The boundary of the Eisenstein symbol

by

Norbert Schappacher[1] *& Anthony J. Scholl*[2]

In the paper [Beilinson 1986], Beilinson defined the "Eisenstein symbol", a universal construction of elements in higher K-theory (motivic cohomology) of self products of elliptic curves. This generalised a construction by Bloch of elements in K_2 of an elliptic curve [Bloch 1980]. A refinement of Beilinson's Eisenstein symbol was given in [Deninger 1989].

The purpose of the present paper is to calculate the boundary of the Eisenstein symbol at a place of bad reduction of the elliptic curve.

In the case of an elliptic curve over a number field, this gives a criterion for the 'integrality' of Eisenstein symbol elements, and thus generalises a formula found by Bloch and Grayson [1986]. In the case of the universal elliptic curve, we obtain the boundary of the Eisenstein symbol at the cusps. (In characteristic zero an equivalent result was proved in [Beilinson 1986] by an analytic method.)

In our presentation the formula involves Bernoulli polynomials. These arise essentially on account of their well-known distribution property — *cf.* 2.6 (i) below.

We now give a precise summary of our main result. Let E/F be an elliptic curve over a field, and $P \subset E$ a finite subgroup scheme of E defined over F. For any integer $n \geq 1$, consider the *Eisenstein symbol map*, following the definition of [Deninger 1989, §8] :

$$\mathcal{E}_P^n : \mathbf{Q}[P]^\circ \longrightarrow H_{\mathcal{M}}^{n+1}(E^n, \mathbf{Q}(n+1))_{\mathrm{sgn}}.$$

Here the following notations are used.

— $\mathbf{Q}[P]^\circ$ is the $\mathbf{Q}$-vector space of $\mathrm{Gal}(\overline{F}/F)$-invariant functions $\beta : P(\overline{F}) \to \mathbf{Q}$ satisfying $\sum_{x \in P(\overline{F})} \beta(x) = 0$ (which we identify with divisors on E in the obvious way).

— $H_{\mathcal{M}}^i(-, \mathbf{Q}(j)) = K_{2j-i}^{(j)}(-)$ is motivic cohomology — *cf.* [Beilinson 1985, 2.2], [Schneider 1988, §3], [Deninger, Scholl, §1].

— for a group scheme A, we identify A^n with the kernel of the sum-mapping $\Sigma : A^{n+1} \to A$. This gives an action of the symmetric group $\mathcal{S}_{n+1}$ on A^n.

— subscript 'sgn' denotes the image under the projector $\Pi_{\mathrm{sgn}} = \frac{1}{(n+1)!} \sum_{\sigma \in \mathcal{S}_{n+1}} \mathrm{sgn}(\sigma).\sigma$.

Now suppose F admits a non trivial discrete valuation v, and let $\mathcal{O}$ and k, be the valuation ring and residue field of v, respectively. We shall assume that k is perfect. Let $E_{/\mathcal{O}}$ be

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a minimal regular model of E, and $E_{/k}$ its special fibre at v. We make the following additional assumptions :

- *i)* $E_{/k}$ is a Néron N-gon (untwisted), for some $N \geq 1$.
- *ii)* P extends to a finite flat subgroup scheme $P_{/\mathcal{O}}$ of the Néron model of E over $\mathcal{O}$. (For example, one could take P to be the N-torsion points of E, with N as in *i.*)

Write $\overset{\circ}{E}$ for the connected component of the Néron model of E over $\mathcal{O}$, and fix an isomorphism $\overset{\circ}{E}_{/k} \cong \mathbf{G}_{m/k}$. This induces an orientation on $E_{/k}$, *i.e.*, a bijection between $\mathbf{Z}/N\mathbf{Z}$ and the set of components of $E_{/k}$. The component corresponding to $\nu \in \mathbf{Z}/N\mathbf{Z}$ will be denoted C_ν. If $\beta \in \mathbf{Q}[P]^0$ and $\nu \in \mathbf{Z}/N\mathbf{Z}$ then we write $d_\beta(\nu)$ for the degree of the restriction of the flat extension of β to the component C_ν.

The boundary map

$$\partial^n : H^{n+1}_{\mathcal{M}}(E^n, \mathbf{Q}(n+1))_{\mathrm{sgn}} \longrightarrow H^n_{\mathcal{M}}(\overset{\circ}{E}{}^n_{/k}, \mathbf{Q}(n))_{\mathrm{sgn}}$$

arises from the localisation sequence of the pair $(\overset{\circ}{E}{}^n_{/\mathcal{O}}, \overset{\circ}{E}{}^n_{/k})$. The target space is a one-dimensional $\mathbf{Q}$-vector space generated by $\Phi^n_n = \Pi_{\mathrm{sgn}}\big(y_0 \cup \cdots \cup y_n\big)$, where $y_0 = (y_1 \cdots y_n)^{-1}$, and for $1 \leq i \leq n$, y_i is a coordinate on the i^{th} copy of $\mathbf{G}_{m/k}$ (*cf.* 1.5 below).

The main result of this paper is :

Theorem.
$$\partial^n \circ \mathcal{E}^n_P(\beta) = C^n_{P,N}\Big(\sum_{\nu \in \mathbf{Z}/N\mathbf{Z}} d_\beta(\nu)\, \mathbf{B}_{n+2}(\langle \frac{\nu}{N} \rangle)\Big).\Phi^n_n,$$

where $C^n_{P,N}$ is an explicit nonzero constant, $\mathbf{B}_k(X)$ is the m-th Bernoulli polynomial, and $0 \leq \langle x \rangle < 1$ is the representative of $x \in \mathbf{Q}/\mathbf{Z}$.

The case $n = 1$ was found by Bloch and Grayson by a somewhat different method. The reader will find applications in their paper, and in the case $n = 2$ in [Mestre, Schappacher 1990, §§3.4, 3.5] — *cf.* section 6 below. In these applications $F = \mathbf{Q}$ and the theorem is used to describe the obstruction to the Eisenstein symbols belonging to the 'integral' motivic cohomology $H^{n+1}_{\mathcal{M}}(E^n, \mathbf{Q}(n+1))_{\mathbf{Z}}$.

The formula of the theorem was discovered by the second author while studying the work of Beilinson on modular curves [Beilinson 1986]. There $\mathcal{E}^n_P$ (which Beilinson denotes $\mathcal{E}^l_{\mathcal{M}}$) is constructed for the universal elliptic curve over the field of modular functions. Beilinson's main result concerning the symbol (Theorem 3.1.7 of *loc. cit.*) is equivalent to 7.4 below, but his proof is analytic, in contrast to our algebraic approach.

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1. The basic formula

We continue to use (and expand upon) the notation of the introduction.

1.1 *The Eisenstein symbol.* We recall the construction of the Eisenstein symbol map, following [Deninger 1989, §8]. For an integer $n \geq 1$, let $p_i : E^n \to E$ $(1 \leq i \leq n)$ denote the projections, and $p_0 = -\sum_{i=1}^{n} p_i$. Write $U = E - P$, and define

$$U^{n\prime} = \bigcap_{0 \leq i \leq n} p_i^{-1}(U).$$

If we need to emphasize the dependence on P we write U_P, etc.

For $i = 0, \ldots, n$, let $\beta_i \in \mathbf{Q}[P]^0$, and choose functions $f_i \in \mathcal{O}(U)^* \otimes \mathbf{Q}$ with divisors β_i. We use the "symbol" notation $\{-, \ldots, -\}$ for the cup product

$$\cup : \otimes^l H^1_{\mathcal{M}}(-, \mathbf{Q}(1)) \longrightarrow H^l_{\mathcal{M}}(-, \mathbf{Q}(l)).$$

Then there is a well-defined map

$$\Theta^n_P : \mathbf{Q}[P]^{0^{\otimes n+1}} \longrightarrow H^{n+1}_{\mathcal{M}}(U^{n\prime}, \mathbf{Q}(n+1))^{P^n}_{\mathrm{sgn}} \tag{1.1.1}$$

given by

$$\beta_0 \otimes \cdots \otimes \beta_n \quad \mapsto \Pi_{P^n} \circ \Pi_{\mathrm{sgn}} \, \{p_0^* f_0, \ldots, p_n^* f_n\}.$$

Here $\Pi_{P^n} = \frac{1}{\#P(\overline{F})^n} \sum_{\mathbf{x} \in P(\overline{F})^n} T^*_{\mathbf{x}}$ is the projector onto the space of $P(\overline{F})^n$-invariants. With the special choices

$$\beta_1 = \cdots = \beta_n = \alpha_P = \sum_{x \in P(\overline{F})} (0) - (x), \tag{1.1.2}$$

it is the first step of the construction of the Eisenstein symbol map $\mathcal{E}^n_P$, and other choices of $\beta_1, \ldots \beta_n$ do not give rise to new elements of motivic cohomology. However we will not make this substitution at once, in order to preserve the symmetry for the subsequent calculation. Note that we are taking the *invariants* under translations by $P(\overline{F})$, rather than the coinvariants considered in [Deninger 1989], in order to calculate explicitly.

The second step in the construction—only needed when $n \geq 2$—is the decomposition of the target space of 1.1.1 into eigenspaces under the L^{-1}-multiplication. This will be discussed in section 4.

1.2 *Varying P.* Let $P \overset{j}{\hookrightarrow} Q$ be (a closed immersion of) two finite subgroup schemes of E defined over F. Then there are commutative diagrams :

(1.2.1)
$$\begin{tikzcd}
\mathbf{Q}[Q]^{0^{\otimes n+1}} \arrow[r, "\Theta^n_Q"] & H^{n+1}_{\mathcal{M}}(U^{n\prime}_Q, \mathbf{Q}(n+1))^{Q^n}_{\mathrm{sgn}} \\
\mathbf{Q}[P]^{0^{\otimes n+1}} \arrow[r, "\Theta^n_P"] \arrow[u, "j_!"] & H^{n+1}_{\mathcal{M}}(U^{n\prime}_P, \mathbf{Q}(n+1))^{P^n}_{\mathrm{sgn}} \arrow[u, "\mathrm{res}"']
\end{tikzcd}$$

and

$$\begin{tikzcd}
\mathbf{Q}[Q]^0 \arrow[r] & H^{n+1}_{\mathcal{M}}(U_Q^{n\prime},\mathbf{Q}(n+1))^{Q^n}_{\mathrm{sgn}} \\
\mathbf{Q}[P]^0 \arrow[r] \arrow[u, "j_!"] & H^{n+1}_{\mathcal{M}}(U_P^{n\prime},\mathbf{Q}(n+1))^{P^n}_{\mathrm{sgn}} \arrow[u, "(Q:P)^n\times\mathrm{res}"']
\end{tikzcd} \tag{1.2.2}$$

where $j_!$ is extension by zero, and the unlabelled horizontal arrows are the maps

$$\beta\mapsto\Theta^n_Q(\beta\otimes\alpha_Q^{\otimes n}) \text{ and } \beta\mapsto\Theta^n_P(\beta\otimes\alpha_P^{\otimes n}) \text{ respectively.}$$

Now let $L\geq 1$ be an integer, and write $\tilde{P}=[\times L]^{-1}(P)\subset E$, $\tilde{U}=E-\tilde{P}$, etc. Write $\pi:\tilde{P}\to P$ for the projection. Multiplication by L induces a Galois covering

$$[\times L]:\tilde{U}^{n\prime}\longrightarrow U^{n\prime}.$$

By Galois descent, this gives a homomorphism

$$[\times L]^*:H^{n+1}_{\mathcal{M}}(U^{n\prime},\mathbf{Q}(n+1))^{P^n}\longrightarrow H^{n+1}_{\mathcal{M}}(\tilde{U}^{n\prime},\mathbf{Q}(n+1))^{\tilde{P}^n},$$

and we have two further commutative diagrams:

$$\begin{tikzcd}
\mathbf{Q}[\tilde{P}]^{0\otimes n+1} \arrow[r, "\Theta^n_{\tilde{P}}"] & H^{n+1}_{\mathcal{M}}(\tilde{U}^{n\prime},\mathbf{Q}(n+1))^{\tilde{P}^n}_{\mathrm{sgn}} \\
\mathbf{Q}[P]^{0\otimes n+1} \arrow[r, "\Theta^n_P"] \arrow[u, "\pi^*"] & H^{n+1}_{\mathcal{M}}(U^{n\prime},\mathbf{Q}(n+1))^{P^n}_{\mathrm{sgn}} \arrow[u, "{[\times L]^*}"']
\end{tikzcd} \tag{1.2.3}$$

and

$$\begin{tikzcd}
\mathbf{Q}[\tilde{P}]^0 \arrow[r] & H^{n+1}_{\mathcal{M}}(\tilde{U}^{n\prime},\mathbf{Q}(n+1))^{\tilde{P}^n}_{\mathrm{sgn}} \\
\mathbf{Q}[P]^0 \arrow[r] \arrow[u, "\pi^*"] & H^{n+1}_{\mathcal{M}}(U^{n\prime},\mathbf{Q}(n+1))^{P^n}_{\mathrm{sgn}} \arrow[u, "{[\times L]^*}"']
\end{tikzcd} \tag{1.2.4}$$

with the unlabelled maps in (1.2.4) being

$$\beta\mapsto\Theta^n_{\tilde{P}}(\beta\otimes\alpha_{\tilde{P}}^{\otimes n}) \text{ and } \beta\mapsto\Theta^n_P(\beta\otimes\alpha_P^{\otimes n}).$$

All of this is straightforward to prove by direct calculation from the formulae in [Deninger 1989, proof of 8.2].

1.3 *Base change.* Let F'/F be a finite extension, v' a discrete valuation of F', and v the restriction of v' to F. Assume that the residue field extension k'/k is separable. Then the following square is commutative:

$$\begin{tikzcd}
H^{n+1}_{\mathcal{M}}(E^n_{/F'},\mathbf{Q}(n+1)) \arrow[r, "\partial_{v'}"] & H^n_{\mathcal{M}}(\mathring{E}_{/k'},\mathbf{Q}(n)) \\
H^{n+1}_{\mathcal{M}}(E^n_{/F},\mathbf{Q}(n+1)) \arrow[r, "\partial_v"] \arrow[u, "\mathrm{res}_{F'/F}"] & H^n_{\mathcal{M}}(\mathring{E}_{/k},\mathbf{Q}(n)) \arrow[u, "e(v'/v)\times\mathrm{res}_{k'/k}"']
\end{tikzcd}$$

Here $\mathrm{res}_{F'/F}$, $\mathrm{res}_{k'/k}$ are the restriction homomorphisms, and $e(v'/v)$ is the ramification index. (Recall that we are assuming k to be perfect.)

In view of 1.2.1 and 1.3, we may now restrict to the following situation.

1.4 *Assumptions.*

— $E_{/k}$ is an untwisted Néron N-gon with $N \geq 3$;

— $P = \boldsymbol{\mu}_N \times \mathbf{Z}/N\mathbf{Z} \subset E(F)$ is a level N structure on E;

— $P_{/k}$ gives the standard level N structure on $(E_{/k})^{\mathrm{smooth}} = \mathbf{G}_m \times \mathbf{Z}/N\mathbf{Z}$.

1.5 Write $U_N = E - P$, and U_1 for the complement of the zero section in $\overset{\circ}{E}_{/\mathcal{O}}$. Consider the Galois covering[*]:

$$\begin{tikzcd}
U^{n\,\prime}_{N\,/k} = \bigcup\limits_{0\leq i\leq n} p_i^{-1}((\mathbf{G}_m - \boldsymbol{\mu}_N)\times \mathbf{Z}/N\mathbf{Z}) \arrow[d, "{[\times N]}"] \\
U^{n\,\prime}_{1\,/k} = \bigcup\limits_{0\leq i\leq n} p_i^{-1}(\mathbf{G}_m - 1)
\end{tikzcd}$$

which by Galois descent gives an isomorphism

$$[\times N]^* : H^{\bullet}_{\mathcal{M}}(U^{n\,\prime}_{1\,/k}, \mathbf{Q}(*)) \xrightarrow{\sim} H^{\bullet}_{\mathcal{M}}(U^{n\,\prime}_{N\,/k}, \mathbf{Q}(*))^{P^n}. \tag{1.5.1}$$

In the next section we shall prove the following basic formula for the composite of Θ^n_P with the boundary map in motivic cohomology

$$\partial_v : H^{\bullet}_{\mathcal{M}}(U^{n\,\prime}_{N\,/F}, \mathbf{Q}(*))^{P^n}_{\mathrm{sgn}} \longrightarrow H^{\bullet-1}_{\mathcal{M}}(U^{n\,\prime}_{N\,/k}, \mathbf{Q}(*-1))^{P^n}_{\mathrm{sgn}}.$$

1.6 Proposition. $\partial_v \Theta^n_P(\otimes\beta_i) = \pm \dfrac{n+1}{N^{2n+1}} \sum\limits_{q=0}^{n} \dbinom{n}{q} \sum\limits_{1\neq\zeta\in\boldsymbol{\mu}_N} \dfrac{\zeta \hat{d}_0(\zeta)\cdots\hat{d}_n(\zeta)}{(\zeta-1)^{q+2}}\, [\times N]^* \Phi^n_q.$

The meanings of the symbols are :

— $d_i(\nu) = d_{\beta_i}(\nu) = \sum\limits_{\zeta\in\boldsymbol{\mu}_N} \beta_i((\zeta,\nu))$ for $\nu \in \mathbf{Z}/N\mathbf{Z}$;

— $\hat{d}_i(\zeta) = \sum\limits_{\nu\in\mathbf{Z}/N\mathbf{Z}} \zeta^{\nu} d_i(\nu)$ is the Fourier transform of d_i;

— Φ^n_q is the element of $H^n_{\mathcal{M}}(U^{n\,\prime}_{1\,/k}, \mathbf{Q}(n))_{\mathrm{sgn}}$ given as follows : let $y = t^{-1}$ be the inverse of the natural coordinate on $\mathbf{G}_m$, and let $y_i = p_i^*(y)$, for the $n+1$ projections $p_0, \ldots, p_n$:

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[*] If $char(k)$ divides N then $[\times N]$ is the composite of a Galois covering and a power of the Frobenius mapping. As the Frobenius induces an automorphism on motivic cohomology, $[\times N]^*$ is an isomorphism in this case also.

$\mathbf{G}_m^n \to \mathbf{G}_m$. Let $\mathcal{S}_{n+1}$ be the symmetric group permuting the coordinates $y_0,\ldots,y_n$. Then

$$\Phi_q^n = \Pi_{\mathrm{sgn}}\{y_1,\ldots,y_q,1-y_{q+1},\ldots,1-y_n\}.$$

1.7 *Remark.* Note in passing that for the special functions f_i, $i=1,\ldots,n$ with divisors $\operatorname{div} f_i = \alpha$ as in 1.1.2, we have that $d_i(\nu) = N^2\delta_{\nu,0} - N$, so that here we find for $\zeta \neq 1$ that $\hat{d}_i(\zeta) = N^2$.

We will see in section 4 that the proposition actually implies the theorem.

2. The calculation

2.1 We begin with some geometry on the arithmetic surface $E_{/\mathcal{O}}$. For the moment, we need only assume that $E_{/k}$ is an untwisted Néron N-gon with $N \geq 3$, and that P is a finite subscheme of E whose flat extension $P_{/\mathcal{O}}$ is contained in the smooth part of $E_{/\mathcal{O}}$. We normalise the orientation of the special fibre $E_{/k} = \cup_{\nu\in\mathbf{Z}/N\mathbf{Z}} C_\nu$ and the coordinate t_ν on C_ν such that $t_\nu = 0$, ∞ are the points of intersection of C_ν with $C_{\nu-1}$, $C_{\nu+1}$ respectively. (There is no ambiguity as $N \geq 3$.)

Let $f \in \mathcal{O}^*(U)\otimes\mathbf{Q}$, and let $a(\nu)$ be the order of f along the ν^{th} component C_ν of $E_{/k}$. Choose once and for all a uniformiser π of the valuation v, and let $g^{(\nu)} = \pi^{-a(\nu)} f \in F(E)^*$. Since $\operatorname{ord}_{C_\nu}(\pi) = 1$, the function $g^{(\nu)}$ is regular outside of P and the C_μ with $\mu \neq \nu$; so its restriction to C_ν is an element of $k(C_\nu)\otimes\mathbf{Q}$ which we also denote $g^{(\nu)}$. Let $D_{/\mathcal{O}}$ be the flat extension of $\operatorname{div} f$ to $E_{/\mathcal{O}}$, and $d(\nu) = \deg\big(D_{/\mathcal{O}}\cap C_\nu\big)$ (*cf.* introduction).

Proposition 2.2.

(i) $\operatorname{div} g^{(\nu)} = (D_{/\mathcal{O}}\cap C_\nu) - b(\nu-1).(0) + b(\nu).(\infty)$, *where* $b(\nu) = a(\nu+1) - a(\nu)$;

(ii) $d(\nu) = b(\nu-1) - b(\nu)$.

Proof. (ii) follows from (i) as $\deg(\operatorname{div} g^{(\nu)}) = 0$. The only remaining non-trivial assertions are the claimed multiplicities at $t_\nu = 0$, ∞. To verify these, represent the completed local ring at 0 as $R = \hat{\mathcal{O}}[[u,v]]/(uv-\pi)$, where $u=0$, $v=0$ are local equations for C_ν, $C_{\nu-1}$ respectively. Then the image of f in the field of quotients of R is of the form:

$$\begin{aligned} f = (\text{unit})\times u^{a(\nu)}v^{a(\nu-1)} &= (\text{unit})\times \pi^{a(\nu)}v^{-b(\nu-1)} \\ &= (\text{unit})\times \pi^{a(\nu-1)}u^{b(\nu-1)} \end{aligned}$$

Therefore the order of $g^{(\nu)}$ at $t_\nu = 0$ is $-b(\nu-1)$, and the order of $G^{(\nu-1)}$ at $t_{\nu-1} = \infty$ is $b(\nu-1)$.

2.3 Now we continue under the assumptions of 1.4. Then $g^{(\nu)} \in \mathcal{O}^*(\mathbf{G}_m - \boldsymbol{\mu}_N)\otimes\mathbf{Q}$, and we write

$$\begin{aligned} G^{(\nu)}(t) = \prod_{\zeta\in\boldsymbol{\mu}_N} g^{(\nu)}(\zeta t) &= (\text{const.})\frac{(t^N-1)^{d(\nu)}}{t^{Nb(\nu-1)}} \\ &= (\text{const.})y^{Nb(\nu)}(1-y^N)^{d(\nu)} \end{aligned}$$

where $y = 1/t$.

2.4 We apply the above with $f = f_i$, $0 \le i \le n$, with the obvious additional subscripts. To calculate the boundary of Θ^n_P we need the following compatibility of the cup-product and the boundary map (see [Loday 1976, 2.3] and [Grayson 1976]).

> *Let* $X/\mathcal{O}$ *be smooth, and* $\partial : H^i_{\mathcal{M}}(X_F, \mathbf{Q}(j)) \to H^{i-1}_{\mathcal{M}}(X_k, \mathbf{Q}(j-1))$ *the boundary map of the localisation sequence. For* $\xi \in H^i_{\mathcal{M}}(X, \mathbf{Q}(j))$*, write* ξ_F*,* ξ_k *for its images in* $H^i_{\mathcal{M}}(X_F, \mathbf{Q}(j))$*,* $H^i_{\mathcal{M}}(X_k, \mathbf{Q}(j))$*. Then for every* ξ*,*
>
> $$\pm(\pi \cup \xi_F) = \xi_k$$
>
> *(the sign depending only on* (i,j)*).*

In particular, up to sign and torsion, the boundary maps in Milnor and Quillen K-theory agree. This gives (up to sign) the following formula for the restriction of $\partial\{p_0^* f_0, \ldots, p_n^* f_n\}$ to the component $C_{\nu_1} \times \ldots \times C_{\nu_n}$:

$$\sum_{r=0}^{n} a_r(\nu_r)\, \{g_0^{(\nu_0)}(y_0), \ldots, \widehat{g_r^{(\nu_r)}(y_r)}, \ldots, g_n^{(\nu_n)}(y_n)\}$$

Here and elsewhere $\nu_0 = -\sum_{i=1}^n \nu_i$. Applying the projector Π_{P^n}—defined in 1.1.1—we obtain

$$N^{-2n} \sum_{\boldsymbol{\nu} \in (\mathbf{Z}/N\mathbf{Z})^n} \sum_{r=0}^{n} a_r(\nu_r)\, \{G_0^{(\nu_0)}(y_0), \ldots, \widehat{G_r^{(\nu_r)}(y_r)}, \ldots, G_n^{(\nu_n)}(y_n)\},$$

where $\boldsymbol{\nu} = (\nu_1, \ldots, \nu_n)$. Applying the inverse of the isomorphism (1.5.1) we write this as the following element of $H^n_{\mathcal{M}}(U^{n\prime}_{1/k}, \mathbf{Q}(n))$:

$$N^{-2n} \sum_{\boldsymbol{\nu} \in (\mathbf{Z}/N\mathbf{Z})^n} \sum_{r=0}^{n} a_r(\nu_r)\, \{y_0^{b_0(\nu_0)}(1-y_0)^{d_0(\nu_0)}, \ldots \widehat{(r)} \ldots, y_n^{b_n(\nu_n)}(1-y_n)^{d_n(\nu_n)}\}.$$

We can expand this in terms of a sum over the symmetric group $\mathcal{S}_{n+1} = \mathrm{Symm}\{0, 1, \ldots, n\}$:

$$\begin{aligned} N^{-2n} \sum_{\boldsymbol{\nu} \in (\mathbf{Z}/N\mathbf{Z})^n} \sum_{q=0}^{n} \sum_{\sigma \in \mathcal{S}_{n+1}} & \frac{\mathrm{sgn}(\sigma)}{(n-q)!q!}\, a_{\sigma 0}(\nu_{\sigma 0}) b_{\sigma 1}(\nu_{\sigma 1}) \cdots b_{\sigma q}(\nu_{\sigma q}) \\ & \times d_{\sigma(q+1)}(\nu_{\sigma(q+1)}) \cdots d_{\sigma n}(\nu_{\sigma n}) \{y_{\sigma 1}, \ldots, y_{\sigma q}, 1 - y_{\sigma(q+1)}, \ldots, 1 - y_{\sigma n}\} \end{aligned}$$

and applying the projector Π_{sgn} we obtain the following expression.

$$\begin{aligned} N^{-2n} \sum_{q=0}^{n} \frac{1}{(n-q)!q!} \sum_{\boldsymbol{\nu} \in (\mathbf{Z}/N\mathbf{Z})^n} \sum_{\sigma \in \mathcal{S}_{n+1}} & a_{\sigma 0}(\nu_0) b_{\sigma 1}(\nu_1) \cdots b_{\sigma q}(\nu_q) \\ & \times d_{\sigma(q+1)}(\nu_{(q+1)}) \cdots d_{\sigma n}(\nu_n) \Phi^n_q. \end{aligned} \tag{2.2.1}$$

2.3 This last expression will be more palpable once it is rewritten in terms of *Fourier transforms*. Recall that we are taking $\hat{\phi}(\zeta) = \sum_{\nu\in\mathbf{Z}/N\mathbf{Z}} \zeta^{\nu}\phi(\nu)$. If $\phi(\nu) = \psi(\nu+a) - \psi(\nu)$, then we have $\hat{\phi}(\zeta) = (\zeta^{-a}-1)\hat{\psi}(\zeta)$. In particular, by 2.1 :

$$\hat{d}_i(\zeta) = (\zeta-1)\hat{b}_i(\zeta) = -\zeta^{-1}(\zeta-1)^2\hat{a}_i(\zeta).$$

Furthermore $\hat{d}_i(1) = \hat{b}_i(1) = 0$. Therefore fixing q, $0 \leq q \leq n$, we have the following identities, valid for any $\sigma \in \mathcal{S}_{n+1}$:

$$-\frac{1}{N}\sum_{1\neq\zeta\in\boldsymbol{\mu}} \frac{\zeta\hat{d}_0(\zeta)\cdots\hat{d}_n(\zeta)}{(\zeta-1)^{q+2}} = \frac{1}{N}\sum_{1\neq\zeta\in\boldsymbol{\mu}_N} \hat{a}_{\sigma 0}(\zeta)\,\hat{b}_{\sigma 1}\cdots\hat{b}_{\sigma q}(\zeta)\cdot\hat{d}_{\sigma(q+1)}(\zeta)\cdots\hat{d}_{\sigma n}(\zeta)$$

$$= \sum_{\boldsymbol{\nu}\in(\mathbf{Z}/N\mathbf{Z})^{n'}} a_{\sigma 0}(\nu_0)b_{\sigma 1}(\nu_1)\cdots b_{\sigma q}(\nu_q)\cdot d_{\sigma(q+1)}(\nu_{(q+1)})\cdots d_{\sigma n}(\nu_n).$$

Consequently, expression 2.2.1 becomes (up to sign)

$$N^{-1-2n}\sum_{q=0}^{n}\frac{(n+1)!}{(n-q)!q!}\Phi_q^n \sum_{1\neq\zeta\in\boldsymbol{\mu}_N}\frac{\zeta\hat{d}_0(\zeta)\cdots\hat{d}_n(\zeta)}{(\zeta-1)^{q+2}}.$$

This proves proposition 1.6.

2.5 *Fourier transforms of Bernoulli polynomials.* Recall the definition of the Bernoulli polynomials $\mathbf{B}_k$:

$$\frac{te^{tX}}{e^t-1} = \sum_{k=0}^{\infty}\mathbf{B}_k(X)\,\frac{t^k}{k!}.$$

Thus, for example,

$$\mathbf{B}_0(X) = 1, \qquad \mathbf{B}_1(X) = X - \frac{1}{2}, \qquad \mathbf{B}_2(X) = X^2 - X + \frac{1}{6}$$

$$\mathbf{B}_3(X) = X^3 - \frac{3}{2}X^2 + \frac{1}{2}X, \qquad \mathbf{B}_4(X) = X^4 - 2X^3 + X^2 - \frac{1}{30}$$

Define, for $\zeta\in\boldsymbol{\mu}_N$, $\hat{\mathbf{B}}_{k,N}(\zeta) = \sum_{\nu\in\mathbf{Z}/N\mathbf{Z}} \mathbf{B}_k(\langle\frac{\nu}{N}\rangle)\zeta^{\nu}$. Then it follows from the definition of the $\mathbf{B}_k$ that

$$\sum_{k=0}^{\infty}\hat{\mathbf{B}}_{k,N}(\zeta)\frac{t^k}{k!} = \frac{t}{e^t-1}\sum_{\nu=0}^{N-1}(\zeta e^{t/N})^{\nu} = \frac{t}{(\zeta e^{t/N}-1)}.$$

Substitute $u = e^{t/N}$ and define

$$\tilde{\mathbf{B}}_k(\zeta) := N^{1-k}\hat{\mathbf{B}}_{k,N}(\zeta) = k\left(u\frac{d}{du}\right)^{k-1}\frac{1}{\zeta u - 1}\,|_{u=1}.$$

¿From this it is elementary to deduce the following proposition the first part of which is a convenient reformulation of the distribution property of the Bernoulli polynomials.

2.6 Proposition. *(i) For every integer $L \geq 1$,*

$$\sum_{\eta^L = \zeta} \tilde{\mathbf{B}}_k(\eta) = L^k \, \tilde{\mathbf{B}}_k(\zeta).$$

(ii) For all $k \geq j \geq 2$, there exist rational numbers $a_{j,k}$ independent of N such that $a_{k,k} = (-1)^{k-1}/k!$ and

$$\frac{\zeta}{(\zeta-1)^k} = \sum_{j=2}^{k} a_{j,k} \tilde{\mathbf{B}}_j(\zeta).$$

For instance, one has

$$\frac{\zeta}{(\zeta-1)^2} = -\frac{1}{2}\, \tilde{\mathbf{B}}_2(\zeta) \qquad \frac{\zeta}{(\zeta-1)^3} = \frac{1}{4}\, \tilde{\mathbf{B}}_2(\zeta) + \frac{1}{6}\, \tilde{\mathbf{B}}_3(\zeta)$$

$$\frac{\zeta}{(\zeta-1)^4} = -\frac{1}{6}\, \tilde{\mathbf{B}}_2(\zeta) - \frac{1}{6}\, \tilde{\mathbf{B}}_3(\zeta) - \frac{1}{24}\, \tilde{\mathbf{B}}_4(\zeta)$$

3. The case $n = 1$ over a number field

We are now already in a position to verify the theorem in the case $n = 1$ [Bloch, Grayson 1986]. In fact we will prove a more general result. We first describe the situation in terms of K-theory to make apparent the relation with *loc. cit.*

Let E be an elliptic curve over a number field F. Consider the localisation sequence:

$$\begin{tikzcd}
0 \arrow[r] & H^2_{\mathcal{M}}(E, \mathbf{Q}(2)) \arrow[r] \arrow[d, equal] & H^2_{\mathcal{M}}(F(E), \mathbf{Q}(2)) \arrow[r, "\mathcal{T}"] \arrow[d, equal] & \coprod_{\xi \in |E|} F(\xi)^* \\
 & K_2(E) \otimes \mathbf{Q} & K_2(F(E)) \otimes \mathbf{Q} &
\end{tikzcd}$$

Here $|E|$ is the set of closed points of E, and the sequence is exact on the left as K_2 of a number field is torsion. The boundary map $\mathcal{T}$ is the "tame symbol".

Let $f_j, g_j \in F(E)^*$ be a finite collection of rational functions on E such that $\sum_j \{f_j, g_j\} \in \ker \mathcal{T}$. Then $\sum_j \{f_j, g_j\}$ defines an element of $H^2_{\mathcal{M}}(E_{/F}, \mathbf{Q}(2)) = K_2(E_{/F}) \otimes \mathbf{Q}$.

Now let v be a finite place of F, with residue field k, at which E has split multiplicative reduction with special fibre a Néron N-gon. We intend to calculate its image under the boundary map

$$\partial : K_2(E) \otimes \mathbf{Q} \longrightarrow K_1'(E_{/k}) \otimes \mathbf{Q}.$$

First note :—

$$K_1'(E_{/k}) \otimes \mathbf{Q} \cong H^1_{\mathcal{M}}(\overset{\circ}{E}_{/k}, \mathbf{Q}(1)) \cong \mathbf{Q}.$$

In fact, since k is finite, the localisation sequence gives a short exact sequence

$$\begin{tikzcd}
0 \arrow[r] & K_1'(E_{/k})\otimes\mathbf{Q} \arrow[r] & K_1(E^{\text{smooth}}_{/k})\otimes\mathbf{Q} \arrow[r] \arrow[d, equal] & K_0(E^{\text{sing}}_{/k})\otimes\mathbf{Q} \arrow[d, equal] \\
 & & \coprod_{\nu\in\mathbf{Z}/N\mathbf{Z}} \mathbf{Q}\cdot t_\nu \arrow[r, "\delta"] & \mathbf{Q}[\mathbf{Z}/N\mathbf{Z}]
\end{tikzcd}$$

with $\delta(t_\nu)=(\nu)-(\nu-1)$. Then the restriction

$$K_1(E^{\text{smooth}}_{/k})\otimes\mathbf{Q}\to K_1(\mathring{E}_{/k})\otimes\mathbf{Q}=H^1_{\mathcal{M}}(\mathring{E}_{/k},\mathbf{Q}(1))=k(\mathbf{G}_m)^*\otimes\mathbf{Q}\cdot t_0$$

induces an isomorphism on the image of $K_1'(E_{/k})\otimes\mathbf{Q}$.

For the calculation we only need the following hypothesis on f_j, f_j':

The closure of the support of the divisors of f_j, f_j' is contained in the smooth part of $E_{/\mathcal{O}}$.

Then, since k is finite, the reduction modulo v of this support is contained in $\boldsymbol{\mu}_M\times\mathbf{Z}/N\mathbf{Z}$ for some M; so by passing to a ramified extension F'/F and using 1.3 we may, and do, assume $M=N$. The first part of the calculation of §2 then gives (up to sign) :

$$\partial\Big(\sum_j\{f_j,f_j'\}\Big)=\frac{2}{N^3}\sum_{1\neq\zeta\in\boldsymbol{\mu}_N}[\times N]^*\left[\frac{\zeta}{(\zeta-1)^2}\Phi_0^1+\frac{\zeta}{(\zeta-1)^3}\Phi_1^1\right]\sum_j\hat{d}_j(\zeta)\,\hat{d}_j'(\zeta)$$

where $d_j(\nu)$, $d_j'(\nu)$ are the degrees of the restriction to C_ν of the closures of the divisors of f_j, f_j'. Using the examples following 2.6 and the relation

$$\Phi_0^1=\frac{1}{2}\sum_{\sigma\in\mathcal{S}_{n+1}}\operatorname{sgn}(\sigma)\{1-y_{\sigma1}\}=\frac{1}{2}\Big\{\frac{1-y_1}{1-y_0}\Big\}=\frac{1}{2}\{y_1\}=\frac{1}{2}\Phi_1^1$$

(*cf.* 5.2 below), we obtain a formula involving only $\mathbf{B}_3$ and Φ_1^1. (It is no accident that $\tilde{\mathbf{B}}_2$ drops out in this way — see section 4 below.) Using $\mathbf{B}_3\big(\langle\frac{-\nu}{N}\rangle\big)=-\mathbf{B}_3\big(\langle\frac{\nu}{N}\rangle\big)$ and the fact that $[\times N]^*\Phi_1^1=N\Phi_1^1$, this gives :

3.2 Proposition. $$\partial\Big(\sum_j\{f_j,f_j'\}\Big)=\pm\frac{1}{3N}\sum_{\mu,\nu\in\mathbf{Z}/N\mathbf{Z}}\sum_j d_j(\mu)\,d_j'(\nu-\mu)\;\mathbf{B}_3\big(\langle\frac{\nu}{N}\rangle\big)\cdot\Phi_1^1.$$

In the special case where all f_j' have the standard divisor this proposition simplifies in view of 1.7 and due to the fact that $\sum_\mu d_j(\mu)=0$.

3.3 Corollary. *Let* $\sum_j\{f_j,g\}\in\ker\mathcal{T}$ *with* $\operatorname{div}g=\sum_{x\in P}(0)-(x)$. *Then*

$$\partial\Big(\sum_j\{f_j,g\}\Big)=\pm\frac{N}{3}\sum_{\nu\in\mathbf{Z}/N\mathbf{Z}}\sum_j d_j(\nu)\;\mathbf{B}_3\big(\langle\frac{\nu}{N}\rangle\big)\cdot\Phi_1^1.$$

3.4 We should remark that if v is a place of F at which the reduction of E is not split multiplicative, then $K_1'(E_{/k}) \otimes \mathbf{Q} = 0$. Thus the restriction to the case where $E_{/k}$ is an untwisted Néron polygon does not miss any interesting cases.

3.5 Now let $\mathcal{O}$ momentarily denote the (global) ring of integers of F. We have the exact sequence

$$(\text{torsion}) \longrightarrow K_2(E_{/\mathcal{O}}) \longrightarrow K_2(E_{/F}) \overset{\partial=\coprod \partial_v}{\longrightarrow} \coprod_v K_1'(E_{/k_v}) \longrightarrow \cdots$$

The fact noted in 3.4, that the target of ∂_v is torsion unless v is a place of split multiplicative reduction for E is in accordance with relative versions of Beilinson's conjectures—*cf.* [Deligne 1985], [Ramakrishnan 1989]. In fact, we have

$$\dim_{\mathbf{Q}} K_1'(E_{/k_v}) \otimes \mathbf{Q} = \operatorname{ord}_{s=0} L_v(E,s)$$

where the L-function of E/F is written $L(E/F,s) = \prod_v L_v(E,s)^{-1}$. But even if the reduction at v is split multiplicative, the tame symbol may nonetheless be trivial on the elements of $K_2(E_{/F})$ we considered here. In fact, if $E_{/k_v}$ is a Néron polygon with one or two sides, then for rational functions f_j, f_j' with reduced divisors supported in $E_{/k}^{\text{smooth}}$, we always have $\partial\left(\sum_j \{f_j, f_j'\}\right) = 0$ because $\mathbf{B}_3(1-x) = -\mathbf{B}_3(x)$.

3.6 *Remark.* When the divisors of f_j, f_j' are supported in torsion points, proposition 3.2 implies the formula of [Bloch, Grayson 1986, p.88]—*cf.* [Mestre, Schappacher 1990, 1.5.1]. But there are also examples of elements $\sum_j \{f_j, f_j'\} \in \ker \mathcal{T}$ when the support of the divisors of f_j, f_j' contains points of infinite order. The first such example, on a curve with complex multiplication, was found by R. Ross (1990 Rutgers Thesis). Recently Jan Nekovář, modifying successfully an earlier attempt by one of the authors (NS), wrote down a one-parameter family of elliptic curves on which non-trivial such elements can be constructed. Some curves in this family have places v with non-trivial $K_1'(E_{/k_v}) \otimes \mathbf{Q}$. They provide concrete applications of the general statement 3.2. But we do not go into this here.

4. The weight decomposition.

4.1 The remaining step in the construction of the Eisenstein symbol is the "weight decomposition" of $H^\bullet_{\mathcal{M}}(U^{n\prime}_{N/F},\mathbf{Q}(*))^{P^n}_{\mathrm{sgn}}$ under the "L^{-1}"-multiplication. Recall [Deninger 1989, §8] that if $L>1$, and $\tilde{P}=[\times L]^{-1}P$ as in 1.2 above, the endomorphism $\mathcal{L}$ is defined by the commutativity of the diagram:

$$\begin{tikzcd}
H^\bullet_{\mathcal{M}}(U^{n\prime},\mathbf{Q}(*))^{P^n}_{\mathrm{sgn}} \arrow[r, hook, "j^*"] & H^\bullet_{\mathcal{M}}(\tilde{U}^{n\prime},\mathbf{Q}(*))^{P^n}_{\mathrm{sgn}} \arrow[r, two heads] & H^\bullet_{\mathcal{M}}(\tilde{U}^{n\prime},\mathbf{Q}(*))^{\tilde{P}^n}_{\mathrm{sgn}} \\
 & \mathcal{L} & H^\bullet_{\mathcal{M}}(U^{n\prime},\mathbf{Q}(*))^{P^n}_{\mathrm{sgn}} \arrow[u, "\wr\,{[\times L]^*}"']
\end{tikzcd} \tag{4.1.1}$$

where j^* is induced by the inclusion $j:\tilde{U}^{n\prime}\hookrightarrow U^{n\prime}$. On the image of $H^\bullet_{\mathcal{M}}(E^n,\mathbf{Q}(*))_{\mathrm{sgn}}$ (which is invariant under P^n), $\mathcal{L}$ coincides with $[\times L]^{*-1}$, and is simply multiplication by L^{-n}.

4.2 Theorem. *[Beilinson 1986], [Deninger 1989]. $H^\bullet_{\mathcal{M}}(U^{n\prime},\mathbf{Q}(*))^{P^n}_{\mathrm{sgn}}$ decomposes into eigenspaces on which $\mathcal{L}$ acts as multiplication by L^{-n-i}, $0\le i\le n-1$; and the inclusion $U^{n\prime}\hookrightarrow E^n$ induces an isomorphism of $H^\bullet_{\mathcal{M}}(E^n,\mathbf{Q}(*))_{\mathrm{sgn}}$ with the L^{-n}-eigenspace of $H^\bullet_{\mathcal{M}}(U^{n\prime},\mathbf{Q}(*))^{P^n}_{\mathrm{sgn}}$.*

The definition of the Eisenstein symbol is now as follows: let $\alpha=\sum_{x\in P}(0)-(x)$. Then $\mathcal{E}^n_P(\beta)$ is the projection of $\Theta^n_P(\beta\otimes\alpha^{\otimes n})$ into the L^{-n} eigenspace, viewed as an element of $H^{n+1}_{\mathcal{M}}(E^n,\mathbf{Q}(n+1))$ under the isomorphism of theorem 4.2.

Let us give a slightly different proof of theorem 4.2. Recall that

$$U^{n\prime}=\bigl\{(x_1,\dots,x_n)\in E^n\bigm|\text{for all }0\le i\le n,\ x_i\notin P\bigr\},$$

where $x_0=-x_1-\ldots-x_n$. We define, for $0\le q\le n$,

$$Y^n_q=\bigl\{(x_1,\dots,x_n)\in E^n\bigm|\text{at least } q \text{ of the } x_i\text{'s are in } P\bigr\};$$
$$\overset{\circ}{Y}{}^n_q=\bigl\{(x_1,\dots,x_n)\in E^n\bigm|\text{exactly } q \text{ of the } x_i\text{'s are in } P\bigr\}.$$

Then $U^{n\prime}=\overset{\circ}{Y}{}^n_0$, $E^n-U^{n\prime}=Y^n_1$ and

$$\overset{\circ}{Y}{}^n_1\xrightarrow{\sim}U^{(n-1)\prime}\times\{0,\dots,n\}\times P. \tag{4.2.1}$$

Moreover we have a decomposition $E^n=\coprod\limits_{0\le q\le n}\overset{\circ}{Y}{}^n_q$ of E^n into locally closed subsets which are invariant under the action of $\mathcal{S}_{n+1}\cdot P^n$. This group acts transitively on the set of components of $\overset{\circ}{Y}{}^n_q$ with isotropy subgroup $(\mathcal{S}_{n+1-q}\times\mathcal{S}_q)\cdot P^{n-q}$. Notice that the subgroup $\mathcal{S}_q$ acts trivially on the component

$$\bigl\{(x_1,\dots,x_n)\in E^n\bigm|x_0,\dots,x_{n-q-1}\notin P,\ x_{n-q}=\ldots=x_n=0\bigr\}$$

from which it follows that if $q \geq 2$ then

$$H^{\bullet}_{\mathcal{M}}(\mathring{Y}^n_q, \mathbf{Q}(*))^{P^n}_{\mathrm{sgn}} = 0.$$

Then by the long exact sequences of motivic cohomology, we deduce that

$$H^{\bullet}_{\mathcal{M}}(E^n, \mathbf{Q}(*))_{\mathrm{sgn}} = H^{\bullet}_{\mathcal{M}}(E^n, \mathbf{Q}(*))^{P^n}_{\mathrm{sgn}} = H^{\bullet}_{\mathcal{M}}(\mathring{Y}^n_0 \cup \mathring{Y}^n_1, \mathbf{Q}(*))^{P^n}_{\mathrm{sgn}}.$$

Moreover, by 4.2.1,

$$H^{\bullet}_{\mathcal{M}}(\mathring{Y}^n_1, \mathbf{Q}(*))^{P^n}_{\mathrm{sgn}_{n+1}} \xrightarrow{\sim} H^{\bullet}_{\mathcal{M}}(U^{(n-1)\prime}, \mathbf{Q}(*))^{P^{n-1}}_{\mathrm{sgn}_n}.$$

We therefore have a long exact sequence:

$$\begin{aligned} &H^{\bullet-2}_{\mathcal{M}}(U^{(n-1)\prime}, \mathbf{Q}(*-1))^{P^{n-1}}_{\mathrm{sgn}_n} \xrightarrow{\delta} H^{\bullet}_{\mathcal{M}}(E^n, \mathbf{Q}(*))^{P^n}_{\mathrm{sgn}_{n+1}} \\ &\longrightarrow H^{\bullet}_{\mathcal{M}}(U^{n\prime}, \mathbf{Q}(*))^{P^n}_{\mathrm{sgn}_{n+1}} \longrightarrow H^{\bullet-1}_{\mathcal{M}}(U^{(n-1)\prime}, \mathbf{Q}(*-1))^{P^{n-1}}_{\mathrm{sgn}_n} \longrightarrow \ldots \end{aligned} \tag{4.2.2}$$

By 4.2.1 the localisation sequence is compatible with the family of endomorphisms which are $\mathcal{L}$ on the middle two terms and $L^{-2}\mathcal{L}$ on the outside ones. By simultaneous induction it follows that:

(4.2.3) The boundary maps δ are zero.

(4.2.4) The eigenvalues of $\mathcal{L}$ on $H^{\bullet}_{\mathcal{M}}(U^{n\prime}, \mathbf{Q}(*))^{P^n}_{\mathrm{sgn}_{n+1}}$ are L^{-n-i}, for $0 \leq i \leq n-1$, and the corresponding eigenspaces are isomorphic to $H^{\bullet-i}_{\mathcal{M}}(E^{n-i}, \mathbf{Q}(*-i))_{\mathrm{sgn}_{n-i+1}}$.

4.3 One would like a similar statement with E replaced by $\mathbf{G}_m$ and P by $\boldsymbol{\mu}_N$. The exact sequence analogous to 4.2.2 still holds. For us the only case of interest is $\bullet = * = n$. Then δ vanishes, since the space

$$H^n_{\mathcal{M}}(\mathbf{G}^n_{m/k}, \mathbf{Q}(n))_{\mathrm{sgn}}$$

is one-dimensional, spanned by the symbol $\{y_1, \ldots, y_n\}$. Hence it will certainly inject into $H^n_{\mathcal{M}}(k(\mathbf{G}^n_m), \mathbf{Q}(n))_{\mathrm{sgn}} = K^M_n(k(y_1, \ldots, y_n)) \otimes \mathbf{Q}$. Therefore the long exact sequence splits into short exact sequences, and by a similar induction argument we see that $H^n_{\mathcal{M}}((\mathbf{G}_m - \boldsymbol{\mu}_N)^{n\prime}, \mathbf{Q}(n))^{P^n}_{\mathrm{sgn}}$ has dimension n, spanned by $\Phi^n_1, \ldots, \Phi^n_n$. (In particular, there is a non trivial relation between $\Phi^n_0, \ldots, \Phi^n_n$ — *cf.* section 5.) However there is no canonical decomposition as it is easy to see that the analogue of $\mathcal{L}$ acts by the scalar L^{-n}, for every $L \geq 1$.

4.4 In order to decompose Θ^n_P according to the weights of $\mathcal{L}$, we must therefore calculate $\mathcal{E}^n_P$ explicitly. Write Ω_P for the composite

$$\Omega_P = [\times L]^* \circ \mathcal{L} : H^{\bullet}_{\mathcal{M}}(U^{n\prime}, \mathbf{Q}(*)) \to H^{\bullet}_{\mathcal{M}}(\tilde{U}^{n\prime}, \mathbf{Q}(*)).$$

By 4.2 we have

$$\mathcal{E}^n_P(\beta)=\left[\prod_{i=1}^{n-1}(L^{-n}-L^{-n-i})^{-1}\bigcirc_{i=1}^{n-1}\left(\mathcal{L}-L^{-n-i}\right)\right]\circ\Theta^n_P(\beta\otimes\alpha^{\otimes n}). \tag{4.4.1}$$

Write $P^{[j]}=L^{-j}P$. We can rewrite the above expression as

$$\left[\prod_{i=1}^{n-1}(L^{-n}-L^{-n-i})^{-1}\,[\times L^{n-1}]^{*-1}\circ\bigcirc_{i=1}^{n-1}\left(\Omega_{P^{[i-1]}}-L^{-n-i}[\times L]^*\right)\right]\circ\Theta^n_P(\beta\otimes\alpha^{\otimes n}). \tag{4.4.2}$$

Note that we may even extend the range of i to, say, $i=n$, making the operator explicitly kill off one more eigenspace which we already know by 4.2 to be zero. We will do this in the computation because it will painlessly suppress the Φ^n_2-component in 1.6. (If we did not do it, this component would have to be shown to cancel out because of relation 5.2—*cf.* the alternative proof we gave for proposition 3.2 which of course represents the simplest case.)

4.5 Let us analyse formula 4.4.1 with a view to computing $\partial^n\circ\mathcal{E}^n_P$ via 1.6. As indicated we modify 4.4.1 by letting i run from 1 to n. This also replaces $[\times L^{n-1}]^{*-1}$ by $[\times L^n]^{*-1}$ in 4.4.2.

4.5.1 Expand

$$\bigcirc_{i=1}^{n}\left(\Omega_{P^{[i-1]}}-L^{-n-i}[\times L]^*\right)=\sum_{I\subseteq\{1,\dots,n\}}(-1)^{|I|}\bigcirc_{i=1}^{n}\Lambda_{I,i}=\sum_I(-1)^{|I|}\,\Lambda_I,$$

where $|I|$ denotes the cardinality of I, and for each $I\subseteq\{1,\dots,n\}$ and $i\in\{1,\dots,n\}$, we define

$$\Lambda_{I,i}=\begin{cases}\Omega_{P^{[i-1]}} & \text{if } i\notin I\\ L^{-n-i}[\times L]^* & \text{if } i\in I.\end{cases}$$

For fixed I, we shall now compute

$$[\times L^nN]^{*-1}\circ\partial^n\circ\Lambda_I\circ\Theta^n_P(\beta\otimes\alpha^{\otimes n}). \tag{4.5.2}$$

By 1.2.2 we find that

$$\Omega_{P^{[i-1]}}\circ\Theta^n_{P^{[i-1]}}(\beta\otimes\alpha^{\otimes n})=\Theta^n_{P^{[i]}}(j_!\beta\otimes j_!\alpha^{\otimes n})$$

and

$$[\times L]^*\circ\Theta^n_{P^{[i-1]}}(\beta\otimes\alpha^{\otimes n})=\Theta^n_{P^{[i]}}(\pi^*\beta\otimes\pi^*\alpha^{\otimes n}).$$

Thus, writing

$$\lambda_I=\bigcirc_{i=1}^{n}\lambda_{I,i}\qquad\lambda_{I,i}=\begin{cases}j_! & \text{if } i\notin I\\ \pi^* & \text{if } i\in I,\end{cases}$$

1.6 allows us—neglecting signs—to transform 4.5.2 into

$$(4.5.3) \qquad \frac{n+1}{L^n(L^nN)^{2n+1}} \sum_{q=0}^{n} \binom{n}{q} \Phi^n_q \sum_{1\neq\eta\in\boldsymbol{\mu}_{L^nN}} \frac{\eta}{(\eta-1)^{q+2}} \left(\hat{d}_{\lambda_I\beta}\,\hat{d}^n_{\lambda_I\alpha}\right)(\eta) \prod_{i\in I} L^{-n-i}.$$

Here the first factor of L^n in the denominator comes from 1.3. In fact, in order to apply 1.6 relative to the group of L^nN-torsion we have to extend the base field to an extension with ramification index L^n.

The following lemma is straightforward. (Notice however that we are using the notation $j_!$ and π^* in two different meanings : on functions $\hat{d}_\gamma$ these operators refer to the groups $\boldsymbol{\mu}_N, \boldsymbol{\mu}_{LN}$; on divisors the notation is relative to $\boldsymbol{\mu}_N \times \mathbf{Z}/N\mathbf{Z}, \boldsymbol{\mu}_{LN} \times \mathbf{Z}/LN\mathbf{Z}$. In each case, j is inclusion and π the natural projection.)

4.5.4 Lemma. *For any $\gamma \in \mathbf{Q}[P]^\circ$, we have*

$$\hat{d}_{j!\gamma} = \pi^*\hat{d}_\gamma \qquad \hat{d}_{\pi^*\gamma} = L^2\, j!\hat{d}_\gamma.$$

This transforms 4.5.3 into

$$(4.5.5) \qquad \frac{n+1}{L^n(L^nN)^{2n+1}} \sum_{q=0}^{n} \binom{n}{q} \Phi^n_q \sum_{1\neq\zeta\in\boldsymbol{\mu}_N} \left(\hat{d}_\beta\,\hat{d}^n_\alpha\right)(\zeta) \prod_{i\in I} L^{n+2-i} \sum_{\eta^{L^{|\overline{I}|}}=\zeta} \frac{\eta}{(\eta-1)^{q+2}},$$

where $\overline{I} = \{1,\ldots,n\} - I$. — Now apply 2.6 and get

$$(4.5.6) \quad \frac{n+1}{L^n(L^nN)^{2n+1}} \sum_{q=0}^{n} \binom{n}{q} \Phi^n_q \sum_{1\neq\zeta\in\boldsymbol{\mu}_N} \left(\hat{d}_\beta\,\hat{d}^n_\alpha\right)(\zeta) \sum_{j=2}^{q+2} a_{j,q+2}\tilde{\mathbf{B}}_j(\zeta)\, L^{jn} \prod_{i\in I} L^{n+2-i-j}.$$

But observe that

$$\sum_{I\subseteq\{1,\ldots,n\}} (-1)^{|I|} \prod_{i\in I} L^{n+2-i-j} = \prod_{i=1}^{n}(1-L^{n+2-i-j}) = \begin{cases} 0 & \text{if } 2\le j\le n+1 \\ \prod_{i=1}^{n}(1-L^{-i}) & \text{if } j=n+2 \end{cases}$$

Thus taking the sum over all $I \subseteq \{1,\ldots,n\}$ in 4.5.6 and inserting this into 4.4.1, all powers of L duly cancel, and we obtain :

$$\partial^n \circ \mathcal{E}^n_P(\beta) = \pm \frac{n+1}{(n+2)!\,N^{2n+1}} \sum_{1\neq\zeta\in\boldsymbol{\mu}_N} \left(\hat{d}_\beta\,\hat{d}^n_\alpha\,\tilde{\mathbf{B}}_{n+2}\right)(\zeta)\ [\times N]^*\Phi^n_n.$$

Since $[\times N]^*\Phi^n_n = N^n\,\Phi^n_n$, the theorem now follows from 1.7 by a trivial computation. The constant comes out to be $C^n_{P,N} = \pm N^n\,(n+1)/(n+2)!$ in the case 1.4.

5. A linear relation.

As observed in 4.3 above, there is a non-trivial relation between the elements Φ^n_q for $0 \le q \le n$. We include it here even though the proof of the theorem we chose to present does not rely on it—*cf.* the remark at the end of 4.4 above.

The relation is derived from the following identity in Milnor K-theory.

5.1 Lemma. *In Milnor K-theory tensored with* $\mathbf{Z}[1/2]$, *we have*

$$\left\{\frac{1-x_1x_2\cdots x_m}{1-x_1}, \frac{x_1(1-x_2)}{1-x_1}, \ldots, \frac{x_{m-1}(1-x_m)}{1-x_{m-1}}\right\} = 0.$$

Proof. By induction: assume true for m, and replace x_m by $x_m x_{m+1}$. Then we get:

$$\begin{aligned}
0 &= \left\{\frac{1-x_1x_2\cdots x_m}{1-x_1}, \frac{x_1(1-x_2)}{1-x_1}, \ldots, \frac{x_{m-2}(1-x_{m-1})}{1-x_{m-2}}, \frac{x_{m-1}(1-x_m x_{m+1})}{1-x_{m-1}}\right\} \\
&= \left\{\frac{1-x_1x_2\cdots x_m}{1-x_1}, \frac{x_1(1-x_2)}{1-x_1}, \ldots, \frac{x_{m-2}(1-x_{m-1})}{1-x_{m-2}}, \frac{1-x_m x_{m+1}}{1-x_m}\right\} \\
&\quad + \left\{\frac{1-x_1x_2\cdots x_m}{1-x_1}, \frac{x_1(1-x_2)}{1-x_1}, \ldots, \frac{x_{m-2}(1-x_{m-1})}{1-x_{m-2}}, \frac{x_{m-1}(1-x_m)}{1-x_{m-1}}\right\}
\end{aligned}$$

Now take the product with

$$\frac{-x_m(1-x_{m+1})}{1-x_m} = 1 - \frac{1-x_m x_{m+1}}{1-x_m}$$

to obtain the desired formula.

Apply this now with $m = n$ and $y_i = x_i$. We get

$$\left\{\frac{y_0(1-y_1)}{1-y_0}, \frac{y_1(1-y_2)}{1-y_1}, \ldots, \frac{y_k(1-y_{k+1})}{1-y_k}, \ldots, \frac{y_{n-1}(1-y_n)}{1-y_{n-1}}\right\} = 0.$$

Expand this using bilinearity. If the $(k+1)^{\text{st}}$ choice is y_k or $(1-y_k)^{-1}$, then for the resulting term to be non-zero the k^{th} choice must be y_{k-1} or $(1-y_{k-1})^{-1}$, and we obtain:

$$\sum_{p=0}^{n}\left\{\frac{y_0}{1-y_0}, \frac{y_1}{1-y_1}, \ldots, \frac{y_{p-1}}{1-y_{p-1}}, 1-y_{p+1}, \ldots, 1-y_n\right\} = 0.$$

Now apply Π_{sgn}. Using the permutation $(012\ldots p)$ the result can be written as

$$0 = \sum_{p=0}^{n}(-1)^p\left\{\frac{y_1}{1-y_1}, \ldots, \frac{y_p}{1-y_p}, 1-y_{p+1}, \ldots, 1-y_n\right\}_{\text{sgn}}$$

$$= \sum_{p=0}^{n}\sum_{q=0}^{p}(-1)^q \sum_{0\le i_1<\cdots<i_q\le p}\left\{1-y_1, \ldots, y_{i_1}, \ldots, y_{i_q}, \ldots, 1-y_{p+1}, \ldots, 1-y_n\right\}_{\text{sgn}}.$$

Here the k^{th} entry is y_k for $k = i_1, \ldots, i_q$ and $1-y_k$ for the remaining $(n-q)$ values of k. We conclude:

5.2 Proposition. $$\sum_{q=0}^{n}(-1)^q\left(\sum_{p=q}^{n}\binom{p}{q}\right)\Phi^n_q = 0.$$

6. The number field case

Let F be a number field, $\mathcal{O}$ its ring of integers, and let v denote finite places of F. The subspace $H^\bullet_{\mathcal{M}}(E^n_{/F},\mathbf{Q}(*))_{\mathbf{Z}}$ of "integral" elements of $H^\bullet_{\mathcal{M}}(E^n_{/F},\mathbf{Q}(*))$ is defined to be the image of

$$H^\bullet_{\mathcal{M}}(\widetilde{E^n_{/\mathcal{O}}},\mathbf{Q}(*)) \longrightarrow H^\bullet_{\mathcal{M}}(E^n_{/F},\mathbf{Q}(*)),$$

where $\widetilde{E^n_{/\mathcal{O}}} \to (E_{/\mathcal{O}})^n$ is a desingularisation of the n-fold power of a global regular minimal model of E. (See 6.6 below.) By the long exact sequence for the pair $\widetilde{E^n_{/\mathcal{O}}}, E^n_{/F}$ this space of integral elements $H^{n+1}_{\mathcal{M}}(E^n_{/F},\mathbf{Q}(n+1))_{\mathbf{Z}}$ equals the kernel of the boundary map

$$H^{n+1}_{\mathcal{M}}(E^n_{/F},\mathbf{Q}(n+1)) \longrightarrow \coprod_{\text{all } v} H^{n+2}_{\mathcal{M},(v)}(\widetilde{E^n_{/\mathcal{O}}},\mathbf{Q}(n+1)).$$

where subscript (v) denotes cohomology with support in the fibre at v.

6.1 Now let v be a place of F satisfying the assumptions 1.4. Write ϵ the projector onto the subspace on which the group $\boldsymbol{\mu}_2^n \cdot \mathcal{S}_n \cdot P^n$ acts as follows : every $\boldsymbol{\mu}_2$ acts by -1, $\mathcal{S}_n$ acts via the sign-character sgn_n, and P^n acts trivially. We then have a commutative diagram :

$$\begin{tikzcd}
H^{n+1}_{\mathcal{M}}(E^n_{/F},\mathbf{Q}(n+1))(\epsilon) \arrow[r] \arrow[d, phantom, "\bigcap"] & H^{n+2}_{\mathcal{M},(v)}(\widetilde{E^n_{/\mathcal{O}}},\mathbf{Q}(n+1))(\epsilon) \arrow[d, "\wr"] \\
H^{n+1}_{\mathcal{M}}(E^n_{/F},\mathbf{Q}(n+1)) \arrow[r] & H^n_{\mathcal{M}}(\overset{\circ}{E}{}^n_{/k_v},\mathbf{Q}(n))
\end{tikzcd}$$

where the isomorphism is between one-dimensional $\mathbf{Q}$-vector spaces. For this isomorphism see [Scholl 1990], proof of 3.1.0(iii); the proof given there applies equally well in the present situation.

6.2 In general, given any finite place v of F, there exists a finite extension F'/F such that, above v, $E_{/F'}$ has either good reduction or situation 6.1 applies. And in the good reduction case one has that $H^n_{\mathcal{M}}(\overset{\circ}{E}{}^n_{/k_v},\mathbf{Q}(n))=0$: see [Soulé 1984, Thm. 3.(iii)].

6.3 Lemma. *Let F'/F be a finite extension. Then*

$$\mathrm{cores}_{F'/F} H^\bullet_{\mathcal{M}}(E^n_{/F'},\mathbf{Q}(*))_{\mathbf{Z}} = H^\bullet_{\mathcal{M}}(E^n_{/F},\mathbf{Q}(*))_{\mathbf{Z}}.$$

This is proved by a slight variation of [Beilinson 1985, 2.4.2] — *cf.* [Schneider 1988, p.13].

6.4 Finally, if v is a place where E has either additive and potentially multiplicative or non split multiplicative reduction, then the target space $H^n_{\mathcal{M}}(\overset{\circ}{E}{}^n_{/k_v},\mathbf{Q}(n))$ is zero if and only if n is odd. This is seen from the Galois action on the generator $t_1 \cup \cdots \cup t_n$ of the corresponding motivic cohomology over a suitable extension field.

We conclude :

6.5 Proposition. *$H^{n+1}_{\mathcal{M}}(E^n_{/F},\mathbf{Q}(n+1))_{\mathbf{Z}}$ is the kernel of the boundary maps*

$$H^{n+1}_{\mathcal{M}}(E^n_{/F},\mathbf{Q}(n+1)) \longrightarrow \coprod_v H^n_{\mathcal{M}}(\overset{\circ}{E}{}^n_{/k_v},\mathbf{Q}(n)),$$

the product being over all (finite) places of F where E has split multiplicative reduction, if n is odd; and over all (finite) places of F where E has potentially multiplicative reduction, if n is even.

Our theorem then allows to calculate explicitly the integrality obstruction for elements of $H^{n+1}_{\mathcal{M}}(E^n_{/F},\mathbf{Q}(n+1))$. This justifies in particular the computations of this obstruction performed in [Mestre, Schappacher 1990].

6.6 Some words regarding the desingularisation $\widetilde{E^n_{/\mathcal{O}}}$ are in order. (Note that in §2.2 of [Mestre, Schappacher 1990], $\widetilde{E^n_{/\mathcal{O}}}$ is incorrectly defined as the normalisation.)

If E has semistable reduction, then the singularities of ${E_{/\mathcal{O}}}^n$ are products of ordinary double points, and can be explicitly resolved [Deligne 1968, Lemme 5.4], [Scholl 1990, §2]. In general, the existence of a desingularisation seems open.

If one does not want to assume the existence of $\widetilde{E^n_{/\mathcal{O}}}$, one may choose F' as in 6.2 and take the left hand side of 6.3 as the definition of $H^{\bullet}_{\mathcal{M}}(E^n_{/F},\mathbf{Q}(*))_{\mathbf{Z}}$.

7. The modular case

7.0 In this section we show how our theorem gives a different proof of one of the main results of [Beilinson, 1986]—Theorem 7.4 below. (In [Deninger, Scholl], this paper is summarised in a language closer to ours.)

7.1 Let N be an integer ≥ 3, and let M_N be the modular curve of level N, and F_N its function field. We consider E/F_N, the universal elliptic curve with level N structure $\alpha : E[N] \xrightarrow{\sim} (\mathbf{Z}/N\mathbf{Z})^2$. Taking $P = (\mathbf{Z}/N\mathbf{Z})^2$ (which we identify with the N-torsion subgroup of E via α) we obtain the Eisenstein symbol map, which we write

$$\mathcal{E}^n_N : \mathbf{Q}[(\mathbf{Z}/N\mathbf{Z})^2]^0 \longrightarrow H^{n+1}_{\mathcal{M}}(E^n,\mathbf{Q}(n+1)).$$

7.2 Write M_N^∞ for the cusps of M_N. Then as is well known, by regarding the cusps as giving level N structures on the standard Néron N-gon, one has an identification of the set of closed points:

$$|M_N^\infty| \xrightarrow{\sim} GL_2(\mathbf{Z}/N\mathbf{Z})/\begin{pmatrix} * & * \\ 0 & \pm 1\end{pmatrix}$$

where $1 \in GL_2(\mathbf{Z}/N\mathbf{Z})$ corresponds to the level N structure

$$\begin{aligned} \mathbf{G}_m \times \mathbf{Z}/N\mathbf{Z} \supset \boldsymbol{\mu}_N \times \mathbf{Z}/N\mathbf{Z} &\xrightarrow{\sim} (\mathbf{Z}/N\mathbf{Z})^2 \\ (\zeta_N^a, b) &\mapsto (a,b) \end{aligned}$$

defined over $\mathbf{Q}(\zeta_N)$.

7.3 The main theorem enables us to calculate the effect of the boundary map

$$\partial : H^{n+1}_{\mathcal{M}}(E^n, \mathbf{Q}(n+1))_{\mathrm{sgn}} \longrightarrow H^n_{\mathcal{M}}(\mathbf{G}_m \times M_N^\infty, \mathbf{Q}(n))_{\mathrm{sgn}} \xrightarrow{\sim} \mathbf{Q}[|M_N^\infty|]$$

on the image of the Eisenstein symbol. Notice that the first arrow depends on the choice of orientation of the special fibre of the Néron model of E, so that as written the composite map is not canonical. To make it canonical we replace the target by the space $V^{(-)^n}$, where

$$V^\pm = \left\{ f : GL_2(\mathbf{Z}/N\mathbf{Z}) \to \mathbf{Q} \,\middle|\, f(g\begin{pmatrix} * & * \\ 0 & 1 \end{pmatrix}) = f(g) = \pm f(-g) \right\}.$$

Then our theorem shows at once that the composite $\partial \circ \mathcal{E}_N^n$ is a nonzero multiple of the $GL_2(\mathbf{Z}/N\mathbf{Z})$-equivariant map $\omega_N^n : \mathbf{Q}[(\mathbf{Z}/N\mathbf{Z})^2]^0 \to V^{(-)^n}$ given by the formula:

$$(\omega_N^n \phi)(g) = \sum_{\underline{x} \in (\mathbf{Z}/N\mathbf{Z})^2} \phi(g \cdot \underline{x}) B_{n+2}(\langle \frac{x_2}{N} \rangle). \tag{7.3.1}$$

Observe that this formula makes sense for any $N \geq 2$.

7.4 Theorem. *[Beilinson 1986, §3] The boundary map $\partial : H^{n+1}_{\mathcal{M}}(E^n, \mathbf{Q}(n+1))_{\mathrm{sgn}} \to V^{(-)^n}$ is an isomorphism on the image of the Eisenstein symbol.*

This is an immediate consequence of (7.3.1) and the properties of the "horospherical isomorphism" (see the paragraph after 3.1.6 in [Beilinson, 1986]). Since we were unable to find a suitable reference for these properties, we give here a direct proof. It is in two steps.

7.5 *Step I: For every $N \geq 2$ and every $n \geq 1$ the map ω_N^n is surjective.*

Clearly one is free to tensor with $\mathbf{C}$. We first show that any function supported on $\begin{pmatrix} * & * \\ 0 & * \end{pmatrix}$ is contained in the image. The subspace of $V^\pm \otimes \mathbf{C}$ composed of such functions has for a basis the set of functions

$$f_\chi : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{cases} 0 & \text{if } c \neq 0 \\ \chi(d) & \text{if } c = 0 \end{cases}$$

where $\chi : (\mathbf{Z}/N\mathbf{Z})^* \to \mathbf{C}^*$ runs over Dirichlet characters with $\chi(-1) = \pm 1$.

Define

$$\phi_\chi(\underline{x}) = \sum_{y \in (\mathbf{Z}/N\mathbf{Z})^*} \chi(y)^{-1} e^{2\pi i x_2 y/N}.$$

Then

$$\omega_N^n \phi_\chi : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{cases} N\chi(d) \displaystyle\sum_{\substack{x \in \mathbf{Z}/N\mathbf{Z} \\ w \in (\mathbf{Z}/N\mathbf{Z})^*}} \chi(w)^{-1} e^{2\pi i w x/N} B_{n+2}(\langle \frac{x}{N} \rangle) & \text{if } c = 0 \\ 0 & \text{if } c \neq 0. \end{cases}$$

Writing the values of the Bernoulli polynomial in terms of Dirichlet L-series and using the character orthogonality relations, the last expression becomes

$$-(n+2)N\varphi(N)\chi(d) \sum_{D|\frac{N}{M}} \frac{\tau(\chi_D)L(\chi_D,-1-n)}{(DM)^{n+1}\varphi(DM)}. \tag{7.5.1}$$

Here M is the conductor of χ, for each $D|\frac{N}{M}$ we have written χ_D for the character modulo DM associated to χ, and $\tau(\chi_D)$ denotes the Gauss sum

$$\sum_{x\in(\mathbf{Z}/DM\mathbf{Z})^*} \chi_D(x)^{-1}e^{2\pi ix/DM}.$$

Rewriting 7.5.1 in terms of the primitive character χ_1 modulo M, we finally obtain

$$\omega_N^n\phi_\chi = \frac{-(n+2)N\varphi(N)}{M^{n+1}\varphi(M)} \prod_{\substack{p|N \\ (p,M)=1}} \left(\frac{p^{n+2}-\chi_1(p)^{-1}}{p^{n+1}(p-1)}\right) \tau(\chi_1)L(\chi_1,-1-n)\, f_\chi.$$

As $\chi(-1)=(-1)^n$, the L-value is nonzero, as are the remaining factors. We therefore have found a nonzero multiple of f_χ in the image of ω_N^n.

Now as a representation of $GL_2(\mathbf{Z}/N\mathbf{Z})$, $V^\pm$ is generated by the functions f_χ. This shows the surjectivity of ω_N^n.

It follows that for every $n\geq 1$ the map

$$(\omega_N^n,\omega_N^{n+1}) : \mathbf{Q}[(\mathbf{Z}/N\mathbf{Z})^2]^0 \longrightarrow V^+\oplus V^-$$

is surjective. Therefore the theorem will be a consequence of the next assertion.

7.6 *Step II: If $N\geq 3$ and $n\geq 1$ then*

$$\dim \operatorname{Im}(\mathcal{E}_N^n)+\dim \operatorname{Im}(\mathcal{E}_N^{n+1}) \leq \dim V^+ + \dim V^-.$$

To prove this we consider (for the moment arbitrary) functions $\phi : \mathbf{Z}^2 \to \mathbf{Q}$, and make the convention that $\phi(x)=0$ whenever $x\in\mathbf{Q}^2-\mathbf{Z}^2$. For a squarefree integer $D=p_1\cdots p_k\geq 1$ define

$$(\Delta_D\phi)(x) = \sum_{E|D}(-1)^{\kappa(E)}E^n\phi(x/E)$$

where $\kappa(E)$ is the number of prime divisors of E. Now,

$$\Delta_D = \Delta_{p_1}\circ\ldots\circ\Delta_{p_k}. \tag{7.6.1}$$

The operators Δ_D have the properties:

(i) Δ_D is injective for every $D\geq 1$;

(ii) If $(D,D')=1$ then $\operatorname{Im}\Delta_D\cap\operatorname{Im}\Delta_{D'}=\operatorname{Im}\Delta_{DD'}$.

The first one of these follows from the elementary identity

$$\phi(x)=\sum_{E|D^\infty} E^n(\Delta_D\phi)(x/E). \tag{7.6.2}$$

To prove (ii), suppose that $\Delta_D\phi=\Delta_{D'}\phi'$. Then setting

$$\psi=\sum_{E|D^\infty} E^n\phi'(x/E)$$

and using (7.6.2) one sees that $\Delta_D\psi=\phi'$ and also $\Delta_{D'}\psi=\phi$.

7.7 Now if $D|N$ then Δ_D induces an injective map

$$\Delta_{D,N}:\mathbf{Q}[(\mathbf{Z}/\frac{N}{D}\mathbf{Z})^2]^0\longrightarrow\mathbf{Q}[(\mathbf{Z}/N\mathbf{Z})^2]^0$$

and from (1.2) and (7.6.1)

$$\mathcal{E}_N^n\circ\Delta_{D,N}=0\quad\text{provided } D>1.$$

We have $\mathcal{E}_N^n(\phi(-x))=(-1)^n\mathcal{E}_N^n(\phi(x))$. Moreover, let $\Delta_{D,N}^n$ denote the composite of $\Delta_{D,N}$ with the projection onto the subspace of $\phi\in\mathbf{Q}[(\mathbf{Z}/N\mathbf{Z})^2]^0$ satisfying $\phi(-\underline{x})=(-1)^n\phi(\underline{x})$. Then $\dim\operatorname{Im}\Delta_{D,N}^n$ depends only on N, D and the parity of n; and

$$\dim\operatorname{Im}\Delta_{D,N}^n+\dim\operatorname{Im}\Delta_{D,N}^{n+1}=(N/D)^2-1$$

for $D>1$. The usual inclusion-exclusion argument then yields

$$\begin{aligned}\dim\operatorname{Im}(\mathcal{E}_N^n)&+\dim\operatorname{Im}(\mathcal{E}_N^{n+1})\\ &\le(N^2-1)-\sum_{p]N}\Big(\Big(\frac{N}{p}\Big)^2-1\Big)+\sum_{p,q]N}\Big(\Big(\frac{N}{pq}\Big)^2-1\Big)-\dots\\ &=N^2\prod_{p]N}\Big(1-\frac{1}{p^2}\Big)\\ &=\#GL_2(\mathbf{Z}/N\mathbf{Z})/\begin{pmatrix}* & *\\ 0 & 1\end{pmatrix}=\dim V^+ +\dim V^-.\end{aligned}$$

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Norbert Schappacher
Max-Planck-Institut für Mathematik
Gottfried-Claren-Str. 26
D-5300 Bonn 2
Germany

Anthony J. Scholl
Department of Mathematics
Science Laboratories, South Road
Durham DH1 3LE
England