BEILINSON'S THEOREM ON MODULAR CURVES.

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0. Introduction.

The purpose of this chapter is to give a detailed account of the known evidence ([**Be1**], §5) for Beilinson's conjecture concerning the values at $s = 2$ of L-functions of modular forms of weight two. We do not discuss here the results concerning the L-values at other integers $s \geq 2$ (for which see [**Be2**]), nor do we treat the case of the product of two modular curves (also to be found in [**Be1**]).

On the whole we follow the method of Beilinson, apart from two differences. Firstly, we give a statement of the main theorem in terms of modular curves, rather than motives of modular forms. In particular, we have tried to work throughout with modular curves of finite level, rather than passing to the inverse limit. Secondly, our proof of the "integrality" statement (Theorem 1.1.2(iii) here; Theorem 5.1.1 in [**Be1**]) is rather different from that proposed by Beilinson; this seems necessary since the "integral" refinement of the Manin-Drinfeld theorem (see [**Be1**], §5.5) does not hold in general. For further remarks on this see 1.1.3(iii) and 7.4 below.

In the course of preparing the talks of which this is an expanded account, Ramakrishnan's expository preprint [**Ra**] was helpful in a number of places. We are also grateful to many people for helpful discussions during the conference, and in particular to G. Harder and R. Weissauer.

1. The theorem.

1.0. We first review the formalism of modular curves and their cohomology. For details, see [**DR**], [**La**].

1.0.0. For any integer $n \geq 3$, there exists a moduli scheme M_n for elliptic curves E with level n structure $(\mathbf{Z}/n\mathbf{Z})^2 \xrightarrow{\sim} E[n]$. We have the following description of its complex points:–

$$M_n(\mathbf{C}) = GL_2(\mathbf{Z})\backslash\mathcal{H}^{\pm} \times GL_2(\mathbf{Z}/n\mathbf{Z})$$

where $\mathcal{H}^{\pm} = \mathbf{C} - \mathbf{R}$. A point on the right represented by $(\tau, \gamma) \in \mathcal{H}^{\pm} \times GL_2(\mathbf{Z}/n\mathbf{Z})$ corresponds to $E = \mathbf{C}/\mathbf{Z} + \mathbf{Z}\tau$ with $\gamma \cdot \begin{pmatrix} 1/n \\ \tau/n \end{pmatrix} \pmod{\mathbf{Z} + \mathbf{Z}\tau}$ as basis for $E[n]$. Writing G for the algebraic group GL_2 over $\mathbf{Q}$, G_f for its points in the finite adèle ring $\mathbf{A}_f$ of $\mathbf{Q}$, and K_n for the compact open subgroup

$$K_n = \ker\{p_n : G_{\hat{\mathbf{Z}}} \to G_{\mathbf{Z}/n\mathbf{Z}}\}$$

one has

$$M_n(\mathbf{C}) = G_{\mathbf{Z}}\backslash\mathcal{H}^{\pm} \times G_{\hat{\mathbf{Z}}}/K_n = G_{\mathbf{Q}}\backslash\mathcal{H}^{\pm} \times G_f/K_n.$$

1.0.1. More generally, for any compact open subgroup K of G_f, there is a modular curve M_K defined over $\mathbf{Q}$ with

$$M_K(\mathbf{C}) = G_{\mathbf{Q}}\backslash\mathcal{H}^{\pm} \times G_f/K.$$

There is a compactification

$$M_K \hookrightarrow \overline{M}_K$$

where $\overline{M}_K$ is a smooth projective (not necessarily geometrically connected) curve over $\mathbf{Q}$, and M_K is the complement of a finite set M_K^∞ of cusps. We denote by $\overline{M}_{K/\mathbf{Z}}$ a regular model of $\overline{M}_K$ over $\mathbf{Z}$ (see §7 below).

Examples of $\overline{M}_K$ include the familiar modular curves

$$X_1(n) \quad \text{when } K = K_1(n) \stackrel{\text{def}}{=} p_n^{-1}\begin{pmatrix} * & * \\ 0 & 1 \end{pmatrix}$$

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and $\overline{M}_n$ itself, usually denoted $X(n)$. Usually we will assume that $K \subseteq G_{\hat{\mathbf{Z}}}$ (the general case may be reduced to this by conjugation).

1.0.2. If $K' \subseteq K$ are open compact subgroups of G_f, there is a natural map $\theta_{K'/K} : \overline{M}_{K'} \to \overline{M}_K$. The projective limit $M = \varprojlim_K M_K$ is the moduli scheme for elliptic curves E with 'universal level structure' $\hat{\mathbf{Z}}^2 \xrightarrow{\sim} \varprojlim_n E[n]$.

1.1. We now turn to Beilinson's conjecture concerning the leading coefficient of $L(H^1(\overline{M}_K), s)$ at $s = 0$.

1.1.0. In Beilinson's formulation, the image of the regulator map

$$r_{\mathcal{D}} = r_{0,1} : H^2_{\mathcal{A}}(\overline{M}_{K/\mathbf{Z}}, \mathbf{Q}(2)) \to H^2_{\mathcal{D}}(\overline{M}_{K/\mathbf{R}}, \mathbf{R}(2))$$

is conjectured to be a $\mathbf{Q}$-structure of

$$\begin{aligned} H^2_{\mathcal{D}}(\overline{M}_K, \mathbf{R}(2)) &= H^1_B(\overline{M}_K(\mathbf{C}), \mathbf{R}(1))^- \\ &= H^1_B(\overline{M}_{K/\mathbf{R}}, \mathbf{R}(1)) \end{aligned}$$

and the determinant ("regulator") of a linear map taking this $\mathbf{Q}$-structure onto $H^1_B(\overline{M}_{K/\mathbf{R}}, \mathbf{Q}(1))$ is conjectured to be, up to a factor in $\mathbf{Q}^*$, the leading coefficient of $L(H^1(\overline{M}_K), s)$ at $s = 0$. (We have denoted by $\overline{M}_{K/\mathbf{R}}$ the analytic space over $\mathbf{R}$ associated to $\overline{M}_K$ —see chapter VIII, §1.) We recall that by definition, $H^2_{\mathcal{A}}(\overline{M}_{K/\mathbf{Z}}, \mathbf{Q}(2))$ is the image of the homomorphism

$$K_2^{(2)}(\overline{M}_{K/\mathbf{Z}}) \longrightarrow K_2^{(2)}(\overline{M}_K)$$

and that it does not depend on the choice of regular model.

Beilinson's theorem exhibits, for each K, a subspace of $H^2_{\mathcal{A}}(\overline{M}_{K/\mathbf{Z}}, \mathbf{Q}(2))$ whose image under $r_{\mathcal{D}}$ is a $\mathbf{Q}$-structure of $H^1_B(\overline{M}_{K/\mathbf{R}}, \mathbf{R}(1))$ with the desired regulator.

1.1.1. This subspace is defined in terms of $\mathcal{O}^*(M_K)$, the so-called modular units of level K. Write $\{\mathcal{O}^*(M_K), \mathcal{O}^*(M_K)\}$ for the $\mathbf{Q}$-subspace of $H^2_{\mathcal{A}}(M_K, \mathbf{Q}(2))$ generated by all symbols $\{u, v\}$ with $u, v \in \mathcal{O}^*(M_K) \otimes \mathbf{Q} = H^1_{\mathcal{A}}(M_K, \mathbf{Q}(1))$. Recall from chapter VIII, 5.1–2 that the map

$$H^2_{\mathcal{A}}(\overline{M}_K, \mathbf{Q}(2)) \longrightarrow H^2_{\mathcal{A}}(M_K, \mathbf{Q}(2))$$

is injective. Define

$$\mathcal{Q}_K = H^2_{\mathcal{A}}(\overline{M}_K, \mathbf{Q}(2)) \cap \{\mathcal{O}^*(M_K), \mathcal{O}^*(M_K)\}$$

and

$$\mathcal{P}_K = \bigcup_{K' \subseteq K} \theta_{K'/K*}(\mathcal{Q}_{K'}) \subset H^2_{\mathcal{A}}(\overline{M}_K, \mathbf{Q}(2))$$

the union being taken over all open subgroups $K' \subseteq K$ (cf. 1.0.2 above).

1.1.2. Theorem.

(i) $r_{\mathcal{D}}(\mathcal{P}_K)$ *is a* $\mathbf{Q}$*-structure of*

$$H^2_{\mathcal{D}}(\overline{M}_{K/\mathbf{R}}, \mathbf{R}(2)) = H^1_B(\overline{M}_{K/\mathbf{R}}, \mathbf{R}(1)).$$

(ii) Let g *be the genus of* $\overline{M}_K$*. Then*

$$\det r_{\mathcal{D}}(\mathcal{P}_K) = L^{(g)}(H^1(\overline{M}_K), 0) \cdot \det H^1_B(\overline{M}_{K/\mathbf{R}}, \mathbf{Q}(1)).$$

(iii)
$$\mathcal{P}_K \subseteq H^2_{\mathcal{A}}(\overline{M}_{K/\mathbf{Z}}, \mathbf{Q}(2)).$$

1.1.3. Remarks. (i) Theorem 1.1.2(i) fails in general if $\mathcal{P}_K$ is replaced by $\mathcal{Q}_K$. For example, when $\overline{M}_K = X_0(p)$ for a prime p there are only two cusps, so that $\mathcal{O}^*(M_K) = \mathbf{Q}^* \cdot u^{\mathbf{Z}}$ for some modular unit u; and in this case it is easy to see that $r_{\mathcal{D}}(\mathcal{Q}_K) = 0$, using the relation $\{u, u\} = 0$.

(ii) In some cases it is known, however, that $\mathcal{Q}_K$ is sufficient. For example, this holds when $\overline{M}_K = X_0(27)$, the elliptic Fermat curve, as can be shown using methods similar to those of chapter VIII—see the concluding example in [**Ro**]. Also, for the elliptic curve $E = \overline{M}_K = X_0(20)$ (which does *not* have complex multiplication) one has $E(\mathbf{Q}) = M_K^\infty$, so that the calculations of Bloch and Grayson [**BG**] for the curve "$20B$" show that again $r_{\mathcal{D}}(\mathcal{Q}_K) \neq 0$.

(iii) The proof of part (iii) of the theorem, which is independent of the other parts, will be given in §7 below. In the talks presented at the conference we were only able to prove a weaker result, namely that

$$r_{\mathcal{D}}(\mathcal{P}_K) = r_{\mathcal{D}}\left(\mathcal{P}_K \cap H^2_{\mathcal{A}}(\overline{M}_{K/\mathbf{Z}}, \mathbf{Q}(2))\right).$$

It seems to be generally believed that $r_{\mathcal{D}}$ is injective on $H^2_{\mathcal{A}}(\overline{M}_K, \mathbf{Q}(2))$, and not just on the integral subspace (see for example [**BG**], where $r_{\mathcal{D}}$ is denoted M). We regard part (iii) of the theorem as evidence for this conjecture, since it is not hard to see that if (iii) did not hold, then the rest of the theorem, together with the weaker statement mentioned above, would imply that the conjecture were false.

1.2. We next decompose the regulator map according to the automorphic representations of G_f, and reduce assertions (i) and (ii) of 1.1.2 to statements involving automorphic forms of weight 2.

1.2.0. Define $\Omega^1(\overline{M}) = \varinjlim_K \Omega^1(\overline{M}_K)$. The following facts are well known (see for example [**La**]).

The natural action of G_f on $\Omega^1(\overline{M})$ gives rise to a decomposition

$$\Omega^1(\overline{M}) \otimes \overline{\mathbf{Q}} = \bigoplus_\pi V_\pi$$

where $\pi : G_f \to GL(V_\pi)$ are irreducible admissible $\overline{\mathbf{Q}}$-representations of G_f, pairwise non-isomorphic.

Let $\mathcal{H}(G_f, K)$ denote the Hecke algebra of compactly supported functions on G_f with values in $\overline{\mathbf{Q}}$ which are biinvariant under K. Write $\mathcal{H}_K$ for the image of $\mathcal{H}(G_f, K)$ in the ring of correspondences on $\overline{M}_K$. Then $\mathcal{H}_K$ acts faithfully on $\Omega^1(\overline{M}_K) \otimes \overline{\mathbf{Q}}$, and

$$\Omega^1(\overline{M}_K) \otimes \overline{\mathbf{Q}} = \bigoplus_\pi V_\pi^K$$

where those of the spaces V_π^K which are nonzero are pairwise non-isomorphic $\mathcal{H}_K$-modules. We define

$$m(\pi, K) = \dim_{\overline{\mathbf{Q}}} V_\pi^K < \infty$$

and we write $e_\pi^K \in \mathcal{H}_K$ for the projector

$$\Omega^1(\overline{M}_K) \otimes \overline{\mathbf{Q}} \longrightarrow V_\pi^K.$$

1.2.1. For each $\sigma : \overline{\mathbf{Q}} \hookrightarrow \mathbf{C}$ and each π as above, we obtain a complex representation π^σ of G_f, and therefore corresponding L-functions and ϵ-factors $L(\pi^\sigma, s)$, $\epsilon(\pi^\sigma, s)$. Following Beilinson we normalise the L-functions so that the functional equation is given by the substitution $s \mapsto 2 - s$ (this represents a shift of $\frac{1}{2}$ from the conventions of [**JL**]).

In verifying Beilinson's conjecture it is convenient not to specify a preferred embedding of $\overline{\mathbf{Q}}$ in $\mathbf{C}$, and accordingly one defines $L(\pi, s)$ to be the function taking values in $\mathbf{C}^{\mathrm{Hom}(\overline{\mathbf{Q}},\mathbf{C})}$ whose σ-component is $L(\pi^\sigma, s)$. Since π can be defined over an algebraic number field, $L(\pi, s)$ actually takes values in the subring

$$\overline{\mathbf{Q}} \otimes \mathbf{C} \subset \mathbf{C}^{\mathrm{Hom}(\overline{\mathbf{Q}},\mathbf{C})}.$$

We similarly define the ϵ-factors $\epsilon(\pi, s)$ as $\overline{\mathbf{Q}} \otimes \mathbf{C}$-valued functions of s.

1.2.2. Theorem.

$$L(H^1(\overline{M}_K), s) = \prod_\pi L(\pi, s)^{m(\pi,K)}.$$

1.2.3. Remarks. (i) Up to a finite number of Euler factors, this result was obtained by Eichler and Shimura. By the work of Igusa, Langlands, Deligne and Carayol, it is now completely proved; see [**Ca**].

(ii) Note that although the individual factors on the right are $\overline{\mathbf{Q}} \otimes \mathbf{C}$-valued functions, their product takes values in $\mathbf{C} \subset \overline{\mathbf{Q}} \otimes \mathbf{C}$.

(iii) We should point out that in the decomposition of the l-adic cohomology $H^1_{\text{ét}}(\overline{M}_K \otimes \overline{\mathbf{Q}}, \mathbf{Q}_l)$ under the action of $\mathcal{H}_K$, the π-isotypical component corresponds to the factors $L(\check{\pi}, s)$, where $\check{\pi}$ is the representation contragredient to π, rather than $L(\pi, s)$. But since $m(\pi, K) = m(\check{\pi}, K)$ this gives the theorem as stated. However, when we decompose the regulator map $r_{\mathcal{D}}$ under the action of G_f in 1.2.6 below, the π-component will actually contribute a factor of $L'(\pi, 0)$ (not $L'(\check{\pi}, 0)$) to the value of the regulator. This can be accounted for by the general form of Beilinson's conjecture, in which it is the L-function of the *dual* of the motive which occurs (see [**Be1**], 3.4(b)).

1.2.4. We next recall the isomorphism (given by Poincaré duality)

$$H^1_B(\overline{M}_{K/\mathbf{R}}, \mathbf{R}(1)) \xrightarrow{\sim} \mathrm{Hom}(\Omega^1(\overline{M}_K), \mathbf{R}).$$

Applying the projector $e^K_{\check{\pi}}$, we obtain a diagram

$$\begin{array}{ccc} e^K_{\check{\pi}} H^1_B(\overline{M}_{K/\mathbf{R}}, \overline{\mathbf{Q}}(1)) & \hookrightarrow & e^K_{\check{\pi}} H^1_B(\overline{M}_{K/\mathbf{R}}, \overline{\mathbf{Q}} \otimes \mathbf{R}(1)) \\ & & \downarrow \wr \\ \mathrm{Hom}_{\overline{\mathbf{Q}}}(V^K_\pi, \overline{\mathbf{Q}}) & \hookrightarrow & \mathrm{Hom}_{\overline{\mathbf{Q}}}(V^K_\pi, \overline{\mathbf{Q}} \otimes \mathbf{R}) \end{array}$$

in which the two groups on the left define $\overline{\mathbf{Q}}$-structures on the free $\overline{\mathbf{Q}} \otimes \mathbf{R}$-modules on the right. Since V_π^K is an irreducible $\mathcal{H}_K$-module, and the right-hand isomorphism is an $\mathcal{H}_K$-isomorphism, we must have

$$e^K_{\check{\pi}} H^1_B(\overline{M}_{K/\mathbf{R}}, \overline{\mathbf{Q}}(1)) = c^+(\pi) \cdot \mathrm{Hom}_{\overline{\mathbf{Q}}}(V^K_\pi, \overline{\mathbf{Q}})$$

for some $c^+(\pi) \in (\overline{\mathbf{Q}} \otimes \mathbf{R})^*/\overline{\mathbf{Q}}^*$, independent of K (provided $V_\pi^K \neq (0)$). This constant is none other than Deligne's period for the motive attached to π whose L-function is $L(\pi, s+1)$ ([**DP**], 2.6, 7.4 and Chapter II of this volume)—see also 2.2.1 below.

1.2.5. We observe that $\mathcal{H}_K$ acts on $H^2_{\mathcal{A}}(\overline{M}_K, \mathbf{Q}(2)) \otimes \overline{\mathbf{Q}}$, leaving stable the subspace $\mathcal{P}_K \otimes \overline{\mathbf{Q}}$. This is immediate from the definition of $\mathcal{P}_K$. Theorem 1.1.2(i) and (ii) are now a consequence of the next theorem.

1.2.6. Theorem. *For every π and every K,*

$$e^K_{\check\pi} r_{\mathcal{D}}(\mathcal{P}_K \otimes \overline{\mathbf{Q}}) = L'(\check\pi, 0)c^+(\pi) \cdot \mathrm{Hom}_{\overline{\mathbf{Q}}}(V^K_\pi, \overline{\mathbf{Q}})$$
$$\subset \mathrm{Hom}_{\overline{\mathbf{Q}}}(V^K_\pi, \overline{\mathbf{Q}} \otimes \mathbf{R}) = e^K_{\check\pi} H^1_B(\overline{M}_{K/\mathbf{R}}, \overline{\mathbf{Q}} \otimes \mathbf{R}(1)).$$

1.2.7. We will actually prove a slightly different statement. Denote by $< , >_K$ the $\overline{\mathbf{Q}}$-linear pairing given by Poincaré duality (1.2.4):

$$H^1_B(\overline{M}_{K/\mathbf{R}}, \overline{\mathbf{Q}} \otimes \mathbf{R}(1)) \times \Omega^1(\overline{M}_K) \otimes \overline{\mathbf{Q}} \xrightarrow{< , >_K} \overline{\mathbf{Q}} \otimes \mathbf{R}.$$

1.2.8. Theorem. *Let π, V_π be as in 1.2.0.*

(i) For every open compact subgroup $K \subset G_f$, for every $\omega \in V^K_\pi$, and for every $\xi \in \mathcal{Q}_K \otimes \overline{\mathbf{Q}}$, we have

$$< r_{\mathcal{D}}(\xi), \omega >_K \in c^+(\pi)L'(\check\pi, 0) \cdot \overline{\mathbf{Q}} \subset \overline{\mathbf{Q}} \otimes \mathbf{R}.$$

(ii) There exist K, ω and ξ as in (i) such that

$$< r_{\mathcal{D}}(\xi), \omega >_K \neq 0.$$

1.2.9. Let us first show that 1.2.8 implies the preceding theorem 1.2.6 and therefore also 1.1.2(i) and (ii).

First, note the adjointness property, for $K' \subseteq K$, and $\theta = \theta_{K'/K}$ as in 1.0.2, 1.1.1: if $\xi' \in H^2_{\mathcal{A}}(\overline{M}_{K'}, \mathbf{Q}(2)) \otimes \overline{\mathbf{Q}}$ and $\omega \in V^K_\pi$, then

$$< r_{\mathcal{D}}(\xi'), \theta^*\omega >_{K'} = < r_{\mathcal{D}}(\theta_*\xi'), \omega >_K \ .$$

From this and the identity $\theta_*\theta^*\mathcal{P}_K = \mathcal{P}_K$, we see that 1.2.8(i)—for all K—implies

$$e^K_{\check\pi} r_{\mathcal{D}}(\mathcal{P}_K \otimes \overline{\mathbf{Q}}) \subseteq L'(\check\pi, 0)c^+(\pi) \cdot \mathrm{Hom}_{\overline{\mathbf{Q}}}(V^K_\pi, \overline{\mathbf{Q}}).$$

Assuming this to be true (for all K) consider the commutative diagram

$$\begin{array}{ccc} e^{K'}_{\check\pi} r_{\mathcal{D}}(\mathcal{P}_{K'} \otimes \overline{\mathbf{Q}}) & \xrightarrow{\theta_*} & e^K_{\check\pi} r_{\mathcal{D}}(\mathcal{P}_K \otimes \overline{\mathbf{Q}}) \\ \Big\downarrow{\scriptstyle i_{K'}} & & \Big\downarrow{\scriptstyle i_K} \\ L'(\check\pi, 0)c^+(\pi) \cdot \mathrm{Hom}_{\overline{\mathbf{Q}}}(V^{K'}_\pi, \overline{\mathbf{Q}}) & \xrightarrow{\theta^*} & L'(\check\pi, 0)c^+(\pi) \cdot \mathrm{Hom}_{\overline{\mathbf{Q}}}(V^K_\pi, \overline{\mathbf{Q}}) \end{array}$$

To prove that i_K is an isomorphism, it suffices to prove that $i_{K'}$ is an isomorphism, for some K'; by the irreducibility of $V^{K'}_\pi$ this follows from $e^{K'}_{\check\pi} r_{\mathcal{D}}(\mathcal{P}_{K'} \otimes \overline{\mathbf{Q}}) \neq 0$, which in turn is a consequence of 1.2.8(ii).

1.3. The final reformulation of Theorem 1.1.2 to be done in this section makes explicit the regulator map $r_{\mathcal{D}}$.

1.3.0. Recall from Chapter VIII, 1.6 the projection

$$pr_{\mathcal{D}} : H^1_B(M_{K/\mathbf{R}}, \mathbf{R}(1)) \longrightarrow H^1_B(\overline{M}_{K/\mathbf{R}}, \mathbf{R}(1)).$$

The commutative diagram

$$\begin{array}{ccc} H^2_{\mathcal{A}}(\overline{M}_K, \mathbf{Q}(2)) & \xrightarrow{r_{\mathcal{D}}} & H^1_B(\overline{M}_{K/\mathbf{R}}, \mathbf{R}(1)) \\ \Big\downarrow & & \Big\downarrow \Big\uparrow {\scriptstyle pr_{\mathcal{D}}} \\ H^2_{\mathcal{A}}(M_K, \mathbf{Q}(2)) & \xrightarrow{r_{\mathcal{D}}} & H^1_B(M_{K/\mathbf{R}}, \mathbf{R}(1)) \end{array}$$

suggests that we compute, for modular units $u, v \in \mathcal{O}^*(M_K) \otimes_{\mathbf{Z}} \overline{\mathbf{Q}}$, the element

$$pr_{\mathcal{D}}(r_{\mathcal{D}}(\{u, v\})) \in H^1_B(\overline{M}_{K/\mathbf{R}}, \mathbf{R}(1)).$$

This is done by the following formula, valid for all $\omega \in \Omega^1(\overline{M}_K \otimes \overline{\mathbf{Q}})$, which is none other than a $\overline{\mathbf{Q}}$-linear extension of VIII, (1.10) (compare also Chapter III, in particular 1.8–1.12):

$$< pr_{\mathcal{D}}(r_{\mathcal{D}}(\{u, v\}), \omega >_K = \frac{1}{2\pi i} \int_{M_K(\mathbf{C})} \log|u| \, \overline{d\log v} \wedge \omega.$$

Note that this is an identity in $\overline{\mathbf{Q}} \otimes \mathbf{C} \ldots$

1.3.1. Let us look again at the commutative diagram in 1.3.0. In view of the Manin-Drinfeld theorem (3.4.0 below) we can apply Bloch's lemma (Lemma 5.2 of Chapter VIII). Thus given u, $v \in \mathcal{O}^*(M_K)$ there is a finite extension F of $\mathbf{Q}$ and an element $\alpha \in \phi_*\{F^*, \mathcal{O}^*(M_K \otimes_{\mathbf{Q}} F)\}$ such that $\xi = \alpha + \{u, v\}$ belongs to the image of $H^2_{\mathcal{A}}(\overline{M}_K, \mathbf{Q}(2))$ in $H^2_{\mathcal{A}}(M_K, \mathbf{Q}(2))$. (Here ϕ is the basechange morphism $M_K \otimes F \longrightarrow M_K$.) From the diagram above we have $pr_{\mathcal{D}}\left(r_{\mathcal{D}}\left(\{u, v\}\right)\right) = r_{\mathcal{D}}(\xi)$. Granted this, the following theorem clearly implies 1.2.8 and therefore also parts (i), (ii) of the main theorem 1.1.2.

1.3.2. Theorem. *Let π, V_π be as in 1.2.0.*

(i) For all K, all $u, v \in \mathcal{O}^*(M_K) \otimes_{\mathbf{Z}} \overline{\mathbf{Q}}$, we have

$$\int_{M_K(\mathbf{C})} \log|u| \, \overline{d\log v} \wedge \omega \in 2\pi i \, c^+(\pi) L'(\check{\pi}, 0) \cdot \overline{\mathbf{Q}} \subset \overline{\mathbf{Q}} \otimes \mathbf{C}.$$

(ii) For some K, some $\omega \in V_\pi^K$, and some u, $v \in \mathcal{O}^*(M_K) \otimes_{\mathbf{Z}} \overline{\mathbf{Q}}$,

$$\int_{M_K(\mathbf{C})} \log|u| \, \overline{d\log v} \wedge \omega \neq 0.$$

2. Transformation of L-values.

2.0. In this section, we are going to rewrite the product

$$2\pi i \, c^+(\pi) L'(\check{\pi}, 0) \in (\overline{\mathbf{Q}} \otimes \mathbf{C})^* / \overline{\mathbf{Q}}^* \tag{2.0.0}$$

which occurs in 1.3.2.

2.0.1. *In this section, every expression is to be regarded as an element of $(\overline{\mathbf{Q}} \otimes \mathbf{C})^*/\overline{\mathbf{Q}}^*$.* We always regard $\mathbf{C}$ as embedded in $\overline{\mathbf{Q}} \otimes \mathbf{C}$ via the second inclusion. Thus if a, b are complex numbers, then in writing $a = b$ we signify that a and b are equal up to a nonzero rational factor.

2.1. The functional equation for the L-function of π implies that

$$L'(\check{\pi}, 0) = \pi^{-2} \epsilon(\pi, 2)^{-1} L(\pi, 2). \tag{2.1.0}$$

2.1.1. Now, the ϵ-factor of a motive is equal (in the sense of 2.0.1) to that of its determinant motive; see [**DP**], 5.5. But the determinant representation of the l-adic realisation of the motive associated to π is the central character ω_π of π, and $\omega_{\check{\pi}}^{-1} = \omega_\pi$—cf. [**DP**], 7.1, 7.4 where the "contragredient" normalisation is used. ω_π is an even Dirichlet character of $\mathbf{Q}$ since π has weight 2.

2.1.2. If χ is any even $\overline{\mathbf{Q}}$-valued Dirichlet character of $\mathbf{Q}$, its ϵ-factor $\epsilon(\chi)$, evaluated at any integer s, is equal to the standard Gauss sum of χ; see [**DP**], 6.4. It is characterised by the following properties—see [**Scha**], II §3, in particular 3.4; compare also [**Bl**].

(i) $\epsilon(\chi) \in (\overline{\mathbf{Q}} \otimes \mathbf{C})^*$.

(ii) If $\mathrm{Aut}(\mathbf{C}/\mathbf{Q})$ acts on $\overline{\mathbf{Q}} \otimes \mathbf{C}$ via the second factor, then for all $\tau \in \mathrm{Aut}(\mathbf{C}/\mathbf{Q})$, one has

$$\epsilon(\chi)^\tau = \chi(\tau) \cdot \epsilon(\chi),$$

where χ is defined on $\mathrm{Aut}(\mathbf{C}/\mathbf{Q})$ via the reciprocal of the cyclotomic character ("geometric Frobenius").

This characterisation applies in particular to $\epsilon(\omega_\pi) = \epsilon(\pi, 2)$, by 2.1.1; as a general consequence, note that

2.1.3. *If χ, χ' are two even Dirichlet characters of $\mathbf{Q}$, then*

$$\epsilon(\chi\chi') = \epsilon(\chi) \cdot \epsilon(\chi').$$

2.2. We now transform the product (2.0.0) into an expression involving a certain auxiliary Dirichlet character χ, which will vary over all even Dirichlet characters of $\mathbf{Q}$. The following fact is essential for the proof of 1.3.2(ii).

2.2.0. Lemma. *Given π, there exists an even Dirichlet character $\chi \neq 1, \omega_\pi^{-1}$ of $\mathbf{Q}$, such that the corresponding value at $s = 1$ of the twisted L-function does not vanish:*

$$L(\pi \otimes \chi, 1) \in (\overline{\mathbf{Q}} \otimes \mathbf{C})^*.$$

The standard proof of this lemma uses the fact that the first homology of the modular curves is generated by modular symbols—see [**Shi**], Theorem 2 and remark, pp. 212–214 for a result which covers all we need here.

2.2.1. Proposition. *Let χ be as in 2.2.0. Then*

$$L(\pi\otimes\chi,1)=c^+(\pi)\cdot\epsilon(\chi),$$

where $c^+(\pi)$ is as in 1.2.4 above.

In fact, Deligne's conjecture is true for motives of modular forms, by [**DP**], §7. Thus one gets

$$L(\pi\otimes\chi,1)=c^+(\pi\otimes\chi).$$

The remaining identity $c^+(\pi\otimes\chi)=\epsilon(\chi)c^+(\pi)$ is essentially a consequence of the fact that the de Rham realisation of the motive of χ is spanned by an element $\epsilon(\chi)$ satisfying (i) and (ii) of 2.1.2—see [**DP**], 6.3; [**Scha**], II,§3; [**Bl**].

2.2.2. In transforming the product (2.0.0), via 2.1.0 and 2.2.1, we shall pick up in the denominator the term

$$\epsilon(\pi,2)\cdot\epsilon(\chi)=\epsilon(\omega_\pi)\cdot\epsilon(\chi)=\epsilon(\omega_\pi\chi),$$

by 2.1.1 and 2.1.3. But Deligne's conjecture is true (due to Siegel) for Artin motives; see [**DP**], 6.7. Thus, since $s=2$ is critical for the L-function of an even Dirichlet character—cf. [**DP**], 5.1.8—we see that

$$\pi^2\epsilon(\omega_\pi\chi)=L(\omega_\pi\chi,2) \tag{2.2.3}$$

where the L-function on the right is the Dirichlet L-function attached to the character $\omega_\pi\chi$.

2.2.4. Remark. We have been using freely the notion of motive and certain special cases of Deligne's conjecture in this section. For the reader who may feel uneasy about this, we want to point out that all the motives and their properties that were needed in this section can be obtained in the category of motives for absolute Hodge cycles—see [**DMOS**] and [**Scha**], in particular Chapter V, §1. But they also exist in the strongest sense required by Grothendieck—cut out by projectors in the category of algebraic correspondences modulo rational equivalence.

Collecting now everything that has been established in this section, we obtain the following result.

2.3. Theorem. *For all χ satisfying $L(\pi\otimes\chi,1)\in(\overline{\mathbf{Q}}\otimes\mathbf{C})^*$, we have in $(\overline{\mathbf{Q}}\otimes\mathbf{C})^*/\overline{\mathbf{Q}}^*$,*

$$c^+(\pi)\cdot L'(\check{\pi},0)=\frac{L(\pi,2)\cdot L(\pi\otimes\chi,1)}{L(\omega_\pi\chi,2)}.$$

3. Eisenstein series and modular units.

3.0. This and the following section prepare the evaluation, in §5, of the regulator integral occurring in 1.3.2(i).

3.0.0. Let us begin with some notations for subgroups of $G = GL_2$:

$$B = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix}; N = \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}; D = \begin{pmatrix} * & 0 \\ 0 & 1 \end{pmatrix}.$$

If H is any of G, B, N, D and R is a $\mathbf{Z}$-algebra, we write H_R for the group of R-valued points of H; H_f for $H_{\mathbf{A}_f}$, where $\mathbf{A}_f$ is the ring of finite adèles of $\mathbf{Q}$; and

$$H^+_{\mathbf{Q}} = \{g \in H_{\mathbf{Q}} : \det g > 0\}.$$

We also write H_p for $H_{\mathbf{Q}_p}$.

If $b = \begin{pmatrix} b_1 & * \\ 0 & b_2 \end{pmatrix} \in B_f$, we write the modulus as

$$|b|_f = \left|\frac{b_1}{b_2}\right|_f,$$

where the right hand side is finite idèle modulus. Thus, for $b \in B^+_{\mathbf{Q}}$, one has $|b| = b_2/b_1$.

3.0.1. The cusps of $\overline{M}_K(\mathbf{C})$ can be written as

$$M^\infty_K(\mathbf{C}) = \pm N_{\hat{\mathbf{Z}}} \backslash G_{\hat{\mathbf{Z}}}/K.$$

For $k \in G_{\hat{\mathbf{Z}}}$, denote the corresponding cusp by $[k]$. The *width* $w(k, K)$ of $[k]$ is the least $w > 0$ such that

$$\begin{pmatrix} 1 & w \\ 0 & 1 \end{pmatrix} \in \pm kKk^{-1} \cap B^+_{\mathbf{Q}}.$$

A uniformising parameter on $\overline{M}_K(\mathbf{C})$ in a neighbourhood of $[k]$ is then

$$(z, k) \mapsto \exp(2\pi iz/w(k, K)),$$

for $\operatorname{Im} z > 0$.

3.0.2. If

$$t = \prod_p t_p : \operatorname{Aut}(\mathbf{C}/\mathbf{Q}) \longrightarrow \hat{\mathbf{Z}}^* = \prod_p \mathbf{Z}^*_p$$

denotes the cyclotomic character, giving the action of $\operatorname{Aut}(\mathbf{C}/\mathbf{Q})$ on $\exp(2\pi i \cdot \mathbf{Q})$, then $\operatorname{Aut}(\mathbf{C}/\mathbf{Q})$ acts on the cusps by the rule

$$[k]^\tau = \left[\begin{pmatrix} t(\tau) & 0 \\ 0 & 1 \end{pmatrix} k\right].$$

3.1.0. Let $\phi : \pm N_{\hat{\mathbf{Z}}} \backslash G_{\hat{\mathbf{Z}}} \to \overline{\mathbf{Q}}$ be a locally constant function. Define

$$\deg \phi : \hat{\mathbf{Z}}^* \longrightarrow \overline{\mathbf{Q}} \subset \overline{\mathbf{Q}} \otimes \mathbf{C}$$

by

$$(\deg \phi)(a) = \int_{SL_2(\hat{\mathbf{Z}})} \phi\left(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} g\right) dg,$$

relative to the Haar measure of total mass 1 on $SL_2(\hat{\mathbf{Z}})$. If ϕ is right invariant under the open subgroup $K \subseteq G_{\hat{\mathbf{Z}}}$, then it just gives a divisor on $\overline{M}_K(\mathbf{C})$, with coefficients in $\overline{\mathbf{Q}}$, supported on $M^\infty_K(\mathbf{C})$; the connected components of $\overline{M}_K(\mathbf{C})$ are indexed by $\hat{\mathbf{Z}}^*/\det(K)$, and $\deg \phi$ measures the degree on each connected component.

If $z \in \mathcal{H}^\pm$, write

(3.1.1)
$$I(z) = \begin{cases} \operatorname{Im}(z) & \text{if } z \in \mathcal{H}, \text{ i.e., } \operatorname{Im}(z) > 0 \\ 0 & \text{if } -z \in \mathcal{H} \end{cases}$$

By the Iwasawa decomposition, we may write any $g \in G_f$ in the form $g = b \cdot k$, with $b \in B^+_{\mathbf{Q}}$, and $k \in G_{\hat{\mathbf{Z}}}$. Then define

$$\hat{\phi}_s : G_f \longrightarrow \overline{\mathbf{Q}} \otimes \mathbf{C}$$

by

$$\hat{\phi}_s(g) = |b|_f^s \phi(k) \tag{3.1.2}$$

for $s \in \mathbf{C}$. This makes the function on $\mathcal{H}^{\pm} \times G_f$

$$(z, g) \mapsto \hat{\phi}_s(g) I(z)^s \tag{3.1.3}$$

well-defined, and invariant under left translation by $B^+_{\mathbf{Q}}$. The *real analytic Eisenstein series* associated with ϕ is then defined, for $\operatorname{Re} s > 1$, by the absolutely convergent series

$$\mathcal{E}_\phi(z, g; s) = -2\pi \sum_{\gamma \in B^+_{\mathbf{Q}} \backslash G_{\mathbf{Q}}} \hat{\phi}_s(\gamma g) I(\gamma z)^s. \tag{3.1.4}$$

This expression is to be regarded as taking values in $\overline{\mathbf{Q}} \otimes \mathbf{C}$.

The following facts are consequences of Selberg's theory—see [**GJ**], or [**Ku**] for a more classical account.

(3.1.5) $\mathcal{E}_\phi$ has a meromorphic continuation to the s-plane, with at worst a simple pole at $s = 1$. The residue of $\mathcal{E}_\phi$ at $s = 1$ is a locally constant function on $\mathcal{H}^{\pm} \times G_f$.

(3.1.6) As a function on $\mathcal{H}^{\pm} \times G_f$, $\mathcal{E}_\phi$ is left $G_{\mathbf{Q}}$-invariant, and right K-invariant if ϕ is. It satisfies the differential equation $\triangle \mathcal{E}_\phi = s(s-1)\mathcal{E}_\phi$, for the Laplacian $\triangle$ on $\mathcal{H}^{\pm}$.

(3.1.7) If $\deg \phi = 0$, then $\mathcal{E}_\phi$ has no pole at $s = 1$. In this case, $\mathcal{E}_\phi(z, g; 1)$ is harmonic on $\mathcal{H}^{\pm}$, and the difference

$$\mathcal{E}_\phi(z, k; 1) - (-2\pi y)\phi(k)$$

is bounded as $y = \operatorname{Im} z$ tends to $+\infty$.

From 3.1.7, it follows that, if $\deg \phi = 0$ and ϕ is right K-invariant, then the differential form on $\mathcal{H}^{\pm} \times G_f$

$$\eta_\phi = 2\partial_z \mathcal{E}_\phi(z, g; 1) \tag{3.1.8}$$

may be viewed as a holomorphic $\overline{\mathbf{Q}} \otimes \mathbf{C}$-valued 1-form on $M_K(\mathbf{C})$, with at worst simple poles at the cusps, and its residue at the cusp $[k] \in M_K^\infty(\mathbf{C})$ is $\phi(k)/w(k, K)$.

3.1.9. Define

$$\mathrm{Eis}^K = \{\eta_\phi \mid \phi \text{ is } K\text{-invariant and } \deg \phi = 0\},$$

and

$$\mathrm{Eis} = \bigcup_K \mathrm{Eis}^K.$$

Notice that *a priori* the space Eis is a $\overline{\mathbf{Q}}$-subspace of the space $\Omega^1(M) \otimes \overline{\mathbf{Q}} \otimes \mathbf{C}$, where $\Omega^1(M) = \varinjlim_K \Omega^1(M_K)$; it is however actually contained in $\Omega^1(M) \otimes \overline{\mathbf{Q}}$, as a consequence of the Manin-Drinfeld theorem—see Theorem 3.4.0 below.

3.2.0. For any ϕ (not necessarily of degree 0) the function

$$\hat{\phi} = \hat{\phi}_1 : G_f \longrightarrow \overline{\mathbf{Q}} \subset \overline{\mathbf{Q}} \otimes \mathbf{C}$$

is a locally constant function satisfying

$$\hat{\phi}(bng) = |b|_f \hat{\phi}(g)$$

for all $b \in B^+_{\mathbf{Q}}$, $n \in N_{\hat{\mathbf{Z}}}$, $g \in G_f$.

Under the action of diagonal matrices, the space of all such functions $\hat{\phi}$ decomposes as

$$\bigoplus_{\chi_1, \chi_2} \mathrm{Ind}(\chi_1, \chi_2), \tag{3.2.1}$$

where (χ_1, χ_2) runs over all pairs of Dirichlet characters

$$\chi_i : \mathbf{A}_{\mathbf{Q}}^*/\mathbf{Q}^* \longrightarrow \overline{\mathbf{Q}}^*$$

whose product $\chi_1\chi_2$ is even, and where $\mathrm{Ind}(\chi_1, \chi_2)$ is the space of all locally constant functions $\hat{\phi} : G_f \to \overline{\mathbf{Q}}$ satisfying

$$\hat{\phi}(bg) = \chi_{1,f}(b_1)\chi_{2,f}(b_2)|b|_f \hat{\phi}(g),$$

for all $b = \begin{pmatrix} b_1 & * \\ 0 & b_2 \end{pmatrix} \in B_f$. In the terminology of [**JL**], we have

$$\mathrm{Ind}(\chi_1, \chi_2) = \mathcal{B}(\mu, \nu) = \bigotimes_p{}' \mathcal{B}(\mu_p, \nu_p),$$

where $\mu, \nu : \mathbf{A}_f^* \to \overline{\mathbf{Q}}^*$ are the characters

$$\mu(x) = \chi_{1,f}(x)|x|_f^{\frac{1}{2}}$$
$$\nu(x) = \chi_{2,f}(x)|x|_f^{-\frac{1}{2}}.$$

3.2.2. The assignment $\eta_\phi \mapsto \hat{\phi}$ defines a G_f-equivariant inclusion

$$\mathrm{Eis} \hookrightarrow \bigoplus_{\chi_1,\chi_2} \mathrm{Ind}(\chi_1, \chi_2).$$

We write

$$\mathrm{Eis}(\chi_1, \chi_2) = \mathrm{Eis} \cap \mathrm{Ind}(\chi_1, \chi_2).$$

3.2.3. Proposition. $\mathrm{Eis} = \bigoplus \mathrm{Eis}(\chi_1, \chi_2)$. *Furthermore,* $\mathrm{Eis}(\chi_1, \chi_2) = \mathrm{Ind}(\chi_1, \chi_2)$ *unless* $\chi_1 = \chi_2 = \chi$*, in which case* $\mathrm{Eis}(\chi, \chi)$ *is the unique* G_f*-invariant subspace of* $\mathrm{Ind}(\chi, \chi)$ *of codimension one.*

Proof. The map $\eta_\phi \mapsto \hat{\phi}$ identifies $\mathrm{Eis}(\chi_1, \chi_2)$ with

$$\left\{ \hat{\phi} \in \mathrm{Ind}(\chi_1, \chi_2) \mid \deg\left(\hat{\phi}|_{G_{\hat{\mathbf{Z}}}}\right) = 0 \right\}.$$

If $\hat{\phi} \in \mathrm{Ind}(\chi_1, \chi_2)$ and $a, b \in \hat{\mathbf{Z}}^*$, then

$$\begin{aligned}
\left(\deg\left(\hat{\phi}|_{G_{\hat{\mathbf{Z}}}}\right)\right)(a) &= \int_{SL_2(\hat{\mathbf{Z}})} \hat{\phi}\left(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} g\right) dg \\
&= \int_{SL_2(\hat{\mathbf{Z}})} \hat{\phi}\left(\begin{pmatrix} ab & 0 \\ 0 & b^{-1} \end{pmatrix} g\right) dg \\
&= \chi_{1,f}(a) \frac{\chi_{1,f}}{\chi_{2,f}}(b) \cdot \left(\deg\left(\hat{\phi}|_{G_{\hat{\mathbf{Z}}}}\right)\right)(1).
\end{aligned}$$

Thus, $\deg\left(\hat{\phi}|_{G_{\hat{\mathbf{Z}}}}\right)$ is determined by its value at 1, and is zero if $\chi_1 \neq \chi_2$. Hence, in this case we have indeed that $\mathrm{Eis}(\chi_1, \chi_2) = \mathrm{Ind}(\chi_1, \chi_2)$.

If, however, $\chi_1 = \chi_2$, then $\mu\nu^{-1} = |\cdot|_f$, and so $\mathcal{B}(\mu_p, \nu_p)$ has a unique nonzero invariant subspace $\mathcal{B}_s(\mu_p, \nu_p)$, the so-called special representation (see [**JL**], Theorem 3.3). It has codimension 1, and is the subspace of functions $\hat{\phi}_p$ on G_p which belong to $\mathcal{B}(\mu_p, \nu_p)$ and satisfy

$$\int_{SL_2(\mathbf{Z}_p)} \hat{\phi}_p(g)\, dg = 0.$$

$\mathrm{Eis}(\chi_1, \chi_2)$ is then identified with the subspace of $\bigotimes_p{}' \mathcal{B}(\mu_p, \nu_p)$ spanned by tensors $\otimes_p \hat{\phi}_p$ such that $\hat{\phi}_p$ belongs to $\mathcal{B}_s(\mu_p, \nu_p)$ for at least one p, and this is the unique invariant subspace of $\mathcal{B}(\mu, \nu)$ of codimension one.

3.3. Remarks. (i) Even if $\deg\phi \neq 0$, we can still define a $G_{\mathbf{Q}}$-invariant 1-form η_ϕ on $\mathcal{H}^\pm \times G_f$; for the residue of $\mathcal{E}_\phi$ at $s=1$ is locally constant on $\mathcal{H}^\pm \times G_f$, and we can write

$$\eta_\phi = \lim_{s\to 1} 2\partial_z \mathcal{E}_\phi.$$

This will be an "almost holomorphic" form, as introduced by Hecke ([**He**], pp. 411–413).

(ii) The representations $\mathrm{Eis}(\chi_1,\chi_2)$ are highly reducible. However, if $\chi_1 \neq \chi_2$, then for infinitely many primes p, the p-components $\chi_{i,p}$ will be unequal, whence the local factors $\mathcal{B}(\mu_p,\nu_p)$ will be irreducible. In particular, for such primes p, there can be no quotient of $\mathrm{Eis}(\chi_1,\chi_2)$ whose restriction to G_p is abelian. We will come back to this in 7.4 below.

(iii) If K is an open subgroup of $G_{\hat{\mathbf{Z}}}$, then for every p such that $G_{\mathbf{Z}_p} \subseteq K$, the Hecke operator $T_p \in \mathcal{H}(G_f, K)$ (cf. 1.2.0 above) is defined as $\dfrac{1}{p}$ times the characteristic function of the double coset $K\begin{pmatrix}1 & 0\\ 0 & \varpi\end{pmatrix}K$, with $\varpi \in \mathbf{A}_f^*$ having component p at the place p, and 1 elsewhere. Then T_p acts as multiplication by $p\chi_1(p)+\chi_2(p)$ on $\mathrm{Ind}(\chi_1,\chi_2)^K$.

3.4. We next recall the well-known theorem of Manin and Drinfeld:

3.4.0. Theorem (Manin-Drinfeld). *Let $\mathcal{C} \subset \mathrm{Pic}^0(\overline{M}_K \otimes \mathbf{C})$ be the subgroup of classes of divisors supported on the cusps of $\overline{M}_K(\mathbf{C})$. Then $\mathcal{C}$ is finite.*

Proof. Choose a prime $p \geq 7$ such that $G_{\mathbf{Z}_p} \subseteq K$. $\mathcal{C}$ is a quotient of

$$\widetilde{H}^0(M_K^\infty(\mathbf{C}),\mathbf{Z}) = \left\{\phi : \pm N_{\hat{\mathbf{Z}}}\backslash G_{\hat{\mathbf{Z}}}/K \to \mathbf{Z} \mid \deg\phi = 0\right\},$$

and, by 3.3(iii), the eigenvalues of T_p on $\mathcal{C}\otimes\overline{\mathbf{Q}}$ have absolute value $\geq p-1$. On the other hand, the characteristic polynomial $\mathcal{X}_p(T)$ of T_p on $\Omega^1(\overline{M}_K)$ has rational coefficients, and all its roots have absolute value $\leq 2\sqrt{p}$. For some positive integer N, the correspondence $N\mathcal{X}_p(T_p)$ annihilates $\mathrm{Pic}^0(\overline{M}_K)$, and since $p-1 > 2\sqrt{p}$ for $p \geq 7$, we get that $\mathcal{C}\otimes\mathbf{Q} = 0$.

3.4.1. In §7 below we shall use a similar argument in the course of proving 1.1.2(iii).

3.5.0. Let $u \in \mathcal{O}^*(M_K\otimes\mathbf{C})$, for some open compact subgroup $K \subseteq G_{\hat{\mathbf{Z}}}$. If $\mathrm{ord}_{[k]}\, u$ denotes the multiplicity of u at the cusp $[k] \in M_K^\infty(\mathbf{C})$, then the function

$$\mathrm{div}(u) : G_{\hat{\mathbf{Z}}} \longrightarrow \overline{\mathbf{Q}}$$
$$k \mapsto \frac{\mathrm{ord}_{[k]}\, u}{w(k,K)}$$

depends only on u, not on the choice of K. It is left $\pm N_{\hat{\mathbf{Z}}}$-invariant, right K-invariant, and has degree zero (see 3.1.0). Conversely, by the Manin-Drinfeld theorem any ϕ with this property is of the form $\mathrm{div}(u)$ for some $u \in \mathcal{O}^*(M_K\otimes\mathbf{C})\otimes_{\mathbf{Z}}\overline{\mathbf{Q}}$.

3.5.1. Proposition. *If $\phi = \mathrm{div}(u) : \pm N_{\hat{\mathbf{Z}}}\backslash G_{\hat{\mathbf{Z}}}/K \to \mathbf{Q}$, then*

$$\eta_\phi = d\log u;$$

the function

$$\mathcal{E}_\phi(z,g;1) - \log|u|$$

is locally constant.

Proof. Since ϕ is $\mathbf{Q}$-valued, $\mathcal{E}_\phi(z,g;s)$ takes values in $\mathbf{C} \subset \overline{\mathbf{Q}}\otimes\mathbf{C}$, and $\mathcal{E}_\phi(z,g;1)$ is real-valued. By 3.1.7, the difference $\mathcal{E}_\phi(z,g;1)-\log|u|$ is harmonic and bounded on $M_K(\mathbf{C})$, whence locally constant. The first claim follows immediately.

3.5.2. Corollary. *Let $\mathcal{O}^*(M\otimes\mathbf{C}) = \varinjlim \mathcal{O}^*(M_K\otimes\mathbf{C})$. Then*

$$d\log\mathcal{O}^*(M\otimes\mathbf{C})\otimes\overline{\mathbf{Q}} = \mathrm{Eis} \subset \Omega^1(M)\otimes\overline{\mathbf{Q}}.$$

We can therefore write the integral occurring in 1.3.2 as

$$\int_{M_K(\mathbf{C})} (\mathcal{E}_\phi(z,g;1)+c)\,\bar{\eta}_\xi \wedge \omega, \tag{3.5.3}$$

where $\xi = \mathrm{div}\, v$, $\phi = \mathrm{div}\, u$ and c is locally constant on $M_K(\mathbf{C})$. Since $\bar{\eta}_\xi\wedge\omega = 2\,d(\mathcal{E}_\xi(z,g;1)\omega)$, Stokes' theorem shows that we may assume that $c=0$.

3.5.4. Proposition. $d\log \mathcal{O}^*(M)\otimes_{\mathbf{Z}}\overline{\mathbf{Q}} = \bigoplus_{\chi\,\text{even}} \mathrm{Eis}(1,\chi).$

Proof. Let $\mathbf{Q}^c$ denote the algebraic closure of $\mathbf{Q}$ in $\mathbf{C}$ (to be distinguished from $\overline{\mathbf{Q}}$). Then

$$\mathcal{O}^*(M) = \mathcal{O}^*(M\otimes\mathbf{C})^{\mathrm{Aut}(\mathbf{C}/\mathbf{Q})} = \mathcal{O}^*(M\otimes\mathbf{Q}^c)^{\mathrm{Gal}(\mathbf{Q}^c/\mathbf{Q})}$$

whence by Hilbert 90

$$d\log\mathcal{O}^*(M) = (d\log\mathcal{O}^*(M\otimes\mathbf{Q}^c))^{\mathrm{Gal}(\mathbf{Q}^c/\mathbf{Q})}.$$

If $\tau\in\mathrm{Gal}(\mathbf{Q}^c/\mathbf{Q})$ and $u\in\mathcal{O}^*(M\otimes\mathbf{Q}^c)$, then by 3.0.2,

$$(\mathrm{div}\, u^\tau)(k) = (\mathrm{div}\, u)\left(\begin{pmatrix} t(\tau) & 0\\ 0 & 1\end{pmatrix} k\right)$$

for all $k\in G_{\hat{\mathbf{Z}}}$. Thus if $u\in\mathcal{O}^*(M)$ and $\phi=\mathrm{div}\, u$, then $\hat\phi\in\bigoplus_\chi \mathrm{Ind}(1,\chi)$.

4. Whittaker functions and L-factors.

4.0. In this section we continue to prepare for the integral evaluation of §5, and also include some facts needed for later sections. Most of the results required from representation theory can be found in [**JL**], [**Ge**] or [**Go**].

4.0.0. For every prime p, let $\psi_p:\mathbf{Q}_p\to\mathbf{C}^*$ be the additive character such that

$$\psi_p(p^{-r}) = \exp(-2\pi i p^{-r}).$$

Then $\psi=\prod_p\psi_p:\mathbf{A}_f\to\mathbf{C}^*$ has the property that $(x_\infty,x_f)\mapsto\exp(2\pi i x_\infty)\cdot\psi(x_f)$ is a non-trivial character of $\mathbf{Q}\backslash\mathbf{A}$.

4.0.1. If $\pi_{\mathbf{C}}$ is any irreducible admissible complex representation of G_f, all of whose local components are infinite-dimensional, then it has a unique realisation $\mathcal{W}(\pi_{\mathbf{C}})$ in the space of Whittaker functions

$$\mathcal{W} = \left\{ f: G_f\to\mathbf{C} \mid f\left(\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix} g\right) = \psi(x)f(g) \text{ for all } x\in\mathbf{A}_f,\ g\in G_f\right\}.$$

If $\pi_{\mathbf{C}}$ occurs in $\Omega^1(\overline{M}\otimes\mathbf{C})$, the space of (complex) cusp forms of weight 2 (cf. 1.2.0 above), then Fourier expansion gives an explicit mapping of the space of $\pi_{\mathbf{C}}$ into $\mathcal{W}$; if $\omega = 2\pi i f(g,z)\,dz$ belongs to the space of $\pi_{\mathbf{C}}$ (here $z\in\mathcal{H}^\pm$, $g\in G_f$), then the mapping is:

$$\omega\mapsto W_\omega(g) = \int_{\mathbf{Q}\backslash\mathbf{A}} f(z+x_\infty, \begin{pmatrix}1 & x_f\\ 0 & 1\end{pmatrix} g)\, e^{-2\pi i(z+x_\infty)}\overline{\psi(x_f)}\,dx \tag{4.0.2}$$

where $x=(x_\infty,x_f)\in\mathbf{R}\times\mathbf{A}_f=\mathbf{A}$. The inverse mapping is given by

$$\omega = \sum_{\substack{a\in\mathbf{Q}\\ a>0}} W_\omega\left(\begin{pmatrix}a & 0\\ 0 & 1\end{pmatrix} g\right)\, d(e^{2\pi i a z}). \tag{4.0.3}$$

4.1. Now suppose that π is an irreducible factor of $\Omega^1(\overline{M})\otimes\overline{\mathbf{Q}}$. Then, as in 1.2.0, for every embedding $\sigma:\overline{\mathbf{Q}}\hookrightarrow\mathbf{C}$, we obtain by extension of scalars a complex representation π^σ of G_f. The resulting Whittaker models $\mathcal{W}(\pi^\sigma)$ fit together to give a "rational" Whittaker model, in a sense to be explained now.

4.1.0. Recall the definition of the cyclotomic character $t:\mathrm{Aut}(\mathbf{C}/\mathbf{Q})\to\hat{\mathbf{Z}}^*$ from 3.0.2, and let $\mathrm{Aut}(\mathbf{C}/\mathbf{Q})$ act on $\overline{\mathbf{Q}}\otimes\mathbf{C}$ via the second factor. In terms of the inclusion $\overline{\mathbf{Q}}\otimes\mathbf{C}\subset\mathbf{C}^{\mathrm{Hom}(\overline{\mathbf{Q}},\mathbf{C})}$, for $\tau\in\mathrm{Aut}(\mathbf{C}/\mathbf{Q})$ and $(x_\sigma)_\sigma\in\overline{\mathbf{Q}}\otimes\mathbf{C}$ we have

$$(x_\sigma)^\tau = (x^\tau_{\tau^{-1}\sigma})_\sigma.$$

(The action of $\mathrm{Aut}(\mathbf{C}/\mathbf{Q})$ is a left action.)

4.1.1. Theorem. *There is a unique realisation of π in a $\overline{\mathbf{Q}}$-subspace $\mathcal{W}^{\mathrm{rat}}(\pi)$ of the $\overline{\mathbf{Q}}\otimes\mathbf{C}$-module*

$$\mathcal{W}^{\mathrm{rat}} = \left\{ f : G_f \to \overline{\mathbf{Q}}\otimes\mathbf{C} \;\middle|\; \begin{array}{l} f\left(\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix} g\right) = \psi(x)f(g) \text{ for all } x\in\mathbf{A}_f \text{ and } g\in G_f \\ f(g)^\tau = f\left(\begin{pmatrix}t(\tau) & 0\\ 0 & 1\end{pmatrix} g\right) \text{ for all } \tau\in\mathrm{Aut}(\mathbf{C}/\mathbf{Q}) \end{array}\right\}.$$

The mapping (4.0.2) gives rises to a (canonical) isomorphism $V_\pi \xrightarrow{\sim} \mathcal{W}^{\mathrm{rat}}(\pi)$.

This theorem is implicit in [**Ha**]. The main point is that the curves $\overline{M}_K$ are defined over $\mathbf{Q}$, and the action of $\tau\in\mathrm{Aut}(\mathbf{C}/\mathbf{Q})$ on the set of cusps

$$M_K^\infty(\mathbf{C}) = \pm N_{\hat{\mathbf{Z}}}\backslash G_{\hat{\mathbf{Z}}}/K$$

is left multiplication by $\begin{pmatrix}t(\tau) & 0\\ 0 & 1\end{pmatrix}$. This enables one to keep track on the action of $\mathrm{Aut}(\mathbf{C}/\mathbf{Q})$ on the Fourier expansions of automorphic forms at the various cusps of $\overline{M}_K$ (and the theorem is no more than a description of this action).

4.1.2. The model $\mathcal{W}(\pi_{\mathbf{C}})$ of 4.0.1 is a restricted tensor product $\bigotimes_p' \mathcal{W}(\pi_{\mathbf{C},p})$ of Whittaker models of the local representations $\pi_{\mathbf{C},p}$ of G_p. The rational Whittaker model also admits such a factorisation; if $\pi = \otimes_p' \pi_p$ is as above, then

$$\mathcal{W}^{\mathrm{rat}}(\pi) = \bigotimes_p{}' \mathcal{W}^{\mathrm{rat}}(\pi_p)$$

where $\mathcal{W}^{\mathrm{rat}}(\pi_p)$ is the unique realisation of π_p in

$$\mathcal{W}_p^{\mathrm{rat}} = \left\{ f : G_p \to \overline{\mathbf{Q}}\otimes\mathbf{C} \;\middle|\; \begin{array}{l} f\left(\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix} g\right) = \psi_p(x)f(g) \text{ for all } x\in\mathbf{Q}_p \text{ and } g\in G_p \\ f(g)^\tau = f\left(\begin{pmatrix}t_p(\tau) & 0\\ 0 & 1\end{pmatrix} g\right) \text{ for all } \tau\in\mathrm{Aut}(\mathbf{C}/\mathbf{Q}) \end{array}\right\}.$$

4.2. The restriction of functions in the Whittaker model to the subgroup $\begin{pmatrix}* & 0\\ 0 & 1\end{pmatrix}$ of G is injective; the resulting space of functions is the Kirillov model. Restricting functions in $\mathcal{W}^{\mathrm{rat}}(\pi)$, resp. $\mathcal{W}^{\mathrm{rat}}(\pi_p)$, gives realisations $\mathcal{K}^{\mathrm{rat}}(\pi)$, $\mathcal{K}^{\mathrm{rat}}(\pi_p)$ of the representations π, π_p in spaces of $\overline{\mathbf{Q}}\otimes\mathbf{C}$-valued functions on $\mathbf{A}_f^*$ and $\mathbf{Q}_p^*$, respectively. In particular, we have

$$\mathcal{K}^{\mathrm{rat}}(\pi_p) \subset \mathcal{K}_p^{\mathrm{rat}} = \left\{ f : \mathbf{Q}_p^* \to \overline{\mathbf{Q}}\otimes\mathbf{C} \;\middle|\; \begin{array}{c} f \text{ locally constant, and } f(x)^\tau = f(t_p(\tau)x) \\ \text{for all } x\in\mathbf{Q}_p^* \text{ and } \tau\in\mathrm{Aut}(\mathbf{C}/\mathbf{Q}) \end{array}\right\}.$$

From [**JL**], Proposition 2.9(i), one obtains:

4.2.0. Lemma. *The space $\mathcal{K}^{\mathrm{rat}}(\pi_p)$ contains all functions in $\mathcal{K}_p^{\mathrm{rat}}$ of compact support.*

4.3. Using the Kirillov model one defines local L-factors. For all p, associated to π_p there is a $\overline{\mathbf{Q}}\otimes\mathbf{C}$-valued function $L(\pi_p, s)$, which is in fact the reciprocal of a polynomial in p^{-s} with coefficients in $\overline{\mathbf{Q}}\subset\overline{\mathbf{Q}}\otimes\mathbf{C}$. The fact that these coefficients are in $\overline{\mathbf{Q}}$ expresses the fact that if π_E is a representation of G_p defined over a subfield $E\subset\mathbf{C}$ which is Galois over $\mathbf{Q}$, then the L-factor $L(\pi_E, s)$—in the usual sense—is a polynomial in p^{-s} with coefficients in E, and if $\alpha\in\mathrm{Gal}(E/\mathbf{Q})$, the L-factor of π^α is obtained by applying α to the coefficients of $L(\pi_E, s)$.

4.3.0. In case π_p is spherical (that is, has a nonzero $G_{\mathbf{Z}_p}$-invariant vector) we have

$$L(\pi_p, s) = \int_{\mathbf{Q}_p^*} |a|_p^{s-1} f(a)\, d^*a,$$

where f is the unique spherical function in $\mathcal{K}^{\mathrm{rat}}(\pi_p)$ such that $f(1)=1$.

4.3.1. The global L-function is defined as the Euler product (for $\mathrm{Re}\, s > 3/2$)

$$L(\pi, s) = \prod_p L(\pi_p, s);$$

it is a $\overline{\mathbf{Q}}\otimes\mathbf{C}$-valued function, whose σ-component is the usual L-function associated to π^σ—cf. 1.2.1 above.

4.4. One also defines Whittaker models for the induced representations $\mathrm{Ind}(\chi_1, \chi_2)$ (see [**Go**], §I.9)—note that these are in general reducible, and are not automorphic representations in the usual sense of the word. The associated L-function is the product of the Dirichlet L-series

$$L(\chi_1, s) \cdot L(\chi_2, s+1).$$

The composite mapping

$$\mathrm{Eis}(\chi_1, \chi_2) \hookrightarrow \mathrm{Ind}(\chi_1, \chi_2) \xrightarrow{\sim} \mathcal{W}^{\mathrm{rat}}(\mathrm{Ind}(\chi_1, \chi_2))$$

can be described by the Fourier expansion map (4.0.2); however in the inversion formula (4.0.3) a constant term must be added.

4.5. We now describe what we need from the theory of L-functions for $GL_2 \times GL_2$—see [**J**].

4.5.0. Suppose that π is an irreducible constituent of $\Omega^1(\overline{M}) \otimes \overline{\mathbf{Q}}$, and that π' is one of the representations $\mathrm{Ind}(\chi_1, \chi_2)$. For $f \in \mathcal{K}^{\mathrm{rat}}(\pi_p)$ and $g \in \mathcal{K}^{\mathrm{rat}}(\pi'_p)$, write

$$I(f, g; s) = \int_{\mathbf{Q}_p^*} |a|_p^{s-1} f(a) \overline{g(a)} \, d^*a \tag{4.5.1}$$

(where complex conjugation acts on $\overline{\mathbf{Q}} \otimes \mathbf{C}$ via the second factor).

4.5.2. Proposition. *(i)* $I(f, g; s) \in \overline{\mathbf{Q}}(p^{-s}) \subset \overline{\mathbf{Q}} \otimes \mathbf{C}(p^{-s})$.

(ii) If f and g are spherical functions with $f(1) = g(1) = 1$, then

$$I(f, g; s) = \frac{L(\pi_p \otimes \overline{\chi}_{1,p}, s+1) \cdot L(\pi_p \otimes \overline{\chi}_{2,p}, s)}{L(\overline{\chi}_{1,p}\overline{\chi}_{2,p}\omega_{\pi,p}, 2s)},$$

where ω_π is the central character of π.

The proof can be extracted from §§14, 15 of [**J**], in particular Proposition 14.4 and its proof, and Lemma 15.9.4. The only point which this reference does not make explicit is that I has coefficients in $\overline{\mathbf{Q}}$. The key observation here is that conjugating the coefficients by $\tau \in \mathrm{Aut}(\mathbf{C}/\mathbf{Q})$ amounts to replacing $f(a)\overline{g(a)}$ with $(f(a)\overline{g(a)})^\tau$ in the integral, hence with $f\left(t_p(\tau)a\right) \overline{g\left(t_p(\tau)a\right)}$. This is the same as substituting $t_p(\tau)a$ for a everywhere in the integral—which however leaves it unchanged.

4.5.3. Now suppose that $f \in \mathcal{W}^{\mathrm{rat}}(\pi)$, $g \in \mathcal{W}^{\mathrm{rat}}(\pi')$ where π and π' are as in 4.5.0, and let ϕ be any function as in 3.1.0 with the additional property that, for every $\tau \in \mathrm{Aut}(\mathbf{C}/\mathbf{Q})$ and $k \in G_{\hat{\mathbf{Z}}}$,

$$\phi\left(\begin{pmatrix} t(\tau) & 0 \\ 0 & 1 \end{pmatrix} k\right) = \phi(k) \in \overline{\mathbf{Q}} \subset \overline{\mathbf{Q}} \otimes \mathbf{C}.$$

Consider the integral

$$J(f, g, \phi; s) = \int_{\mathbf{A}_f^* \times G_{\hat{\mathbf{Z}}}} |a|_f^{s-1} \phi(k) f\left(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} k\right) \overline{g\left(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} k\right)} \, d^*a \, dk$$

where dk denotes the Haar measure on $G_{\hat{\mathbf{Z}}}$ with total mass one. By restricting to sufficiently small open subgroups of $G_{\hat{\mathbf{Z}}}$ (leaving f, g, ϕ invariant under right translation) and using 4.5.2 and its proof one sees that J can be meromorphically continued to the s-plane (cf. [**J**], pp. 124ff.), and that, for some $A = A(f, g, \phi) \in \overline{\mathbf{Q}}$,

$$J(f, g, \phi; 1) = A \cdot \frac{L(\pi \otimes \overline{\chi}_1, 2) L(\pi \otimes \overline{\chi}_2, 1)}{L(\overline{\chi}_1 \overline{\chi}_2 \omega_\pi, 2)}.$$

4.5.4. We need one further local fact; by Lemma 4.2.0, we see that if S is any nonempty compact open subset of $\mathbf{Q}_p^*$, there exist $f \in \mathcal{K}^{\mathrm{rat}}(\pi_p)$ and $g \in \mathcal{K}^{\mathrm{rat}}(\pi'_p)$ such that

$$\mathrm{supp}(f) \cup \mathrm{supp}(g) \subseteq S \quad \text{and} \quad I(f, g; 1) = 1.$$

5. Evaluation of the regulator integral.

5.0. We now begin the calculation of the integral of 1.3.2,

$$\int_{M_K(\mathbf{C})} \log|u|\overline{d\log v}\wedge\omega$$

where π, V_π is fixed as in 1.2.0, K is an open compact subgroup of G_f, $\omega\in V_\pi^K$, and u, $v\in\mathcal{O}^*(M_K)$.

5.0.0. Write $\phi=\operatorname{div}(u)$ and $\xi=\operatorname{div}(v)$. By virtue of (3.5.3) this integral equals

$$\int_{M_K(\mathbf{C})} \mathcal{E}_\phi(z,g;1)\overline{\eta}_\xi\wedge\omega. \tag{5.0.1}$$

Note that η_ξ is a $\overline{\mathbf{Q}}$-linear combination of elements of $\operatorname{Eis}(1,\chi)$, where χ runs over all even Dirichlet characters, by 3.5.4.

5.1. The main tool in the evaluation of (5.0.1) is Rankin's trick. Let (π,V_π) be an irreducible constituent of $\Omega^1(\overline{M})\otimes\overline{\mathbf{Q}}$, and let $(\pi',V_{\pi'})$ be a representation of G_f which is either also an irreducible constituent of $\Omega^1(\overline{M})\otimes\overline{\mathbf{Q}}$, or else one of the representations $\operatorname{Ind}(\chi_1,\chi_2)$. In the former case write $V^0_{\pi'}=V_{\pi'}$, and in the latter set $V^0_{\pi'}=\operatorname{Eis}(\chi_1,\chi_2)\subseteq V_{\pi'}$. If $\omega\in V_\pi^K$ and $\eta\in(V^0_{\pi'})^K$, the associated Whittaker functions W_ω, W_η are defined (by 4.0 and 4.4 above).

5.1.0. Proposition ("Rankin's trick"). *For any $\phi:\pm N_{\hat{\mathbf{Z}}}\backslash G_{\hat{\mathbf{Z}}}/K\to\overline{\mathbf{Q}}$, and any s with $\operatorname{Re}s>1$,*

$$\int_{M_K(\mathbf{C})}\mathcal{E}_\phi(z,g;s)\,\overline{\eta}\wedge\omega=\pi i\frac{\Gamma(s+1)}{(4\pi)^{s-1}}\left[G_{\hat{\mathbf{Z}}}:\pm K\right]$$
$$\times\int_{\mathbf{A}_f^*\times G_{\hat{\mathbf{Z}}}}\phi(k)W_\omega\left(\begin{pmatrix}a&0\\0&1\end{pmatrix}k\right)\overline{W_\eta\left(\begin{pmatrix}a&0\\0&1\end{pmatrix}k\right)}\,|a|_f^{s-1}\,d^*a\,dk$$

where dk is the Haar measure on $G_{\hat{\mathbf{Z}}}$ with total mass 1.

Proof. The integral is

$$\int_{G_{\mathbf{Q}}\backslash\mathcal{H}^\pm\times G_f/K}-2\pi\sum_{\gamma\in B^+_{\mathbf{Q}}\backslash G_{\mathbf{Q}}}\hat{\phi}_s(\gamma g)I(\gamma z)^s\,\overline{\eta}\wedge\omega$$
$$=\int_{B^+_{\mathbf{Q}}\backslash\mathcal{H}^\pm\times G_f/K}-2\pi\hat{\phi}_s(g)I(z)^s\,\overline{\eta}\wedge\omega$$

(since $\overline{\eta}\wedge\omega$ is $G_{\mathbf{Q}}$-invariant). Now if $\operatorname{Im}(z)>0$ then

$$\overline{\eta}\wedge\omega=\overline{\left(\sum_{b>0}W_\eta\left(\begin{pmatrix}b&0\\0&1\end{pmatrix}g\right)d\left(e^{2\pi ibz}\right)+C\right)}$$
$$\wedge\sum_{a>0}W_\omega\left(\begin{pmatrix}a&0\\0&1\end{pmatrix}g\right)d\left(e^{2\pi iaz}\right)$$

where C denotes the "constant term" in the Fourier expansion of η. The only terms contributing to the integral are those for which $a=b$, and so it may be rewritten as

$$\int_{B^+_{\mathbf{Q}}\backslash\mathcal{H}\times G_f/K}-2\pi\hat{\phi}_s(g)y^s\sum_{a>0}\overline{W_\eta\left(\begin{pmatrix}a&0\\0&1\end{pmatrix}g\right)}W_\omega\left(\begin{pmatrix}a&0\\0&1\end{pmatrix}g\right)\overline{d\left(e^{2\pi iaz}\right)}\wedge d\left(e^{2\pi iaz}\right)$$
$$=16\pi^3 i\int_{Z_{\mathbf{Q}}N_{\mathbf{Q}}\backslash\mathcal{H}\times G_f/K}\hat{\phi}_s(g)y^s\overline{W_\eta(g)}W_\omega(g)e^{-4\pi y}\,dx\wedge dy$$
$$=\pi i\frac{\Gamma(s+1)}{(4\pi)^{s-1}}\left[G_{\hat{\mathbf{Z}}}:\pm K\right]\int_{Z_{\mathbf{Q}}N_f\backslash G_f}\hat{\phi}_s(g)\overline{W_\eta(g)}W_\omega(g)\,dg$$

where dg is the measure on $Z_{\mathbf{Q}} N_f \backslash G_f$ derived from the Haar measure on G_f for which $G_{\hat{\mathbf{Z}}}$ has measure 1. Observing that

$$Z_{\mathbf{Q}} N_f \backslash G_f = \pm N_{\hat{\mathbf{Z}}} \backslash D_f G_{\hat{\mathbf{Z}}}$$

and that, in terms of the decomposition

$$G_f = N_f A_f G_{\hat{\mathbf{Z}}}, \qquad g = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix} k$$

the Haar measure is $dg = \left| \dfrac{a_1}{a_2} \right|_f^{-1} dx\, d^* a_1\, d^* a_2\, dk$, we obtain the required formula.

5.1.1. Remark. When $\deg(\phi) = 0$, the above yields

$$\text{(5.1.2)} \qquad \begin{aligned} &\int_{M_K(\mathbf{C})} \mathcal{E}_\phi(z, g; 1) \overline{\eta} \wedge \omega = \pi i \left[G_{\hat{\mathbf{Z}}} : \pm K \right] \\ &\quad \times \lim_{s \to 1} \int_{\mathbf{A}_f^* \times G_{\hat{\mathbf{Z}}}} \phi(k) W_\omega \left(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} k \right) \overline{W_\eta \left(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} k \right)} |a|_f^{s-1}\, d^* a\, dk. \end{aligned}$$

For the nonvanishing argument in §6 it is useful to note that the limit of the integrals in 5.1.0 exists for every ϕ (even though in general $\mathcal{E}_\phi$ has a pole at $s = 1$). Indeed, since the residue of $\mathcal{E}_\phi$ at $s = 1$ is locally constant, the same reasoning as in (3.5.3) above shows that it does not contribute to the integral.

5.2. It remains to collect up the loose ends of the argument. As observed in 5.0 above, the integral (5.0.1) is a $\overline{\mathbf{Q}}$-linear combination of integrals of the form (5.1.2), with $\eta \in \left(V_{\pi'}^0 \right)^K$ where $\pi' = \mathrm{Ind}(1, \chi)$ for even characters χ. Furthermore $\phi = \mathrm{div}(u)$ satisfies the Galois invariance property required in 4.5.3. This paragraph tells us that the integral (5.0.1) lies in the $\overline{\mathbf{Q}}$-subspace of $\overline{\mathbf{Q}} \otimes \mathbf{C}$ generated by the elements

$$2\pi i \frac{L(\pi, 2) \cdot L(\pi \otimes \overline{\chi}, 1)}{L(\overline{\chi} \omega_\pi, 2)} \in \overline{\mathbf{Q}} \otimes \mathbf{C},$$

as χ varies over all even Dirichlet characters of $\mathbf{Q}$. By 2.3, these elements are $\overline{\mathbf{Q}}$-multiples of

$$2\pi i \cdot c^+(\pi) \cdot L'(\check{\pi}, 0).$$

This concludes the proof of part (i) of Theorem 1.3.2.

6. Nonvanishing of the regulator integral.

6.0. In this section we complete the proof of part (ii) of Theorem 1.3.2. Fix π as in 1.2.0, and let $\chi \neq 1, \overline{\omega}_\pi$ be an even Dirichlet character of $\mathbf{Q}$ such that $L(\pi \otimes \overline{\chi}, 1) \neq 0$ (see 2.2.0). In view of 3.5.1 and 3.5.4, in order to establish 1.3.2(ii) it suffices to prove the following.

6.0.0 Proposition. *There exist* $\phi : \pm N_{\hat{\mathbf{Z}}} D_{\hat{\mathbf{Z}}} \backslash G_{\hat{\mathbf{Z}}} \longrightarrow \mathbf{Q}$ *of degree zero (3.1.0),* $\omega \in V_\pi$*, and* $\eta \in \mathrm{Eis}(1, \chi)$*, such that, for some* $K \subseteq G_{\hat{\mathbf{Z}}}$ *fixing* ϕ*,* ω *and* η *under right translation, the integral*

$$\int_{M_K(\mathbf{C})} \mathcal{E}_\phi(z, g; 1)\, \bar{\eta} \wedge \omega \tag{6.0.1}$$

is nonzero.

6.1.0. Let us first prove 6.0.0 *without* the requirement that $\deg(\phi)$ be zero. Choose ω and η in such a way that their Whittaker functions factorise:

$$W_\omega = \bigotimes_p{}' \, W_{\omega,p}$$
$$W_\eta = \bigotimes_p{}' \, W_{\eta,p},$$

and such that for a suitable finite set S of primes, one has:

- for $p \notin S$, $W_{\omega,p}$ and $W_{\eta,p}$ are right $G_{\hat{\mathbf{Z}}}$-invariant; and
- for $p \in S$, we have (cf. 4.5.4 above)

$$I\left(W_{\omega,p}\left(\begin{pmatrix} \cdot & 0 \\ 0 & 1 \end{pmatrix}\right), W_{\eta,p}\left(\begin{pmatrix} \cdot & 0 \\ 0 & 1 \end{pmatrix}\right); 1\right) = 1.$$

For $p \in S$, denote by K_p an open subgroup of $G_{\mathbf{Z}_p}$ such that $W_{\omega,p}$ and $W_{\eta,p}$ are right K_p-invariant. Choose ϕ to be the characteristic function of

$$N_{\hat{\mathbf{Z}}}\Big(\prod_{p \notin S} G_{\mathbf{Z}_p} \times \prod_{p \in S} K_p \cdot D_{\mathbf{Z}_p}\Big).$$

Then in view of the choice of χ (from 6.0 above), it follows from Propositions 5.1.0 and 4.5.2 that the integral (6.0.1) does not vanish, as claimed.

6.1.1. Finally, given any ϕ (not necessarily of degree zero), ω and η such that the integral (6.0.1) is nonzero, denote by ψ the function on $G_{\hat{\mathbf{Z}}}$ such that $\hat{\psi}$ (in the notation of 3.2.0) is the $(1,1)$-component of $\hat{\phi}$ in the decomposition 3.2.1. Then $\deg(\phi) = \deg(\psi)$, and the integral (6.0.1), with ϕ replaced by ψ, vanishes, since the product of the central characters of the three representations involved is $1 \cdot \overline{\chi} \cdot \omega_\pi \neq 1$. Thus replacing ϕ by $\phi - \psi$ leaves (6.0.1) unchanged, while $\deg(\phi - \psi) = 0$.

7. Integrality.

7.0. In this section we prove Theorem 1.1.2(iii). The main ingredient is the analysis of the reduction mod p of modular units using supersingular points.

7.0.0. Throughout the section, we fix a prime p and an integer $m \geq 3$ which is prime to p. Define

$$G^{(p)} = \prod_{l \neq p}{}' G_l.$$

We denote by $\mathcal{H}(m,p)$ the Hecke algebra of compactly-supported functions on $G^{(p)}$ with values in $\overline{\mathbf{Q}}$ which are biinvariant under $G^{(p)} \cap K_m$.

7.0.1. We need to recall the structure of the reduction mod p of certain modular curves. If $M_{K/\mathbf{Z}}$ is a model for M_K over $\operatorname{Spec}\mathbf{Z}$, we will denote $M_{K/\mathbf{Z}} \otimes_{\mathbf{Z}} \mathbf{F}_p$ by $M_{K/\mathbf{F}_p}$. In the cases

$$K = K_m \cap K_0(p) \quad \text{or} \quad K(mp^k),$$

we take $M_{K/\mathbf{Z}}$ to be the regular model described in [**DR**], V.1.18, V.4 and [**KM**], Chapter 13.

7.0.2. There is a natural map $M_{m/\mathbf{Z}} \longrightarrow \operatorname{Spec}\mathbf{Z}[\boldsymbol{\mu}_m]$ induced by the Weil pairing, whose fibres in characteristic p are smooth and geometrically connected. We write $\Sigma_{m,p}$ for the set of supersingular points of $M_{m/\mathbf{Z}} \otimes_{\mathbf{Z}} \overline{\mathbf{F}}_p$, and $S_{m,p}$ for the set of primes of $\mathbf{Q}(\boldsymbol{\mu}_m)$ lying over p (which is thus the same as the set of connected components of $M_{m/\mathbf{Z}} \otimes \mathbf{F}_p$). If A is any abelian group, let $A[\Sigma_{m,p}]$, $A[S_{m,p}]$ be the groups of A-valued functions on $\Sigma_{m,p}$, $S_{m,p}$. There is a natural inclusion $A[S_{m,p}] \subset A[\Sigma_{m,p}]$, and also a surjection $\gamma : A[\Sigma_{m,p}] \longrightarrow A[S_{m,p}]$ given by summing along the fibres; write $A[\Sigma_{m,p}]^0 \stackrel{\text{def}}{=} \ker\gamma$.

7.0.3. Consider the modular curve $\overline{M}_K$ for $K = K_0(p) \cap K_m$. There is a short exact sequence

$$0 \longrightarrow T \longrightarrow \operatorname{Pic}^0(\overline{M}_{K/\mathbf{F}_p}) \longrightarrow \operatorname{Pic}^0(\overline{M}_{m/\mathbf{F}_p})^2 \longrightarrow 0 \tag{7.0.4}$$

in which T is a torus over $\mathbf{F}_p$ whose character group is $\mathbf{Z}[\Sigma_{m,p}]^0$ (cf. [**DR**] V.1.18 and I.3.7).

7.1.0. If $(\pi, V_\pi) = \bigotimes_l{}'(\pi_l, V_{\pi,l})$ is an irreducible representation of G_f occurring in $\Omega^1(\overline{M}) \otimes \overline{\mathbf{Q}}$, let

$$V_\pi^{(p)} = \bigotimes_{l \neq p}{}' V_{\pi,l}\,;$$

it is an irreducible admissible representation of $G^{(p)}$.

We denote by $sp(1)$ the (special) representation of G_p on the space of locally constant functions on $\mathbf{P}^1(\mathbf{Q}_p)$ modulo constants.

7.1.1. Theorem. *There is an isomorphism of $\mathcal{H}(m,p)$-modules*

$$\overline{\mathbf{Q}}[\Sigma_{m,p}]/\overline{\mathbf{Q}}[S_{m,p}] \xrightarrow{\sim} \bigoplus_\pi \big(V_\pi^{(p)}\big)^{K_m},$$

where the sum is taken over all irreducible $(\pi, V_\pi) \subset \Omega^1(\overline{M}) \otimes \overline{\mathbf{Q}}$ for which π_p is isomorphic to $sp(1)$.

This result is well known, and in principle is due to Eichler. The result in the case $m = 1$ has been used by Mestre and Oesterlé to find Weil curves with prime conductor (see [**M**]). Since there does no seem to be an easily accessible proof of the general case in the literature, we sketch one here. First note that the right-hand space is isomorphic, as an $\mathcal{H}(m,p)$-module, to the sum of spaces $V_\pi^{K_m \cap K_0(p)}$, taken over all (π, V_π) for which p divides the conductor of π. This in turn is precisely the space of all holomorphic cusp forms of weight two on $K(m) \cap K_0(p)$ which are "p-new". Now the reduction mod p of the corresponding part of the Jacobian of $\overline{M}_{K_m \cap K_0(p)}$ is the torus T. Since $\operatorname{End} T \xrightarrow{\sim} \operatorname{End}(\operatorname{Hom}(\mathbf{Z}[\Sigma_{m,p}]^0, \mathbf{Z}))$ by (7.0.4), the result follows.

7.2.0. In what follows, the reductions mod p of modular curves and their components will always be assumed to be endowed with the reduced subscheme structure.

7.2.1. Consider $M_{n/\mathbf{Z}}$ for $n = mp^k$, $k \geq 1$. We recall the structure of $M_{n/\mathbf{F}_p}$ and the covering $M_{n/\mathbf{F}_p} \longrightarrow M_{m/\mathbf{F}_p}$.

7.2.2. The irreducible components of $M_{n/\mathbf{F}_p}$ are indexed by $S_{m,p} \times \mathbf{P}^1(\mathbf{Z}/p^k\mathbf{Z})$. The action of $G_{\mathbf{Z}/n\mathbf{Z}} = G_{\mathbf{Z}/m\mathbf{Z}} \times G_{\mathbf{Z}/p^k\mathbf{Z}}$ on them is the product of the action of $G_{\mathbf{Z}/m\mathbf{Z}}$ on $S_{m,p}$ given by determinant followed by the Artin map, and the usual action of $G_{\mathbf{Z}/p^k\mathbf{Z}}$ on $\mathbf{P}^1(\mathbf{Z}/p^k\mathbf{Z})$ by linear fractional transformations.
7.2.3. If $\mathcal{D}$ is a $G_{\mathbf{Z}/m\mathbf{Z}}$-orbit of an irreducible component of $M_{n/\mathbf{F}_p}$, then $\mathcal{D}$ is a smooth curve over $\mathbf{F}_p$ (but not in general irreducible). $\mathcal{H}(m,p)$ acts on D by correspondences. The covering $\mathcal{D} \longrightarrow M_{m/\mathbf{F}_p}$ is finite and flat, everywhere of degree $p^k\phi(p^k)$, and is totally ramified over the supersingular points. Thus we can identify the supersingular points of $\mathcal{D}$ with $\Sigma_{m,p}$, and this identification is compatible with the action of $\mathcal{H}(m,p)$. By the previous paragraph, the irreducible components of $\mathcal{D}$ can be identified with $S_{m,p}$.
7.2.4. Let $u \in \mathcal{O}^*(M_n)$, and let $\mathcal{C}$ be an irreducible component of $M_{n/\mathbf{F}_p}$. Use the superscript ^{h} to denote the complement of the supersingular points. u gives rise to a unit (modulo roots of unity) on $\mathcal{C}^h$ as follows. Let $\mathrm{ord}_{\mathcal{C}}$ denote the normalised valuation on the function field $\mathbf{Q}(M_n)$ at the generic point of $\mathcal{C}$, and write $e = \mathrm{ord}_{\mathcal{C}}(p) = p^k\phi(p^k)$. Then if $k(\mathcal{C})$ is the function field of $\mathcal{C}$,

$$\begin{aligned} u_{\mathcal{C}} &= \partial_{\mathcal{C}}\{u,p\} \\ &= u^e/p^{\mathrm{ord}_{\mathcal{C}}(u)} \end{aligned}$$

may be viewed as belonging to $k(\mathcal{C})^*$. Its only poles or zeros can be at singular points of the fibre $M_{n/\mathbf{F}_p}$, so in fact $u_{\mathcal{C}} \in \mathcal{O}^*(\mathcal{C}^h)$.

7.2.5. Theorem. *$u_{\mathcal{C}}$ has the same order of pole or zero at each supersingular point of $\mathcal{C}$.*

Proof. It is convenient to consider, rather than $\mathcal{C}$, the orbit $\mathcal{D} = \coprod \mathcal{C}_i$ of $\mathcal{C}$ under $G_{\mathbf{Z}/m\mathbf{Z}}$; thus $\mathcal{D}^h = \coprod \mathcal{C}_i^h$. Then define $u_{\mathcal{D}} \in \mathcal{O}^*(\mathcal{D}^h)$ to be the unit whose restriction to $\mathcal{C}_i^h$ is $u_{\mathcal{C}_i}$. Consider the composite homomorphism:

$$\mathcal{O}^*(M_n) \longrightarrow \mathcal{O}^*(\mathcal{D}^h) \longrightarrow \mathbf{Z}[\Sigma_{m,p}]$$

where the first map is given by $u \mapsto u_{\mathcal{D}}$, and the second is the divisor map. The assertion of the theorem is that the image of this composite lies in $\mathbf{Z}[S_{m,p}]$. But by composing with the quotient map and extending scalars we obtain a map

$$(\mathcal{O}^*(M_n)/\mathbf{Q}(\boldsymbol{\mu}_n)^*) \otimes_{\mathbf{Z}} \overline{\mathbf{Q}} \longrightarrow \overline{\mathbf{Q}}[\Sigma_{m,p}]/\overline{\mathbf{Q}}[S_{m,p}]$$

which is a homomorphism of $\mathcal{H}(m,p)$-modules. The first space occurs in the space of Eisenstein series (by 3.5.2), and the second in the space of cusp forms (by Theorem 7.1.1)—so by the same principle as in the proof of the Manin-Drinfeld theorem (3.4.0 above) this map is zero.

7.3.0. Fix a positive integer n, which we assume is the product of two coprime integers, each ≥ 3. Recall that $M_{n/\mathbf{Z}}$ is the complement in $\overline{M}_{n/\mathbf{Z}}$ of the cuspidal subscheme $M^\infty_{n/\mathbf{Z}}$, itself a disjoint union of copies of Spec $\mathbf{Z}[\boldsymbol{\mu}_n]$. Denote the set of connected components of $M^\infty_{n/\mathbf{Z}}$ simply by "cusps". We have boundary maps in absolute cohomology/homology (see [**Be1**], 2.2.3 and Chapter V)

$$\partial_\infty : H^2_{\mathcal{A}}(M_n, \mathbf{Q}(2)) \longrightarrow H^1_{\mathcal{A}}(M_n^\infty, \mathbf{Q}(1)) = \coprod_{\text{cusps}} \mathbf{Q}(\boldsymbol{\mu}_n)^* \otimes_{\mathbf{Z}} \mathbf{Q}$$

$$\begin{aligned} \partial_p : H^2_{\mathcal{A}}(M_n, \mathbf{Q}(2)) \longrightarrow H'^{-1}_{\mathcal{A}}(M_{n/\mathbf{F}_p}, \mathbf{Q}(0)) &\subset H^1_{\mathcal{A}}(M^h_{n/\mathbf{F}_p}, \mathbf{Q}(1)) \\ &= \mathcal{O}^*(M^h_{n/\mathbf{F}_p}) \otimes_{\mathbf{Z}} \mathbf{Q}. \end{aligned}$$

From the localisation sequence, we have (cf. 1.1.1 above)

$$\begin{aligned} \mathcal{Q}_n &\stackrel{\text{def}}{=} \{\mathcal{O}^*(M_n), \mathcal{O}^*(M_n)\} \cap H^2_{\mathcal{A}}(M_n, \mathbf{Q}(2)) \\ &= \{\mathcal{O}^*(M_n), \mathcal{O}^*(M_n)\} \cap \ker \partial_\infty. \end{aligned}$$

7.3.1. Theorem. $\mathcal{Q}_n \subseteq H^2_{\mathcal{A}}(\overline{M}_{n/\mathbf{Z}}, \mathbf{Q}(2))$.

7.3.2. Remark. This is sufficient to prove Theorem 1.1.2(iii). Indeed, from the definition of $\mathcal{P}_K$ and the remarks in 1.2.9 above, if $\xi \in \mathcal{P}_K$ then for some $n \geq 3$ with $K_n \subseteq K$, there exists $\alpha \in \mathcal{Q}_n$ satisfying $\xi = \theta_{K_n/K*}\alpha$. Now $\theta_{K_n/K*}$ maps $H^\bullet_{\mathcal{A}}(\overline{M}_{n/\mathbf{Z}}, *)$ into $H^\bullet_{\mathcal{A}}(\overline{M}_{K/\mathbf{Z}}, *)$, since the graph of $\theta_{K_n/K}$ extends to a correspondence on the regular models over $\mathbf{Z}$.

Proof. We have to show that $\partial_p(\mathcal{Q}_n) = 0$ for every p. If $p \nmid n$ then this follows at once from the localisation sequence, since in this case $H'^{-1}_{\mathcal{A}}(M_{n/\mathbf{F}_p}, \mathbf{Q}(0)) = \mathcal{O}^*(M_{n/\mathbf{F}_p}) \otimes_{\mathbf{Z}} \mathbf{Q}$. So let us restrict to the case $n = mp^k$, $k \geq 1$. We have the following diagram with exact rows, in which the solid arrows commute up to sign:

$$\begin{array}{ccccccc}
0 & \longrightarrow & \mathcal{Q}_n & \longrightarrow & \{\mathcal{O}^*(M_n), \mathcal{O}^*(M_n)\} & \xrightarrow{\partial_\infty} & \coprod_{\text{cusps}} \mathbf{Q}(\boldsymbol{\mu}_n)^* \otimes_{\mathbf{Z}} \mathbf{Q} \\
 & & \alpha_p & & \downarrow \partial_p & & \downarrow \theta_p \\
0 & \longrightarrow & \mathcal{O}^*(\overline{M}^h_{n/\mathbf{F}_p}) \otimes_{\mathbf{Z}} \mathbf{Q} & \longrightarrow & \mathcal{O}^*(M^h_{n/\mathbf{F}_p}) \otimes_{\mathbf{Z}} \mathbf{Q} & \xrightarrow{\epsilon_p} & \coprod_{\text{cusps}} \coprod_{S_{m,p}} \mathbf{Q}
\end{array}$$

Here the maps ∂_p, ∂_∞ are those in the localisation sequence. The map

$$\epsilon_p : \mathcal{O}^*(M^h_{n/\mathbf{F}_p}) \otimes_{\mathbf{Z}} \mathbf{Q} \longrightarrow \coprod_{M^\infty_{n/\mathbf{F}_p}} \mathbf{Q}$$

is the infinite part of the divisor map (noting that the points of $M^\infty_{n/\mathbf{F}_p}$ are just (cusps) $\times S_{m,p}$). Finally θ_p is the "content" map

$$\mathbf{Q}(\boldsymbol{\mu}_n)^* \xrightarrow{(\mathrm{ord}_\wp)_\wp} \coprod_{\wp \in S_{m,p}} \mathbf{Q}.$$

We therefore obtain a map α_p as indicated making the left-hand square commutative. It is enough to show that $\alpha_p(\mathcal{Q}_n) = 0$. But for any component $\mathcal{C}$, we have in $\mathcal{O}^*(\mathcal{C}^h) \otimes_{\mathbf{Z}} \mathbf{Q}$

$$\partial_{\mathcal{C}}\{f, g\}^e = \frac{f_{\mathcal{C}}^{\mathrm{ord}_{\mathcal{C}}\, g}}{g_{\mathcal{C}}^{\mathrm{ord}_{\mathcal{C}}\, f}}.$$

Applying 7.2.5 above we see that if $\xi \in \mathcal{Q}_n$ then $\alpha_p(\xi)$, when restricted to any irreducible component $\bar{\mathcal{C}}^h$ of $\overline{M}^h_{n/\mathbf{F}_p}$, has the same order at each supersingular point. But every such $\bar{\mathcal{C}}^h$ is the complement, in the complete curve $\bar{\mathcal{C}}$, of the set of supersingular points, and the total degree of the divisor of $\alpha_p(\xi)$ on $\bar{\mathcal{C}}$ is of course zero. Therefore $\alpha_p(\xi) = 0$ as required.

7.4. Remark. In [**Be1, Be2**], Beilinson suggests another proof of 7.3.1 above. This relies however on the assumption that

$$\mathcal{O}^*(M_n) \otimes_{\mathbf{Z}} \mathbf{Q} = \mathbf{Q}(\boldsymbol{\mu}_n)^* \cdot \mathcal{O}^*(M_{n/\mathbf{Z}}) \otimes_{\mathbf{Z}} \mathbf{Q}$$

(which would imply an "integral" version of the Manin-Drinfeld theorem, see [**Be2**], 5.2.4). This does not hold in general; for example, if p divides n, the modular unit $\Delta(pz)/\Delta(z)$ belongs to the space on the left, but not to that on the right. The difficulty arises because the Eisenstein representations $\mathrm{Eis}(\chi_1, \chi_2)$ are highly reducible, and can be intertwined with many different irreducible representations of G_f. The argument of 5.5 of [**Be1**] produces certain maps

$$\mathrm{Eis}(\chi_1, \chi_2) \longrightarrow U_p$$

where U_p is a G_f-module on which, for every $l \neq p$, G_l acts via a sum of abelian characters. Using remark 3.3(ii) above, one can then obtain the relation (compare 3.5.4 above)

$$d \log \mathcal{O}^*(M_{\mathbf{Z}}) \otimes_{\mathbf{Z}} \overline{\mathbf{Q}} \supseteq \bigoplus_{\substack{\chi \neq 1 \\ \chi \text{ even}}} \mathrm{Eis}(1, \chi).$$

This is enough to prove the weaker version of 1.1.2(iii) described in 1.1.3(iii). One can also obtain a precise description of $d \log \mathcal{O}^*(M_{\mathbf{Z}}) \otimes_{\mathbf{Z}} \overline{\mathbf{Q}}$ analogous to 3.5.4, but we shall not go into this here.

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