1 Let $f\colon \mathbb{R}_{>0} \to \mathbb{C}$ be a continuous function such that for every $N > 0$, $y^N f(y) \to 0$ as $y \to \infty$. What is the *Mellin transform* $M(f,s)$ of f? Find the Mellin transform of the function $1/(e^y - 1)$.

Suppose there is an increasing sequence of real numbers $\sigma_1 < \sigma_2 < \dots$ with $\sigma_j \to \infty$, and nonzero constants $c_j \in \mathbb{C}$ such that for every integer $k \geq 0$,

$$f(y) = c_1 y^{\sigma_1} + \cdots + c_k y^{\sigma_k} + y^{\sigma_{k+1}} g_k(y)$$

where g_k is continuous on $[0,\infty)$. Show that $M(f,s)$ has a meromorphic continuation to $\mathbb{C}$, holomorphic apart from a simple pole at $s = -\sigma_j$ with residue c_j for each $j \geq 1$.

$$M(f,s) = \int_0^\infty f(y)\, y^s \frac{dy}{y}$$ (assuming that for some $m \in \mathbb{Z}$, $y^m f(y) \to 0$ as $y \to 0$).

$$M\left(\frac{1}{e^y-1}, s\right) = \int_0^\infty \frac{1}{e^y-1}\, y^s \frac{dy}{y}$$

$$= \int_0^\infty \sum_{n\geq 1} e^{-ny}\, y^s \frac{dy}{y} = \sum_{n \geq 1} \int_0^\infty e^{-y} n^{-s} y^s \frac{dy}{y} = \Gamma(s)\zeta(s)$$

(assuming $\mathrm{Re}(s) > 1$).

Write $$M(f,s) = \int_1^\infty f(y)\, y^s \frac{dy}{y} + \int_0^1 f(y)\, y^s \frac{dy}{y}.$$

By hypothesis that f is rapidly decreasing at ∞, first integral converges and defines an entire function of s. For every k,

$$\int_0^1 f(y)\, y^s \frac{dy}{y} = \int_0^1 \sum_{j=1}^k c_j\, y^{s+\sigma_j-1}\, dy + \int_0^1 y^{s+\sigma_k-1} g_k(y)\, dy$$

$$= \sum_{j=1}^k \frac{c_j}{s+\sigma_j} + \int_0^1 y^{s+\sigma_k-1} g_k(y)\, dy \quad \text{if } \mathrm{Re}(s) + \sigma_1 > 0.$$

As g_k is cts at $s=0$, the integral is analytic for $\mathrm{Re}(s) + \sigma_{k+1} - 1 \geq 0$ (in fact if $\mathrm{Re}(s) + \sigma_{k+1} > 0$)

so this formula defines an analytic ctn. of $M(f,s)$ to

$$\{ s \in \mathbb{C} \mid \mathrm{Re}(s) \geq 1 - \sigma_{k+1} \}$$

with simple poles at $s = -\sigma_j$, residues c_j.

As this holds $\forall k$ and $\sigma_k \to \infty$, get result.

2 Let

$$G_2(z) = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty}{}' \frac{1}{(mz+n)^2}$$

where Σ' denotes that the term $(m,n) = (0,0)$ is omitted.

a) Show that $G_2(z) = \frac{\pi^2}{3} E_2(z)$, where

$$E_2(z) = 1 - 24 \sum_{n=1}^{\infty} \sigma_1(n) q^n \qquad (q = e^{2\pi i z}).$$

b) Let $\theta = q\frac{q}{dq} = \frac{1}{2\pi i}\frac{d}{dz}$. Assuming the relation $E_2(-1/z) = z^2 E_2(z) + 12z/2\pi i$, show that if $f \in M_k$, the space of modular forms of weight k, then

$$g = (\theta - \frac{k}{12} E_2) f \in M_{k+2}$$

and that g is a cusp form if and only if f is.

c) Let $\Delta = q \prod_{n \geq 1} (1-q^n)^{24} = \sum_{n \geq 1} \tau(n) q^n$ be the normalised cusp form of weight 12. Use the fact that there is no nonzero cusp form of weight 14 to show that

$$(1-n)\tau(n) = 24 \sum_{r=1}^{n-1} \sigma_1(r)\tau(n-r).$$

(a) $\displaystyle\sum_{n=-\infty}^{\infty} \frac{1}{(z+n)^k} = \frac{(2\pi i)^k}{(k-1)!} \sum_{d=1}^{\infty} d^{k-1} e^{2\pi i d z}$ for $k \geq 2$ even

So $\displaystyle G_2(z) = 2\sum_{n=1}^{\infty} n^{-2} + 2\sum_{m=1}^{\infty}\sum_{n=-\infty}^{\infty} \frac{1}{(mz+d)^2}$

$\displaystyle = \pi^2/3 + 2\sum_{m\geq 1} (-4\pi^2) \sum_{d \geq 1} d\, q^{md}$

$\displaystyle = \pi^2/3 \left(1 - 24 \sum_{n \geq 1} \sigma_1(n) q^n\right) = \frac{\pi^2}{3} E_2(z).$

b) Since $f(z+1) = f(z)$ and $E_2(z+1) = E_2(z)$, $g(z+1) = g(z)$.

Now $\frac{d}{dz} f(\frac{-1}{z}) = \frac{1}{z^2} f'(-\frac{1}{z}) = \frac{d}{dz}(z^k f(z)) = kz^{k-1} f(z) + z^k f'(z)$

$\therefore f'(-1/z) = z^{k+2} f'(z) + k z^{k+1} f(z)$.

$\therefore g(-1/z) = \frac{1}{2\pi i} f'(-1/z) - \frac{k}{12} E_2(-1/z) f(-1/z)$

$= \frac{1}{2\pi i} z^{k+2} f'(z) + \frac{k}{2\pi i} z^{k+1} f(z) - \frac{k}{12}\left(z^2 E_2(z) + \frac{12 z}{2\pi i}\right) z^k f(z)$

$= z^{k+2} g(z)$.

(3)

Evidently g is holomorphic (as Fourier series for E_2 is), and its q-expansion is

$$q\frac{d}{dq}\sum_{n\geq 0} a_n(f)\,q^n - \frac{k}{12}\Big(1-24\sum_{n\geq 1}\sigma_1(n)\,q^n\Big)\sum_{n\geq 0} a_n(f)\,q^n$$

$$= -\frac{k}{12}a_0(f) + \sum_{n\geq 1}\Big[\Big(n-\frac{k}{12}\Big)a_n(f) + 2k\sum_{r=1}^{n}\sigma_1(r)\,a_{n-r}(f)\Big]q^n$$

So $a_n(g) = 0 \quad \forall n<0$, and $\quad a_0(g) = 0 \Leftrightarrow a_0(f) = 0$

$(\Rightarrow g \in M_{k+2}) \qquad\qquad (\text{so } g \in S_{k+2} \Leftrightarrow f \in S_{k+2})$.

c) As $S_{14} = 0$, if $f = \Delta$ then $g = 0$. So by (b),

$$(n-1)\,\tau(n) + 24\sum_{r=1}^{n}\sigma_1(r)\,\tau(n-r) = 0.$$