Modular forms and motives

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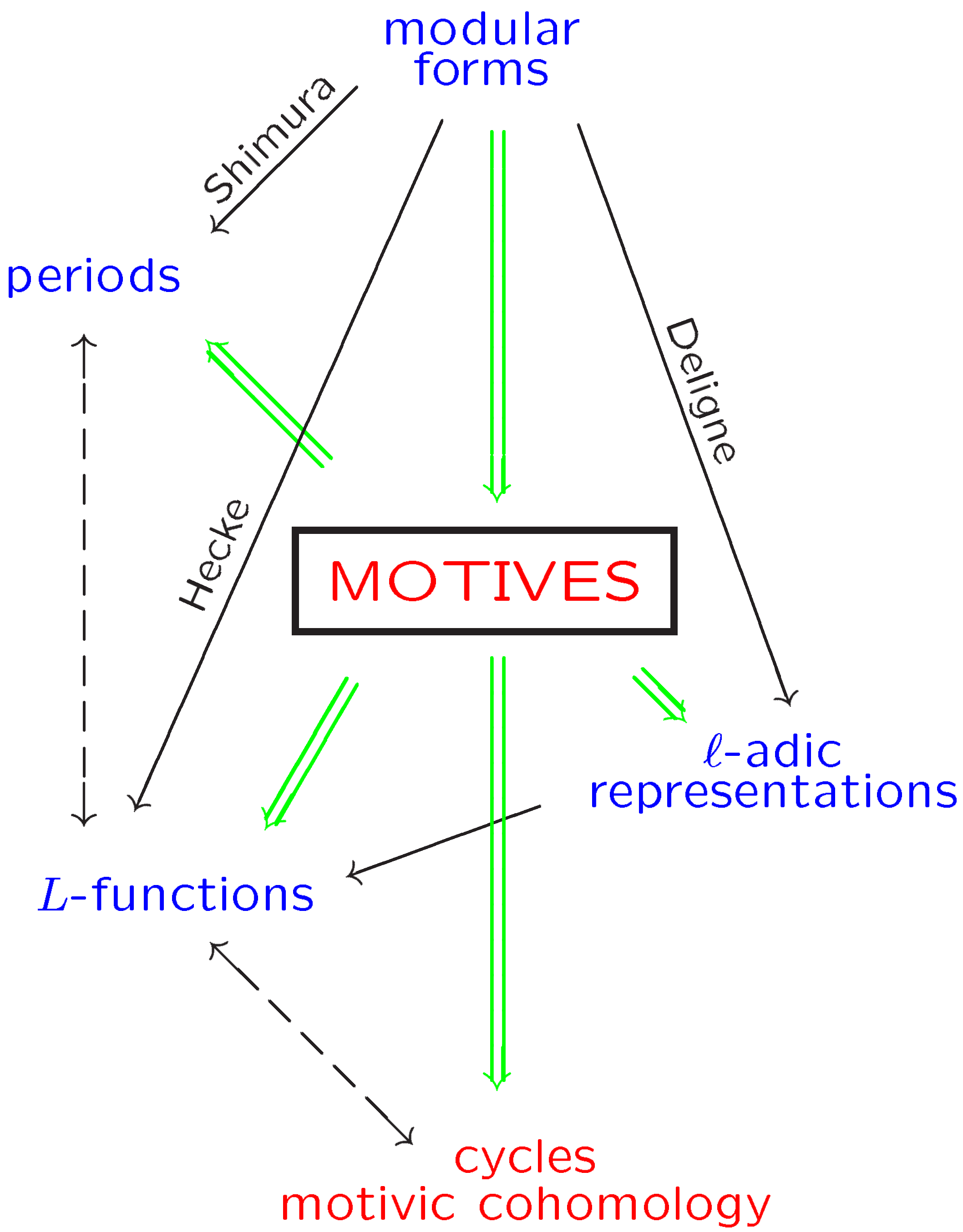


modular
forms

Shimura

periods

Hecke

Deligne

MOTIVES

ℓ-adic
representations

L-functions

cycles
motivic cohomology

Cusp forms of weight $w \geq 0$
on $\Gamma \subset SL_2(\mathbb{Z})$:
finite index

$S_w(\Gamma) =$ all $f : \mathfrak{H} \xrightarrow{\text{holom.}} \mathbb{C}$ such that:

- $f\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^w f(\tau) \ \forall \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$

- $\mathrm{Im}(\tau)^{w/2} \, |f(\tau)|$ bounded

$$\Longrightarrow \ f = \sum_{n \geq 1} A_n \, q^{n/\mu}, \quad q = e^{2\pi i \tau}, \quad \begin{pmatrix} 1 & \mu \\ 0 & 1 \end{pmatrix} \in \Gamma$$

Deligne (1968) (or $S_{k+2}(\Gamma_0(N), \chi)$)

$f \in S_w(SL_2(\mathbb{Z}))$, eigenform for Hecke T_n

$(A_1 = 1,\ A_{mn} = A_m A_n$ if $(m,n) = 1 \ \ldots\ )$

$K = \mathbb{Q}(\{A_n\})$, $\lambda | \ell$ prime of K.

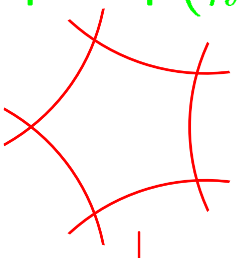
Thm $\exists \rho_{f,\lambda} : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \longrightarrow GL_2(K_\lambda)$

- unram. outside ℓ $(N\ell)$
- $p \neq \ell \Rightarrow$ $\mathrm{tr}\rho_{f,\lambda}(\mathrm{Frob}_p) = A_p$
  det $= p^{w-1}$ $(\chi(p)p^{w-}$

Kuga-Sato varieties

$n = 3$

$\Gamma = \Gamma(n)$



$$\begin{tikzcd} X_n \arrow[r, hook] \arrow[d, "\pi"'] & \overline{X}_n \arrow[d] & & \\ M_n \arrow[r, hook] & \overline{M}_n & M_n^\infty \arrow[l, hook'] & \bullet \arrow[l] \end{tikzcd}$$

elliptic surface

affine smooth curve/$\mathbb{Q}(\zeta_n)$

projective

$$\begin{tikzcd} \dfrac{\mathbb{C}}{\mathbb{Z}+\tau\mathbb{Z}} \arrow[d] & \subset X_n(\mathbb{C}) \arrow[d] = \Gamma\backslash\mathfrak{H}\times\mathbb{C}/\mathbb{Z}^2 \\ {[\tau]} & \in M_n(\mathbb{C}) = \Gamma\backslash\mathfrak{H} \end{tikzcd}$$

$$\begin{tikzcd} (\tau, z) \arrow[d, "\begin{pmatrix} a & b \\ c & d \end{pmatrix}"] \\ \left(\dfrac{a\tau+b}{c\tau+d}, \dfrac{z}{c\tau+d}\right) \end{tikzcd}$$

projective

singular if $k \geq 2$

$$X_n^k = \underbrace{X_n \times_{M_n} \cdots \times X_n}_{k} \subset \overline{X}_n^k \twoheadleftarrow \overline{\overline{X}}_n^k$$

“nice” resolution

$$f \longleftrightarrow \omega_f = (2\pi i)^{k+1} f\, d\tau \wedge dz_1 \cdots \wedge dz_k$$

gives:

$$S_{k+2}(\Gamma(n)) \xrightarrow{\sim} H^0(\overline{\overline{X}}_n^k, \Omega^{k+1}) \otimes \mathbb{C}$$

Automorphisms of X_n^k

$$X_n^k \ni (x_1, \dots, x_k):$$

		character→ ε
$\mathfrak{S}_k$	(permuting factors)	sgn
$\{\pm 1\}^k \ni$	$(e_i)\colon (x_i) \mapsto (e_i x_i)$	$\prod e_i$
$(\mathbb{Z}/n\mathbb{Z})^{2k}$	(translation by sections)	trivial

$$\begin{tikzcd}
\Gamma^k = \{\pm 1\}^k \rtimes \mathfrak{S}_k \subset \Gamma_n^k = \Gamma^k \ltimes (\mathbb{Z}/n\mathbb{Z})^{2k} \arrow[d, "\varepsilon"] \\
\{\pm 1\}
\end{tikzcd}$$

$$\gamma \in \Gamma_n^k \hookrightarrow \mathrm{Aut}(\overline{\overline{X}}{}_n^k)$$

$$\gamma^* \omega_f = \varepsilon(\gamma) \omega_f$$

$$\begin{array}{ccc}
S_{k+2}(\Gamma(n)) \oplus \overline{S_{k+2}(\Gamma(n))} & \hookrightarrow & H^*(\overline{\overline{X}}{}_n^k, \mathbb{C})(\varepsilon) \\
(f, g) & \longmapsto & \omega_f + \overline{\omega_g}
\end{array}$$

$\boxed{\text{Thm}}$ This map is an $\xrightarrow{\sim}$ $(k > 0)$

Shimura's isomorphism

Leray for $X_n^k \xrightarrow{\pi^k} M_n \xleftarrow{\pi} X_n$

$$E_2^{ab} = H^a(M_n, R^b\pi^k_*\mathbb{Q}) \Longrightarrow H^{a+b}(X_n^k, \mathbb{Q})$$

(also for H_c^* and $H_p^* := \mathrm{im}(H_c^* \to H^*)$)

ε-eigenspaces:—

$$(R^*\pi_k^*\mathbb{Q})(\varepsilon) \underset{\substack{\uparrow \\ \text{Künneth}}}{=} (\otimes^k R^*\pi_*\mathbb{Q})(\varepsilon) \underset{\substack{\uparrow \\ [\times -1]^* = (-1)^i \text{ on } R^i\pi_*\mathbb{Q}}}{=} \mathrm{Sym}^k R^1\pi_*\mathbb{Q} =: \mathcal{F}_k$$

$\therefore H^*_?(X_n^k)(\varepsilon) = H^1_?(M_n, \mathcal{F}_k)$ in degree $k+1$

($\Gamma =$ (nothing), c, p)

$$S_{k+2} \oplus \overline{S_{k+2}} \xrightarrow[\text{(Shimura)}]{\sim} H^1_p(M_n, \mathcal{F}_k) \otimes \mathbb{C}$$

so enough to show:

$$H^*(\overline{\overline{X}}{}^k_n)(\varepsilon) \xrightarrow{\sim} H^*_p(X_n^k)(\varepsilon)$$

$\boxed{\text{Structure at } \infty\text{, I}}$

$$\begin{tikzcd}
& \overline{\overline{X}}^k_n \arrow[r, two heads] \arrow[dr, phantom, "\cup"] & \overline{X}^k_n & (P_n)^k \\
& & \overline{X}^{k,reg}_n \arrow[u, phantom, "\cup"] & \left\{ (z_i) \,\middle|\, \begin{array}{l} \text{at most one} \\ z_i \in P_n^{sing} \end{array} \right\} \arrow[u, phantom, "\cup"] \\
\text{smooth}/M_n & & \overline{X}^{k,sm}_n \arrow[u, phantom, "\cup\ (1)"] & \{(z_i) \mid \text{no } z_i \in P_n^{sing}\} \arrow[u, phantom, "\cup"] \\
& & X^k_n \arrow[u, phantom, "\cup\ (2)"] &
\end{tikzcd}$$

$$\boxed{(1)\ H^*(\overline{X}^{k,sm}_n)(\varepsilon) \xrightarrow{\sim} H^*_p(X^k_n)(\varepsilon)}$$

$$\boxed{(2)\ H^*(\overline{X}^{k,reg}_n)(\varepsilon) \xrightarrow{\sim} H^*(\overline{X}^{k,sm}_n)(\varepsilon)}$$

So enough to show:

$$H^*(\overline{\overline{X}}^k_n)(\varepsilon) \xrightarrow{\sim} H^*(\overline{X}^{k,reg}_n)(\varepsilon)$$

or equivalently

$$H_*(\underbrace{\overline{\overline{X}}^k_n - \overline{X}^{k,reg}_n}_{\text{exceptional divisor}})(\varepsilon) = 0$$

(1):

$$\overline{X}_n^{k,sm} - X_n^k \simeq M_n^\infty \times \mathbb{G}_m^k \times (\mathbb{Z}/n)^k$$

$$H^*(\mathbb{G}_m^k)(\varepsilon) = \mathbb{Q}(-k-1) \quad \text{in degree } k$$

$$H_c^* \qquad = \mathbb{Q}(0) \qquad \text{——"——}$$

$\Longrightarrow$ (1) by exact sequences of H and H_c.

(2):

$$\overline{X}_n^{k,reg} - \overline{X}_n^{k,sm} \simeq M_n^\infty \times \left(kn^k \text{ copies of } \mathbb{G}_m^{k-1} \right)$$

↻

permuted trans^{ly} by $?_n^k$

$$\Delta = \text{stab}(\mathbb{G}_m^{k-1}) \supset \Gamma^{k-1} \times \{\pm 1\}$$

acting trivially ↗

$\Longrightarrow$

$$H^*(-) = \text{Ind}_\Delta^{?_n^k} H^*(\mathbb{G}_m^{k-1} \times M_n^\infty) \text{ is } \underline{\text{induced}}$$

$\Longrightarrow$ (Frobenius reciprocity)

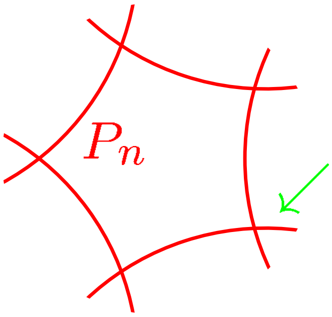
$$H^*(-)(\varepsilon) = H^*(\mathbb{G}_m^{k-1} \times M_n^\infty)(\varepsilon\big|_\Delta) = 0$$

↑

as $\varepsilon\big|_{\{\pm 1\}} \neq 1$

Structure at ∞, II

$\overline{X}_n \supset \bar{\pi}^{-1}$(cusp) :



locally $xy = t$

$\Longrightarrow$ $\overline{X}_n^k$ locally $\simeq$

$$\begin{tikzcd} \text{cone } V^k = \{x_1y_1 = \cdots = x_ky_k\} \subset \mathbb{A}^{2k} \arrow[dr, "t=x_iy_i"] & \\ \circlearrowleft \; \Gamma^k & \mathbb{A}^1 \end{tikzcd}$$

Q^k = projectivisation of V^k

= $\bigcup$ of 2^k copies of V^{k-1}

(smooth for $k \leq 2$)

Singular $\Leftrightarrow$ $x_i = y_i = 0 = x_j = y_i$, $i \neq j$.

$$\begin{tikzcd} V^k\langle 1\rangle \arrow[r, "\text{blow up } \underline{0}"] \arrow[d, "\text{line bdl.}"', "\cup"] & V^k \\ Q^k & \end{tikzcd}$$

$\boxed{\therefore \text{ sequence of blowups:}}$

$$\begin{tikzcd}
\widetilde{V^k} = V^k\langle k-1\rangle \arrow[d] \arrow[dd, bend right, "\psi_2"'] \arrow[ddd, bend right=60, "\psi_1"'] \\
\vdots \\
V^k\langle 2\rangle \arrow[d] \\
V^k\langle 1\rangle \arrow[d] \\
V^k = V^k\langle 0\rangle
\end{tikzcd}$$

$\boxed{\Longrightarrow \text{ stratification:}}$

$$\widetilde{V^k} = F_{k-1} \supset \quad \dots \; F_j \; \dots \quad \supset F_0 = V^{k,reg}$$

↑

largest open on
which ψ_j is $\simeq$

$$F_{j+1} - F_j = Q^{(k-j),reg} \times (\text{linear variety})$$

$\circlearrowleft$

Γ^{k-j}

$\boxed{\text{Lemma}}$ for any smooth S,

$$H^*(Q^{k,reg} \times S)(\varepsilon) = 0.$$

$\boxed{\text{Motives}}$ (for rational equiv . . .)

$$\mathcal{W}_n^k := \left(\overline{\overline{X}}{}_n^k, \Pi_\varepsilon\right)$$

$$\|$$

$$(\#\Gamma_n^k)^{-1} \sum \varepsilon(\gamma)\gamma \in \mathbb{Z}\left[\tfrac{1}{2nk!}\right]\left[\mathrm{Aut}\,\overline{\overline{X}}{}_n^k\right]$$

Realisations (Betti, ℓ-adic):

$$\mathcal{W}_{n,B}^k = H^*(\overline{\overline{X}}{}_n^k(\mathbb{C}), \mathbb{Q})(\varepsilon)$$

$$= \text{Eichler-Shimura parab. coh.}$$

$$\mathcal{W}_{n,\ell}^k = H^*(\overline{\overline{X}}{}_n^k \otimes \overline{\mathbb{Q}}, \mathbb{Q}_\ell)(\varepsilon)$$

$$= \text{Deligne parabolic cohomology}$$

$$\mathcal{W}_n^k \longrightarrow h(X_n^k)(\varepsilon) \xrightarrow{\curvearrowleft} \bigoplus_{M_n^\infty} \mathbb{Q}(-k-1)[-k-1]$$

split by Eisenstein symbol: realisation

$$S_{k+2} \oplus \overline{S_{k+2}} \longrightarrow M_{k+2} \oplus \overline{S_{k+2}} \xrightarrow{\overset{\text{Eis. series}}{\curvearrowleft}} \bigoplus_{M_n^\infty} \mathbb{C}$$

Motivic cohomology

$D > 1$, $(D, 6) = 1$, E/S elliptic curve
$x \in E(S)$, $y = Dx$ disjoint from 0-section.

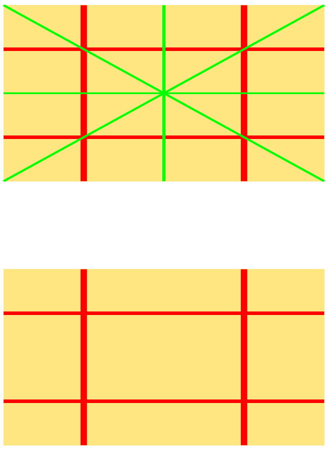
Propn. (Siegel/Robert/Kato) $\exists$ canonical function ϑ_D on E with divisor $D^2(0) - [\times D]$.

$$i_x \colon E^k \longrightarrow E^{k+1} \xrightarrow{pr_j} E \quad (0 \leq j \leq k)$$

$$(u_i) \longmapsto (x - u_1, u_1 - u_2, \ldots, u_{k-1} - u_k, u_k)$$

$$\text{Image} = \left\{ (z_i) \,\middle|\, \textstyle\sum_0^k z_i = x \right\}.$$

$$\begin{tikzcd}
\Gamma^k \circlearrowright \left\{ \begin{matrix} Du_i \neq 0, \pm x \\ D(u_i \pm u_j) \neq 0 \end{matrix} \right\} \arrow[r, "i_x"] \arrow[d, "{[\times D]}"'] & (E - \ker[\times D])^{k+1} \\
U_y^k = \left\{ \begin{matrix} u_i \neq 0, \pm y \\ u_i \neq u_j \end{matrix} \right\} \arrow[d, "j"'] & \\
(E - \{\pm y\})^k &
\end{tikzcd}$$



Lemma

$j^* =$ isom$^{\mathrm{n}}$ on $H_{\mathcal{M}}(-,\,)(\varepsilon)\otimes\mathbb{Z}[\frac{1}{2k!}]$.

(each component of complement is fixed by some γ with $\varepsilon(\gamma) = -1$.)

$$\begin{array}{ccc} [\times D]_* i_x^*(pr_0^*\vartheta_D \cup \cdots \cup pr_k^*\vartheta_D)(\varepsilon) & \mapsto & \sigma_y \\ \Cap & & \Cap \\ H^{k+1}_{\mathcal{M}}(U_y^k, k+1)(\varepsilon) & \simeq & H^{k+1}_{\mathcal{M}}((E-\{\pm y\})^k, k+1)(\varepsilon) \end{array}$$

Eisenstein symbol (Б)

$ny = 0$, some $n > 1$

$$\begin{tikzcd} {[\times n]_*\sigma_y} & \in & H^{k+1}_{\mathcal{M}}((E-0)^k, k+1)(\varepsilon) \arrow[d, "\wr"] \\ & & H^{k+1}_{\mathcal{M}}(E^k, k+1)(\varepsilon) \end{tikzcd}$$

Elliptic polylog (Б,Л,Г,W . . .)

$$h((E-\{\pm y\})^k)(\varepsilon) = \mathrm{Sym}^k\,\underbrace{h(E-\{\pm y\})^-}_{\substack{\parallel \\ h^1(E)\to\mathcal{H}_y\to\mathbb{Q}(-1)}}$$

$$\Longrightarrow \qquad \sigma_y \in \mathrm{Ext}^1(\mathbb{Q}(-1), \mathrm{Sym}^k\mathcal{H}_y)$$