$ (in the notations of 4.3).

Lemma 5.12. $[\times n]^* \circ p_1 = np_1 = p_1 \circ [\times n]^*$

Proof. By definition $p_1 = (1/m)\widetilde{\beta} \circ \bar{\xi}^{g-1}$ for a certain isogeny $\beta\colon X \to \widehat{X}$. We therefore have, using 3.10

$$[\times n]^* \circ \widetilde{\beta} = \Omega(P_{[\times n]} \circ \beta) = \Omega([\times n] \circ \beta) = n\Omega(\beta).$$

This gives the first equality. For the second we have

$$p_1 \circ [\times n]^* = \frac{1}{m}\widetilde{\beta} \circ \bar{\xi}^{g-1} \circ [\times n]^* = \frac{n^{2-2g}}{m}\widetilde{\beta} \circ [\times n]^* \circ \bar{\xi}^{g-1}.$$

Now $[\times n]^* \circ [\times n]_* = n^{2g} \in \operatorname{End} h(X)$, so if we write $\widetilde{\beta} \circ [\times n]^* = \Omega(\beta')$ for some $\beta' \in \operatorname{Hom}(X, \widehat{X}) \otimes \mathbf{Q}$ then

$$n^{2g}\widetilde{\beta} = \widetilde{\beta} \circ [\times n]^* \circ [\times n]_* = \Omega(\beta' \circ J_{[\times n]}) = \Omega(\beta' \circ [\times n]) = n\Omega(\beta').$$

Therefore $\widetilde{\beta} \circ [\times n]^* = n^{2g-1}\widetilde{\beta}$ and thus $p_1 \circ [\times n]^* = np_1$, proving the lemma.

Now from 5.12 we see that p_1, p_1^{can} commute and that $p_1 \circ p_1^{\mathrm{can}} = p_1$. Therefore $h^1(X) = (X, p_1, 0)$ is a direct factor of $h^1(X)^{\mathrm{can}} = (X, p_1^{\mathrm{can}}, 0)$. But by 4.5 and 5.10, $\operatorname{End} h^1(X) = \operatorname{End} h^1(X)^{\mathrm{can}} = \operatorname{End} X \otimes \mathbf{Q}$. Therefore $h^1(X) = h^1(X)^{\mathrm{can}}$.

Remark. One can also check that β can be chosen to be the isogeny $\phi_{\mathcal{L}} : X \to \widehat{X}$ determined by $\xi = c_1(\mathcal{L})$. This gives an alternative verification of 5.3.

6. Further topics

6.1. (*Relative motives.*) Let S be a smooth (not necessarily projective) variety, and let $\mathcal{V}_S$ be the category of smooth projective S-schemes. Then one can introduce the notion of a relative correspondence between two objects X, Y of $\mathcal{V}_S$ as a cycle class of codimension $\dim(X/S)$ on $X \times_S Y$. In [**6**] this is used to define the category $\mathcal{M}_S$ of motives over the base S. Deninger and Murre show there that if X/S is an abelian scheme of relative dimension d, then the relative motive $h(X/S)$ has a canonical decomposition $h(X/S) = \bigoplus_{i=0}^{2d} h^i(X/S)$.

If S itself is projective, then the functor $\mathcal{V}_S \to \mathcal{V}_k$ induces a functor $h(S, -) : \mathcal{M}_S \to \mathcal{M}_k$. In other words, relative motives should be thought of as (complexes of) sheaves on S. For an abelian scheme X/S one can then form various motives $h(S, h^i(X/S))$.

In an ideal world one would like a more general notion of relative motive: for example, if $j : S \hookrightarrow S'$ is the smooth compactification of a curve S, and $X/S \in \mathcal{V}_S$, then one would like to be able to give a meaning to $j_* h(X/S)$, and define motives such as $h^p(S', j_* h^q(X/S))$.

There is one simple case in which it is possible to do this. Let $S/\mathbf{Q}$ be the standard modular curve of level $n \geq 3$ (which parameterises elliptic curves together with a chosen basis for the subgroup of n-division points), and let $\pi : E \to S$ be the universal elliptic curve. Let $j : S \to S'$ be the smooth compactification of S. Write $M = h^1(E/S)$, a relative motive over S, and let M_ℓ be the corresponding ℓ-adic sheaf $R^1\pi_* \mathbf{Q}_\ell$ on S. The parabolic cohomology groups

$$_k W_\ell = H^1(S' \otimes \overline{\mathbf{Q}}, j_* \operatorname{Sym}^k M_\ell)$$

are representations of $\operatorname{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$ which are pure of weight $k+1$. Since $\operatorname{Sym}^k M$ is a submotive of $h(X)$, where X is the k-fold fibre product of E over S, $_k W_\ell$ is in fact a subquotient of the cohomology of X', a smooth compactification of X. One can in this case prove [**25**] by brutal construction that there is a submotive $_k W \subset h(X')$ (in the category of Chow motives) such that the ℓ-adic cohomology of $_k W$ is the parabolic cohomology. It is tempting to write this motive as $h^1(S', j_* \operatorname{Sym}^k M)$, but there is as yet no way to make similar constructions in a general situation. We do not give any details as the proof uses arguments which are rather dissimilar to what has gone before.

6.2. In this final section we will try to give some vague hints of how the results described above fit in to a general (mainly conjectural) picture. To begin with we discuss the effect of choosing different equivalence relations on cycles.

• Numerical equivalence. This is the coarsest adequate equivalence relation. In [**9**] Jannsen proves, without using the standard conjectures, that the category of motives over a field is abelian and semisimple if and only if the equivalence relation used is numerical equivalence.

• Homological equivalence (with respect to a fixed Weil cohomology theory H^*). According to the standard conjectures (see [**11**] and [**13**]) $\mathcal{M}_k^{\mathrm{num}}$ and $\mathcal{M}_k^{\mathrm{hom}}$ should coincide. At present, it is only for $\mathcal{M}_k^{\mathrm{hom}}$ that we can define the realisation functors—but see also remark (4) in [**9**].

The standard conjectures also predict that in $\mathcal{M}_k^{\mathrm{hom}}$ the motive $h(X)$ has a direct sum decomposition

$$h(X) = \bigoplus_{i=0}^{2\dim X} h^i(X) \tag{6.2.1}$$

such that the cohomology functor H^i factors through the projection $h(X) \to h^i(X)$. Now in the usual cohomology theories any homomorphism $H^i(X) \to H^j(Y)$ induced by an algebraic cycle is zero unless $i = j$—in the ℓ-adic theory this is by Deligne's proof of the Weil conjectures. Therefore the decomposition (6.2.1) is unique.

• Rational equivalence. In general $\mathcal{M}_k^{\mathrm{rat}}$ is not abelian, by 3.5 above. One expects however (for reasons explained below) that every $h(X)$ has in $\mathcal{M}_k^{\mathrm{rat}}$ a direct sum decomposition (6.2.1) in which the motives $h^i(X)$ are well defined up to unique isomorphism. Moreover the filtration on $h(X)$ by subobjects

$$h^{\leq i}(X) = \bigoplus_{j \leq i} h^j(X)$$

should be uniquely determined and functorial with respect to inverse image and duality. The direct sum decomposition itself will not however be uniquely determined. The corresponding filtration on the Chow groups $A^d(X) = \operatorname{Hom}(\mathbb{L}^d, h(X))$ by the subgroups $\operatorname{Hom}(\mathbb{L}^d, h^{\leq i}(X))$ would be the conjectural filtration discussed in [**10**]. See [**24**], where the idea of using such a "Chow-Künneth" decomposition to study the filtration is introduced and elaborated in detail.

If $\xi \in A^1(X)$ is the class of an ample divisor, then the hard Lefschetz theorem should hold in the following sense: $\bar{\xi}^i : h(X) \otimes \mathbb{L}^i \to h(X)$ respects the filtration up to a shift by $2i$, and induces an isomorphism between the subquotients $h^{\dim X - i}(X) \otimes \mathbb{L}^i$ and $h^{\dim X + i}(X)$.

In (say) ℓ-adic cohomology the ring of correspondences $\operatorname{Corr}^0(X, X)$ acts not only on $H^*(\bar{X}, \mathbf{Q}_\ell)$ but also on the object $R\Gamma(\bar{X}, \mathbf{Q}_\ell)$ in the derived category $\mathcal{D}^b(\operatorname{Spec} k, \mathbf{Q}_\ell)$ of complexes of ℓ-adic representations of $\operatorname{Gal}(\bar{k}/k)$. In this category there is the canonical filtration by truncation. Moreover a theorem of Deligne ([**3**], Prop. 2.4; see also [**4**]) states that there is an isomorphism (which is not unique) in $\mathcal{D}^b(\operatorname{Spec} k, \mathbf{Q}_\ell)$

$$R\Gamma(\bar{X}, \mathbf{Q}_\ell) \xrightarrow{\sim} \bigoplus_{i=0}^{2\dim X} H^i(\bar{X}, \mathbf{Q}_\ell)[-i].$$

One should therefore think of the object $h^i(X)$ in $\mathcal{M}_k^{\mathrm{rat}}$ as a complex concentrated in degree i, and of $\operatorname{Hom}(h^i(X), h^j(Y))$ as an analogue of

$$\operatorname{Hom}(H^i(\bar{X}, \mathbf{Q}_\ell)[-i], H^j(\bar{Y}, \mathbf{Q}_\ell)[-j]) = \operatorname{Ext}^{i-j}(H^i(\bar{X}, \mathbf{Q}_\ell), H^j(\bar{Y}, \mathbf{Q}_\ell)).$$

The image of the Lefschetz motive $\mathbb{L}$ in the ℓ-adic setting is $\mathbf{Q}_\ell(-1)[-2]$, so $A^d(h^i(X)) = \operatorname{Hom}(\mathbb{L}^d, h^i(X))$ should be analogous to

$$\operatorname{Hom}(\mathbf{Q}_\ell(-d)[-2d], H^i(\bar{X}, \mathbf{Q}_\ell)[-i]) = \operatorname{Ext}^{2d-i}(\mathbf{Q}_\ell(-d), H^i(\bar{X}, \mathbf{Q}_\ell)).$$

This fits in well with the formalism of mixed motives, and suggests that there exists a "derived category of mixed motives". It would contain $\mathcal{M}_k^{\mathrm{num}}$ as the (abelian, semisimple!) subcategory formed of direct sums of "pure complexes" (of any weight) concentrated in degree zero, and would contain $\mathcal{M}_k^{\mathrm{rat}}$ as the subcategory of "pure complexes of weight zero". In particular this would imply that $\operatorname{Hom}(h^i(X), h^i(Y))$ was the same in $\mathcal{M}_k^{\mathrm{num}}$ and $\mathcal{M}_k^{\mathrm{rat}}$. For further indications along these lines we refer to the papers of Jannsen [**10**] and Levine [**18**], as well as other papers in these Proceedings.

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