Extensions of motives and higher Chow groups

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Introduction

This note has two purposes: the first is to give a somewhat different description of the higher cycle class map defined by Bloch [**3**] for his higher Chow groups. The second is to construct some extensions of motives related to elements of higher Chow groups and to show that the extension classes of their ℓ-adic realisations can be computed using the cycle map.

Slightly more precisely, let X be smooth and proper over a field k, and consider the motivic cohomology $H^j_{\mathcal{M}}(X, \mathbf{Q}(n))$, one candidate for which is $CH^n(X, 2n-j) \otimes \mathbf{Q}$. Conjecturally there should be a category of mixed motives $\mathcal{MM}_k$ and a spectral sequence

$$E_2^{ij} = \mathrm{Ext}^i_{\mathcal{MM}_k}(\mathbf{Q}(-n), h^j(X)) \Rightarrow H^{i+j}_{\mathcal{M}}(X, \mathbf{Q}(n))$$

There should also be a morphism from this spectral sequence to the Hochschild-Serre spectral sequence of continuous ℓ-adic cohomology [**6**]:

$$E_2^{ij} = \mathrm{Ext}^i_{\mathrm{Gal}(\bar{k}/k)}(\mathbf{Q}_\ell(-n), H^j(\overline{X}, \mathbf{Q}_\ell)) \Rightarrow H^{i+j}_{\mathrm{cont}}(X, \mathbf{Q}_\ell(n))$$

which on the E_2 terms is the realisation functor, and on the abutment is the Chern class map.

Here we exhibit a fragment of this structure. Assume that k is of characteristic zero so as to be able to use the category of mixed motives constructed unconditionally by Jannsen in [**7**] (see also [**4**]), and that X is projective. Then for $j \neq 2n$ we construct "geometrically" a map

$$H^j_{\mathcal{M}}(X, \mathbf{Q}(n)) \to \mathrm{Ext}^1_{\mathcal{MM}_k}(\mathbf{Q}(-n), h^{j-1}(X)). \tag{0.1}$$

The cycle class carries $H^j_{\mathcal{M}}(X, \mathbf{Q}(n))$ into the subspace

$$\ker[H^j_{\mathrm{cont}}(X, \mathbf{Q}_\ell(n)) \to H^j(\overline{X}, \mathbf{Q}_\ell(n))]$$

(since for $j \neq 2n$ the right hand group has no Galois invariants). Therefore composing with the edge homomorphism of the Hochschild-Serre spectral sequence one obtains a map

$$H^j_{\mathcal{M}}(X, \mathbf{Q}(n)) \to H^1_{\mathrm{cont}}(k, H^{j-1}(\overline{X}, \mathbf{Q}_\ell(n))) = \mathrm{Ext}^1_{\mathrm{Gal}(\bar{k}/k)}(\mathbf{Q}_\ell(-n), H^{j-1}(\overline{X}, \mathbf{Q}_\ell))). \tag{0.2}$$

We show that the Galois extension class (0.2) is the image under the realisation functor of the "geometric" extension class (0.1).

The analogous construction for $j = 2n$ was done in §9 of [**7**]. Indeed, $H^{2n}_{\mathcal{M}}(X, \mathbf{Q}(n))$ is the Chow group $CH^n(X) \otimes \mathbf{Q}$, and to each cycle there is associated a morphism of motives $\mathbf{Q}(-n) \to h^{2n}(X)$. The kernel of this assignment is $CH^n(X)^0 \otimes \mathbf{Q}$, the group of classes of cycles homologically equivalent to zero. The cohomology of the complement of the support of a cycle gives rise to an extension of $\mathbf{Q}(-n)$ by $h^{2n-1}(X)$, and Jannsen shows that the extension class of its ℓ-adic realisation is given by the composite of the cycle class and the edge homomorphism of the Hochschild-Serre spectral sequence.

The construction of extensions given here was sketched in the appendix to [**5**], but without using simplicial schemes. The work originated in an earlier, unpublished (but see [**7**], Appendix C3) construction of extensions coming from the K-cohomology group $H^1(X, \mathcal{K}_2)$ (essentially the case of $H^3_{\mathcal{M}}(X, \mathbf{Q}(2))$). That the cycle map can be described as in §1 is probably known to some experts (see for example the penultimate paragraph of [**2**]), but there is no proof of the compatibility in the literature.

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Notations and conventions.

0.1. In what follows X will be a smooth and projective scheme over a field k of characteristic zero which is of finite type over $\mathbf{Q}$. As usual we write $\bar{k}$ for the algebraic closure of k and $\overline{X}$ for $X \otimes \bar{k}$.

0.2. We use the notation $[p, q]$ for the set of strictly increasing maps $\alpha : \{0, 1, \ldots, p\} \to \{0, 1, \ldots, q\}$, which we usually write as increasing sequences $(\alpha_0 < \alpha_1 < \ldots < \alpha_p)$.

0.3. Recall [**2**] the definition of the algebraic "simplex"

$$\Delta^q = \{(x_0, \ldots, x_q) \in \mathbf{A}^{q+1} \mid \textstyle\sum x_i = 1\}.$$

There are face maps $\partial_i : \Delta^q \hookrightarrow \Delta^{q+1}$ given by $\partial_i(t_0, \ldots, t_q) = (t_0, \ldots, t_{i-1}, 0, t_i, \ldots, t_q)$, as well as degeneracy maps $\sigma_i : \Delta^{q+1} \to \Delta^q$. Write $\partial\Delta^q$ for the union of the codimension 1 faces of Δ^q. View $\partial\Delta^q$ as a (strict) simplicial scheme

$$\coprod^{[0,q]} \Delta^{q-1} \leftleftarrows \coprod^{[1,q]} \Delta^{q-2} \;\cdots\; \begin{smallmatrix}\leftarrow\\ \vdots \\ \leftarrow\end{smallmatrix} \coprod^{[q-1,q]} \Delta^0.$$

The j^{th} face map (for $0 \le j \le p$)

$$\coprod^{[p,q]} \Delta^{q-p-1} \to \coprod^{[p-1,q]} \Delta^{q-p}$$

maps the $(\alpha_0 < \ldots < \alpha_p)^{\text{th}}$ copy of Δ^{q-p-1} to the $(\alpha_0 < \ldots \hat{\alpha}_j \ldots < \alpha_p)^{\text{th}}$ copy of Δ^{q-p} by $\partial_{\alpha_j - j}$.

0.4. ℓ-adic cohomology of schemes over non-algebraically closed fields always denotes continuous étale cohomology [**6**].

1. The cycle map

1.1. Recall from [**2**] the definition of the cycle group $z^n(X,q)$: it is the free abelian group generated by integral subschemes of $\Delta^q \times X$ of codimension n, which meet each face (in every dimension) properly. Then $z^n(X,\bullet)$ is a simplicial abelian group with face and degeneracy maps ∂_i^*, σ_i^*. The higher Chow groups $CH^n(X,q)$ are by definition the homotopy groups of $z^n(X,\bullet)$, or equivalently the homology groups of the complex

$$(z^n(X,\bullet), \sum(-1)^i\partial_i^*)$$

By the normalisation theorem, they can be calculated from the complex $(\bar{z}^n(X,\cdot),\partial_0^*)$ where

$$\bar{z}^n(X,q) = \{y \in z^n(X,q) \mid \partial_i^*(y) = 0 \quad \text{for } 1 \leq i \leq q\}.$$

1.2. In terms of this it is possible to give a slightly different description of Bloch's cycle class map. Consider the strict augmented simplicial scheme $\Sigma^q X$:

$$\Delta^q X \leftarrow \coprod\nolimits^{[0,q]} \Delta^{q-1}X) \leftleftarrows \coprod\nolimits^{[1,q]} \Delta^{q-2}X \ \cdots \ \vdots \coprod\nolimits^{[q-1,q]} \Delta^0 X.$$

Here the scheme $\coprod^{[p,q]} \Delta^{q-p-1}X$ is regarded as living in degree p, with $-1 \leq p \leq q-1$. The face maps are the same as in 0.3 above.

1.3. Let $H^*(-,n)$ be any cohomology theory satisfying the axioms of [**3**], §4. This means, amongst other things, that there are functorial complexes $K(n)^\bullet$ whose cohomology is $H^*(-,n)$. The cohomology of the simplicial scheme $\Sigma^q X$ is calculated by means of the double complex

$$E_0^{ab} = \Gamma(\Delta^{q-a}X, K(n)^b)^{[a-1,q]}$$

in which the second differential is $d_{K(n)}$ and the first is given as follows: if $f : [a-1,q] \to \Gamma(\Delta^{q-a}X, K(n)^b)$ then

$$(d'f)(\alpha_0 < \ldots < \alpha_a) = \sum_{j=0}^{a} (-1)^j \partial^*_{\alpha_j - j} f(\alpha_0 < \ldots \hat{\alpha}_j \ldots < \alpha_a).$$

This gives rise to the spectral sequence

$$E_1^{ab} = H^b(\Delta^{q-a}X, n)^{[a-1,q]} \Rightarrow H^{a+b}(\Sigma^q X, n). \tag{1.3.1}$$

Similarly, if H_{supp} denotes cohomology with respect to the family of codimension n subvarieties which meet every face properly, there is a spectral sequence

$$_\bullet E_1^{ab} = H^b_{\text{supp}}(\Delta^{q-a}X, n)^{[a-1,q]} \Rightarrow H^{a+b}_{\text{supp}}(\Sigma^q X, n). \tag{1.3.2}$$

1.4. By the homotopy axiom, the complex $E_1^{\bullet b}$ is isomorphic to the normalisation of the complex

$$H^b(X,n)\otimes[\mathbf{Z}\to\mathbf{Z}^{[0,q]}\rightrightarrows\mathbf{Z}^{[1,q]}\dots\vdots\;\mathbf{Z}^{[q-1,q]}] \tag{1.4.1}$$

and so $E_2^{ab}=H^b(X,n)$ if $a=q$ and zero otherwise. For the other spectral sequence, the weak purity axiom implies that ${}_\bullet E_1^{ab}=0$ for $b<2n$. This yields an edge homomorphism

$${}_\bullet E_2^{a,2n}\to{}_\bullet E_\infty^{a+2n}$$

which we may compose with the morphism of spectral sequences (induced by $H_{\text{supp}}\to H$) to get a map

$${}_\bullet E_2^{0,2n}\to{}_\bullet E_\infty^{2n}\to E_\infty^{2n}=H^{2n-q}(X,n). \tag{1.4.1}$$

1.5. If $y\in\bar z^n(X,q)$ has $\partial_0^*(y)=0$ then its class $cl_{n,q}(y)\in{}_\bullet E_1^{0,2n}$ is visibly killed by d_1, hence maps to ${}_\bullet E_2^{0,2n}$. In other words the composition of the cycle class map and (1.4.1) defines a homomorphism

$$\{y\in\bar z^n(X,q)\mid\partial_0^*(y)=0\}\longrightarrow H^{2n-q}(X,n). \tag{1.5.1}$$

1.6. We compare this with Bloch's map (for which we use the notation of [**5**], §2.8). It is convenient to renumber the spectral sequences (1.3.1), (1.3.2) to become:

$$\begin{aligned}E_1^{-a,b}&=H^b(\Delta^aX,n)^{[q-a-1,q]}\Rightarrow H^{q-a+b}(\Sigma^qX)\\ {}_\bullet E_1^{-a,b}&=H^b_{\text{supp}}(\Delta^aX,n)^{[q-a-1,q]}\Rightarrow H^{q-a+b}_{\text{supp}}(\Sigma^qX).\end{aligned} \tag{1.6.1}$$

In terms of this, the first differential on the original double complex

$$E_0^{-a,b}=\Gamma(\Delta^aX,K(n)^b)^{[q-a-1,q]}$$

is given by

$$(d'f)(\alpha_0<\dots<\alpha_{q-a})=\sum_{j=0}^{q-a}(-1)^j\partial^*_{\alpha_j-j}f(\alpha_0<\dots\hat\alpha_j\dots<\alpha_{q-a})$$

and similarly for the complexes with supports.

1.7. Bloch's map is defined by a similar procedure as above, but using the double complexes (truncated outside the range $0\le a\le 2N$ for some $N>>0$)

$$\tilde E_0^{-a,b}=\Gamma(\Delta^aX,K(n)^b)$$

with first differential

$$\tilde d'=\sum_{i=0}^{a}(-1)^i\partial_i^*.$$

and the analogous complex ${}_\bullet\tilde{E}_0$ with supports. Define morphisms $E_0 \to \tilde{E}_0$, ${}_\bullet E_0 \to {}_\bullet\tilde{E}_0$ by the formula

$$f \mapsto \sum_{\alpha\in[q-a-1,q]} (-1)^{|\alpha|} f(\alpha) \quad \text{where } |\alpha| = \sum_{j=0}^{q-a-1} \alpha_j.$$

One checks easily that this map commutes with the differentials, hence gives morphisms of spectral sequences $E \to \tilde{E}$, ${}_\bullet E \to {}_\bullet\tilde{E}$. It is then clear that the map (1.5.1) is the same as Bloch's map.

1.8. It follows that the map (1.5.1) factors through the quotient $CH^n(X,q)$ (since Bloch's map does). We will indicate a direct proof of this in 3.6 below.

2. Construction of extensions in cohomology

2.1. We continue with the notation of the preceding section, but now assume that the cohomology theory takes values in an abelian tensor category $\mathcal{C}$, with unit object $\mathbf{1}_\mathcal{C}$; and that the cycle class of a subvariety $Y \subset X$ of codimension p is a map $\mathbf{1}_\mathcal{C} \to H^{2p}_Y(X,p)$. To distinguish this from the "absolute" cohomology theory we denote it henceforth by $\underline{H}$. Then the higher cycle class is a map

$$cl_{n,q} : CH^n(X,q) \longrightarrow \mathrm{Hom}(\mathbf{1}_\mathcal{C}, \underline{H}^{2n-q}(X,n)).$$

A typical example is $\underline{H}^i_Y(X,j) = H^i_{\overline{Y}}(\overline{X}, \mathbf{Z}_\ell(j))$ with $\mathcal{C}$ the category of $\mathbf{Z}_\ell$-modules with a continuous action of $\mathrm{Gal}(\bar{k}/k)$. Here the higher cycle class has finite image for $q>0$, since $H^{2n-q}(\overline{X}, \mathbf{Q}_\ell(n))$ has no Galois invariants except when $q=0$.

2.2. We now construct a map

$$cl^{(1)}_{n,q} \colon \ker cl_{n,q} \longrightarrow \mathrm{Ext}^1_\mathcal{C}(\mathbf{1}_\mathcal{C}, \underline{H}^{2n-q-1}(X,n)).$$

Let $y \in z^n(X,q)$ be a cycle with $\partial_i^*(y) = 0$ for $0 \le i \le q$, and write $Y = \mathrm{supp}(y) \subset \Delta^q X$. Let Y_i be the disjoint union of the intersections of Y with the codimension i faces of $\Delta^q X$, and $Y_0 = Y$. Thus $Y_\bullet$ is a closed simplicial subscheme of $\Sigma^q X$. Consider the long exact cohomology sequence

$$\underline{H}^{2n-1}_{Y_\bullet}(\Sigma^q X, n) \to \underline{H}^{2n-1}(\Sigma^q X, n) \to \underline{H}^{2n-1}(\Sigma^q X - Y_\bullet, n) \to \underline{H}^{2n}_{Y_\bullet}(\Sigma^q X, n) \to \underline{H}^{2n}(\Sigma^q X, n).$$

2.3 Lemma. $\underline{H}^i_{Y_\bullet}(\Sigma^q X, n) = 0$ *for* $i < 2n$.

Proof. As in 1.4 above, there is a spectral sequence

$$E_1^{ab} = \underline{H}^b_{Y_a}((\Sigma^q X)_a, n) \Rightarrow \underline{H}^{a+b}_{Y_\bullet}(\Sigma^q X, n).$$

Since by hypothesis Y_a has codimension $\geq n$ in $(\Sigma^q X)_a$, $E_1^{ab} = 0$ for every $a \geq 0$ and $b < 2n$ by the purity axiom.

2.4. By definition, $cl_{n,q}$ is the adjoint of the composite

$$\mathbf{1}_{\mathcal{C}} \otimes CH^n(X,q) \to \underline{H}^{2n}_{Y_\bullet}(\Sigma^q X, n) \to \underline{H}^{2n}(\Sigma^q X, n) \xrightarrow{\sim} \underline{H}^{2n-q}(X,n).$$

Therefore if $cl_{n,q}(y) = 0$ one obtains by pullback an extension in $\mathcal{C}$

$$\begin{array}{ccccccccc} 0 & \to & \underline{H}^{2n-1}(\Sigma^q X, n) & \to & \underline{H}^{2n-1}(\Sigma^q X - Y_\bullet, n) & \to & \underline{H}^{2n}_{Y_\bullet}(\Sigma^q X, n) & \to & \underline{H}^{2n}(\Sigma^q X, n) \\ & & \| & & \uparrow & & \uparrow{\scriptstyle cl_{n,q}(y)} & & \uparrow \\ 0 & \to & \underline{H}^{2n-1}(\Sigma^q X, n) & \to & (*) & \to & \mathbf{1}_{\mathcal{C}} & \to & 0 \end{array}$$

whose class is by definition $cl^{(1)}_{n,q}(y)$.

2.5. Suppose that $\mathcal{C}$ has enough injectives, and that the cohomology groups $\underline{H}^*$ can be calculated from functorial complexes $\underline{K}(n)^\bullet \in K^+(\mathcal{C})$ which are bounded below. Then there is a "Hochschild-Serre" spectral sequence

$$E_2^{ij} = \mathrm{Ext}^i_{\mathcal{C}}(\mathbf{1}_{\mathcal{C}}, \underline{H}^j(-,n)) \Rightarrow H^{i+j}(-,n)$$

where the "absolute" cohomology groups H^* are those of $K(n)^\bullet = \mathbf{R}\,\mathrm{Hom}_{\mathcal{C}}(\mathbf{1}_{\mathcal{C}}, \underline{K}(n)^\bullet)$. (In the example of ℓ-adic cohomology H^* is $H^*(X, \mathbf{Z}_\ell(n))$.) In particular there is an edge homomorphism

$$\ker[H^{2n-q}(X,n) \to \mathrm{Hom}(\mathbf{1}_{\mathcal{C}}, \underline{H}^{2n-q}(X,n))] \to \mathrm{Ext}^1_{\mathcal{C}}(\mathbf{1}_{\mathcal{C}}, \underline{H}^{2n-q-1}(X,n)) \qquad (2.5.1)$$

2.6 Proposition. *Assume that $q > 0$. The composite of the cycle class map $cl_{n,q}\colon CH^n(X,q) \to H^{2n-1}(X,n)$ with the edge homomorphism (2.5.1) is equal to $cl^{(1)}_{n,q}$.*

Proof. The analogous result for $q = 0$ is proved in [**7**]. Both that and the present case rely on a compatibility in homological algebra:

2.7 Proposition ([7] 9.4). *Let $\Phi : \mathcal{A} \to \mathcal{B}$ be a left-exact functor between abelian categories, and assume that $\mathcal{A}$ has enough injectives. Let $0 \to A^\bullet \to B^\bullet \to C^\bullet \to 0$ be a short exact sequence of complexes in $\mathcal{A}$ which are bounded below. Then the following diagram is commutative:*

$$\begin{array}{ccccc} \ker[\mathbf{R}^i\Phi A^\bullet \to \Phi H^i(B^\bullet)] & \longrightarrow & \ker[\mathbf{R}^i\Phi B^\bullet \to \Phi H^i(B^\bullet)] & & \\ \downarrow & & & \searrow^{\sigma} & \\ \ker[\Phi H^i(A^\bullet) \to \Phi H^i(B^\bullet)] & & & & R^1\Phi H^{i-1}(B^\bullet) \\ \| & & & \swarrow & \\ R^0\Phi \ker[H^i(A^\bullet) \to H^i(B^\bullet)] & \xrightarrow{\tau} & R^1\Phi\,\mathrm{coker}[H^{i-1}(A^\bullet) \to H^{i-1}(B^\bullet)] & & \end{array}$$

Here the arrows of the form $\mathbf{R}^i\Phi K^\bullet \to \Phi H^i(K^\bullet)$ are the edge homomorphisms in the hypercohomology spectral sequence for $\mathbf{R}^*\Phi K^\bullet$. The only remaining non-obvious arrows are those labelled σ, which is the next edge homomorphism in the same spectral sequence, and τ, which is the boundary map in the long exact cohomology sequence attached to the short exact sequence

$$0\longrightarrow \mathrm{coker}[H^{i-1}(A^\bullet) \to H^{i-1}(B^\bullet)]\longrightarrow H^{i-1}(C^\bullet)\longrightarrow \ker[H^i(A^\bullet) \to H^i(B^\bullet)]\longrightarrow 0.$$

2.8. Apply this with $A^\bullet$, $B^\bullet$, $C^\bullet$ suitable injective resolutions of the complexes

$$R\Gamma_{Y_\bullet}(\Sigma^q X, \underline{K}(n)), \quad R\Gamma(\Sigma^q X, \underline{K}(n)), \quad R\Gamma(\Sigma^q X - Y_\bullet, \underline{K}(n)),$$

and 2.6 follows at once.

3. Construction of extensions of motives

3.1. To begin with recall that Jannsen [**7**] and Deligne [**4**] have defined the tensor category $\mathcal{MR}_k$ of *mixed realisations.* In [**7**] 6.11, it is shown how to define the (cohomological) mixed realisations of simplicial schemes with smooth components, and the analogues with supports. Denote these by $h^*_-(-)$. This is enough to apply the ideas of §2 to the mixed realisation functor: there is an exact sequence

$$\begin{array}{ccccccccc} 0 & \to & h^{2n-1}(\Sigma^q X) & \to & h^{2n-1}(\Sigma^q X - Y_\bullet) & \to & h^{2n}_{Y_\bullet}(\Sigma^q X) & \to & h^{2n}(\Sigma^q X) \\ & & \| & & & & & & \| \\ & & h^{2n-q-1}(X) & & & & & & h^{2n-q}(X) \end{array}$$

and a higher cycle $y \in z^n(X,q)$ with $\partial_i^* y = 0$ for all i defines a map $\mathbf{Q}(-n) \to h^{2n}_{Y_\bullet}(\Sigma^q X)$. But as $q > 0$ the image of the class of y in $h^{2n-q}(X)$ vanishes, so its pullback gives an extension E_y

$$\begin{array}{ccccccccc} 0 & \to & h^{2n-1}(\Sigma^q X) & \to & h^{2n-1}(\Sigma^q X - Y_\bullet) & \to & h^{2n}_{Y_\bullet}(\Sigma^q X) & \to & h^{2n}(\Sigma^q X) \\ & & \| & & \uparrow & & \uparrow{\scriptstyle cl_{n,q}(y)} & & \uparrow \\ 0 & \to & h^{2n-1}(\Sigma^q X) & \to & E_y & \to & \mathbf{Q}(-n) & \to & 0 \end{array}$$

in the category $\mathcal{MR}_k$.

3.2. One possible definition of the categrory $\mathcal{MM}_k$ of *mixed motives* is the Tannakian subcategory of $\mathcal{MR}_k$ generated by $h^i(V)$ for k-varieties V. Now $h^{2n-1}(\Sigma^q X - Y_\bullet)$ is the same as $h^{2n-1}(\Delta^q X - Y \operatorname{rel} \partial\Delta^q X)$. Then the following simple result shows that $h^{2n-1}(\Sigma^q X - Y_\bullet)$, and therefore also E_y, belong to $\mathcal{MM}_k$:

3.3 Proposition ([8] 1.4). *Let U be quasiprojective over k, and $j\colon Z \hookrightarrow U$ a closed subset. Then there is for each i a mixed motive $h^i(U \operatorname{rel} Z)$ over k whose realisations are isomorphic to the relative cohomology groups of (U, Z).*

3.4. To show that the assignment $y \mapsto E_y$ factors through $CH^n(X, q)$, the results of §2 are not enough; one would need functorial complexes in the derived category of mixed realisations. One way to get round this is to use the notion of MAH-complex introduced in [**7**], 6.11.8. Alternatively the extensions may be split geometrically as follows.

3.5. Let $z \in \bar{z}^n(X, q+1)$ with $\partial_0^* z = y$.Write $\Lambda^{q+1}X$ for the strict simplicial scheme whose r^{th} component is the disjoint union of all codimension $r+1$ faces of $\Delta^{q+1}X$ which are not contained in codimension 1 face $\partial_0(\Delta^q X)$. Let $C^{q+1}X$ be the augmented simplicial scheme $\Delta^{q+1}X \leftarrow \Lambda^{q+1}X$. Let Z be the support of z, and $Z_\bullet \subset C^{q+1}X$ the corresponding simplicial subscheme. Explicitly, $C^{q+1}X$ is

$$\Delta^{q+1}X \leftarrow \coprod^{[0,q]} \Delta^q X \leftleftarrows \coprod^{[1,q]} \Delta^{q-1}X \ldots \vdots \coprod^{[q,q]} \Delta^0.$$

Hence its mixed realisation is

$$h^*(C^{q+1}X) = h^*(X) \otimes H^*[\mathbf{Z} \to \mathbf{Z}^{[0,q]} \to \ldots \to \mathbf{Z}^{[q,q]}] = 0.$$

Therefore the boundary maps $h^{i-1}(C^{q+1}X - Z_\bullet) \to h^i_Z(C^{q+1}X)$ are isomorphisms. The cycle z therefore defines a map $\mathbf{Q}(-n) \to h^{2n-1}(C^{q+1} - Z_\bullet)$. Restricting this with respect to the inclusion $\Sigma^q X \hookrightarrow C^{q+1}X$ gives a map $\mathbf{Q}(-n) \to h^{2n-1}(\Sigma^q X - Y_\bullet)$ which splits E_y.

3.6. *Remark.* Essentially the same argument as above shows that the cycle map (1.5.1) vanishes on the cycle $\partial_0^* z$.

3.7. In summary, we have defined a homomorphism

$$\begin{array}{ccc} CH^n(X,q) & \to & \mathrm{Ext}_{\mathcal{MM}_k}(\mathbf{Q}(-n), h^{2n-q-1}(X)) \\ y & \mapsto & E_y \end{array}$$

such that the diagram

$$\begin{array}{ccc} CH^n(X,q) & \xrightarrow{y \mapsto E_y} & \mathrm{Ext}^1_{\mathcal{MM}_k}(\mathbf{Q}(-n), h^{2n-q-1}(X)) \\ \downarrow \text{cycle} & & \downarrow \ell\text{−adic realisation} \\ H^{2n-q}_{\text{cont}}(X, \mathbf{Q}_\ell(n)) & \xrightarrow{\text{edge}} & \mathrm{Ext}^1_{\mathrm{Gal}(\bar{k}/k)}(\mathbf{Q}_\ell(-n), H^{2n-q-1}(\overline{X}, \mathbf{Q}_\ell)) \end{array}$$

commutes. The analogous statement will also hold if $\mathbf{Q}_\ell$-cohomology over $\bar{k}$ is replaced by rational cohomology over $\mathbf{C}$ or $\mathbf{R}$, with its Hodge structure, and ℓ-adic cohomology over k by Beilinson-Deligne (absolute Hodge) cohomology [**1**].

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