Dedekind ζ-function, applications to arithmetic.

Quick review of basic facts about (quadratic) number fields.

K = no. field of degree $n \geq 1$

$\mathcal{O}_K$ ring of integers ($\simeq \mathbb{Z}^n$ as $\mathbb{Z}$-module)

norm $N_{K/\mathbb{Q}} : K \to \mathbb{Q}$

$\mathcal{O}_K \supset I$ non-zero ideal

$NI \overset{\text{def}}{=} (\mathcal{O}_K : I) < \infty$; $N(IJ) = NI . NJ$

$I = (x) \Rightarrow NI = |N_{K/\mathbb{Q}}(x)|$.

Every ideal is ! product of prime ideals
$$I = \prod \mathfrak{p}^{m_\mathfrak{p}}$$

p rational prime $\Rightarrow (p) = \prod \mathfrak{P}_i^{e_i}$

and $N((p)) = p^n \Rightarrow \prod (N\mathfrak{P}_i)^{e_i} = p^n$.

$Cl(K) = \{\text{non-0 ideals} \subset \mathcal{O}_K\}/\sim$

where $I \sim J \Leftrightarrow \exists x \in K^\times$ st. $J = xI$

– finite group under multiplication of ideals.

$\exists n$ distinct embeddings $K \overset{\sigma}{\hookrightarrow} \mathbb{C}$

$n = r_1 + 2r_2$ where

$r_1 = \#\{\text{embeddings } K \hookrightarrow \mathbb{R}\}$

$r_2 = \#\{\text{pairs } \sigma, \bar\sigma \neq \sigma : K \hookrightarrow \mathbb{C}\}$

$$\left(N_{K/\mathbb{Q}}(x) = \prod_{\sigma : K \hookrightarrow \mathbb{C}} \sigma(x). \right)$$

$\mathcal{O}_K^\times \cong \underbrace{(\text{finite})}_{\text{order} \overset{=}{\text{def}} w_K} \times \mathbb{Z}^{r_1+r_2-1}$ (Dirichlet's unit theorem)

$n=2$: $K = \mathbb{Q}(\sqrt{d})$, $d \in \mathbb{Z} \setminus \{0,1\}$ squarefree

$\mathcal{O}_K = \mathbb{Z}[\theta]$
$= \mathbb{Z} + \mathbb{Z}\theta$, $\quad \theta = \begin{cases} \sqrt{d} & d \not\equiv 1 \bmod 4 \\ \frac{1+\sqrt{d}}{2} & d \equiv 1 \bmod 4 \end{cases}$

$N_{K/\mathbb{Q}}(x) = xx' = a^2 - db^2$, $x = a + b\sqrt{d}$, $x' = a - b\sqrt{d}$.

Discriminant $d_K = \begin{cases} 4d \\ d \end{cases} = \begin{vmatrix} 1 & \theta \\ 1 & \theta' \end{vmatrix}^2$.

$d<0$: $K \subset \mathbb{C}$ $\quad (r_1, r_2) = (0,1)$
$\mathcal{O}_K^\times = \{\pm 1\}, \{\pm 1, \pm i\}, \{\pm 1, \pm\omega, \pm\omega^2\}$ $\quad \omega = e^{2\pi i/3}$.

$d>0$: $K \subset \mathbb{R}$, $(r_1, r_2) = (0,1)$
$\mathcal{O}_K^\times = \{\pm 1\} \times \langle \varepsilon \rangle$, ε = fundamental unit.
(from Pell's equation)

Factorisation of (p) :-

$\boxed{p \text{ odd:}}$ $\begin{cases} p \mid d \Rightarrow & (p) = \mathfrak{P}^2, \ \mathfrak{P} = (p, \sqrt{d}) \\ \left(\frac{d}{p}\right) = +1 \Rightarrow & (p) = \mathfrak{P}\mathfrak{P}', \ \mathfrak{P} = (p, a - \sqrt{d}) \text{ if } a^2 \equiv d \pmod p \\ \left(\frac{d}{p}\right) = -1 \Rightarrow & (p) = \mathfrak{P} \text{ prime.} \end{cases}$

$\boxed{p = 2:}$

- $d \not\equiv 1 \pmod 4$: then $(2) = \mathfrak{P}^2$, $\mathfrak{P} = (2, \sqrt{d})$ d even
  or $(2, 1+\sqrt{d})$ d odd
- $d \equiv 1 \pmod 4$: then $d \equiv 1 \pmod 8 \Rightarrow (2) = \mathfrak{P}\mathfrak{P}'$, $\mathfrak{P} = (2, \frac{1+\sqrt{d}}{2})$
  $5 \pmod 8 \Rightarrow (2) = \mathfrak{P}$ is prime.

This, plus quadratic reciprocity, implies :–

(Best reference: Borevich–Shafarevich "Number Theory", Ch 3, §§8.1–8.2 and Ch 5, § 4.2)

Thm $K = \mathbb{Q}(\sqrt{d})$, discriminant d_K. Then :–

(i) $(p) = \mathfrak{P}^2$ in $\mathcal{O}_K \Leftrightarrow p \mid d_K$ [p ramified]

(ii) $\exists!$ Dirichlet character $\chi_K : (\mathbb{Z}/|d_K|\mathbb{Z})^\times \longrightarrow \{\pm 1\}$

such that for all $p \nmid d_K$, $\chi_K(p) = \begin{cases} +1 & \Leftrightarrow (p) = \mathfrak{P}\mathfrak{P}', \ \mathfrak{P} \neq \mathfrak{P}' \ [p \text{ splits}] \\ -1 & \Leftrightarrow (p) \text{ is prime } [p \text{ is inert}] \end{cases}$

Moreover, $\chi_K(-1) = \operatorname{sgn}(d)$, and χ_K is primitive (does not factor through $(\mathbb{Z}/e\mathbb{Z})^\times$ for any $e < |d_K|$). $\square$

20-2

Def^{n}: K an alg. no. field, ring of integers $\mathcal{O}_K$. The Dedekind ζ-fn. of K is

$$\zeta_K(s) = \sum_{\substack{\text{ideals } I \subset \mathcal{O}_K \\ I \neq (0)}} \frac{1}{(NI)^s} \qquad NI = \text{norm of } I = (\mathcal{O}_K : I) < \infty$$

Prop^{n}: $\zeta_K(s)$ converges for $Re(s) = \sigma > 1$ and has Euler product

$$\zeta_K(s) = \prod_{\substack{\text{prime ideals} \\ \{0\} \neq P \subset \mathcal{O}_K}} \frac{1}{1-(NP)^{-s}}.$$

Proof. Every ideal is a ! product of prime ideals. So at least formally,

$$\zeta_K(s) = \prod_P \left(1 + \frac{1}{NP^s} + \frac{1}{N(P^2)^s} + \cdots\right)$$

$$= \prod_P \left(1 - \frac{1}{NP^s}\right)^{-1} \qquad \text{since } N(IJ) = NI.NJ.$$

Now for given rational prime p, # of $P \mid (p)$ is $\leq n$, and $NP \geq p$.

So $\prod_P$ converges for $Re(s)>1$ by comparison with $\prod_p \left(1 - \frac{1}{p^\sigma}\right)^{-n} = \zeta(\sigma)^n$. $\square$

K a quadratic field, discr $= d_K$, quadratic character χ_K mod $|d_K|$.

Prop^{n}: $\zeta_K(s) = \zeta(s)\, L(\chi_K, s)$

Proof. Euler product $\zeta_K(s) = \prod_P (1-(NP)^{-s})^{-1}$ equals product of 3 terms:–

$$\prod_{p \mid d_K} (1-p^{-s})^{-1} \qquad \text{since } (p) = P^2, \quad NP = p$$

$$\prod_{p \nmid d_K,\ \chi_K(p) = -1} (1-p^{-2s})^{-1} = \prod_{p \nmid d_K,\ \chi_K(p) = -1} (1-p^{-s})^{-1}(1+p^{-s})^{-1} \qquad \text{since } (p) = P, \quad NP = p^2$$

$$\prod_{p \nmid d_K,\ \chi_K(p) = +1} (1-p^{-s})^{-2} \qquad \text{as } (p) = PP', \quad P \neq P', \quad NP = NP' = p.$$

$$\therefore \zeta_K(s)/\zeta(s) = \prod_{\substack{p \nmid d_K \\ \chi_K(p) = -1}} (1+p^{-s})^{-1} \prod_{\substack{p \nmid d_K \\ \chi_K(p) = +1}} (1-p^{-s}) = \prod_{p \nmid d_K} (1-\chi_K(p)p^{-s})^{-1}$$

$$= L(\chi_K, s). \qquad \square$$

21-1

Thm. Let $K = \mathbb{Q}(\sqrt{-d})$ be imaginary quadratic. Then $\zeta_K(s)$ has a meromorphic continuation to $\mathbb{C}$, simple pole at $s=1$. We have

$$\zeta_K(0) = -\frac{h_K}{w_K} \quad \text{and} \quad \operatorname{Res}_{s=1} \zeta_K(s) = \frac{2\pi}{|d_K|^{1/2}} \cdot \frac{h_K}{w_K}.$$

Proof. First write $\zeta_K(s) = \sum_{\mathcal{C} \in Cl(K)} \zeta_K(\mathcal{C}, s)$

where $\zeta_K(\mathcal{C}, s) = \sum_{I \in \mathcal{C},\, I \subset \mathcal{O}_K} (NI)^{-s}$ $\quad Cl(K)$ = ideal class gp of K.

If $I_0 \subset \mathcal{O}_K$ with $I_0^{-1} \in \mathcal{C}$ then $\mathcal{C} = \{ x I_0^{-1} \mid 0 \neq x \in I_0 \}$

and $N(x I_0^{-1}) = (N I_0)^{-1} \cdot |N_{K/\mathbb{Q}}(x)|$, so

$$\zeta_K(\mathcal{C}, s) = (N I_0)^s \cdot \sum_{0 \neq x \in I_0}^{*} |N_{K/\mathbb{Q}}(x)|^{-s}$$

where $\sum^*$ means sum over non-zero elements of I_0 modulo units $\mathcal{O}_K^\times$

This much is true for any K. In our case $K = \mathbb{Q}(\sqrt{-d})$:-

- $\mathcal{O}_K^\times$ is finite, order $= w_K$.
- $N_{K/\mathbb{Q}}(x) = x\bar{x}$.

So $$\zeta_K(\mathcal{C}, s) = (N I_0)^s \cdot \frac{1}{w_K} \sum_{0 \neq x \in I_0} \frac{1}{(x\bar{x})^s} = (N I_0)^s \frac{1}{w_K} G(I_0, s)$$

$G(I_0, s)$ = Epstein ζ-fn of $I_0 \subset \mathbb{C}$ with Euclidean norm. So :-

- $\zeta_K(\mathcal{C}, s)$ has mero. ctn. to $\mathbb{C}$
- $\zeta_K(\mathcal{C}, 0) = -\frac{1}{w_K}$
- $\operatorname{Res}_{s=1} \zeta_K(s) = \frac{N I_0}{w_K} \cdot \operatorname{Res}_{s=1} G(I_0, s) = \frac{N I_0}{w_K} \operatorname{Res}_{s=1} \left[\pi^s \Gamma(s)^{-1} Z(I_0, s) \right]$

$$= \frac{N I_0}{w_K} \cdot \pi\, m(I_0)^{-1} = \pi \frac{m(\mathcal{O}_K)^{-1}}{w_K}$$

If $K = \mathbb{Q}(\sqrt{-d})$ then $\mathcal{O}_K = \mathbb{Z}[\sqrt{-d}]$ $\quad d \not\equiv 3 \pmod 4$, $\quad d_K = -4d$

$\qquad\qquad\qquad\qquad\qquad\quad\;\; \mathbb{Z}\left[\frac{1 + \sqrt{-d}}{2}\right]$ $\quad d \equiv 3 \pmod 4$, $\quad d_K = -d$

$$\Rightarrow m(\mathcal{O}_K) = \left\{ \begin{matrix} \sqrt{d} \\ \frac{1}{2}\sqrt{d} \end{matrix} \right\} = \frac{1}{2}\sqrt{|d_K|} \quad \text{in both cases}$$

$$\therefore \operatorname{Res}_{s=1} \zeta_K(\mathcal{C}, s) = \frac{2\pi}{|d_K|^{1/2}} \cdot \frac{1}{w_K}$$

$\square$

This leads to beautiful formulae for class numbers!

<u>Thm</u>. K imaginary quadratic. Then

$$h_K = -\frac{w_K}{2|d_K|} \sum_{\substack{1 \le n < |d_K| \\ (n, d_K) = 1}} n \chi_K(n)$$

<u>Proof</u> $-\frac{h_K}{w_K} = \zeta_K(0) = \zeta(0) \cdot L(\chi_K, 0) = \left(-\frac{1}{2}\right) \cdot \sum_{\substack{1 \le n \le |d_K| \\ (n, d_K) = 1}} \frac{-n}{|d_K|} \chi_K(n)$ $\square$

Let's simplify this in the case d_K odd (ie. $d_K = -d$, $d \equiv 3 \pmod 4$, $K = \mathbb{Q}(\sqrt{-d})$) and $d > 3$ (so $w_K = 2$)

Then $h = -\frac{1}{d} \sum_{\substack{0 < n < d \\ (n, d) = 1}} \chi_K(n) n = -\frac{1}{d} \sum_{\substack{0 < n < d/2 \\ (n, d) = 1}} \chi_K(n) n + \underbrace{\chi_K(d-n)}_{= -\chi_K(n)} (d-n)$ as $\chi_K(-1) = -1$

$$= \sum_{\substack{0 < n < d/2 \\ (n, d) = 1}} \chi_K(n) \left(1 - \frac{2n}{d}\right) \qquad (1)$$

But also we can write $h_K = -\frac{1}{d} \sum_{\substack{0 < n < d \\ (n, d) = 1,\ n \text{ even}}} \chi_K(n) n + \chi_K(d-n)(d-n)$ since n even $\Leftrightarrow$ $d-n$ odd

$$= \sum_{\substack{0 < n < d \\ (n, d) = 1,\ n \text{ even}}} \chi_K(n) \left(1 - \frac{2n}{d}\right) = \sum_{\substack{0 < m < d/2 \\ (m, d) = 1}} \chi_K(2) \chi_K(m) \left(1 - \frac{4m}{d}\right) \qquad (2)$$

Combining (1) and (2) $\Rightarrow$ $(2 - \chi(2)) h_K = \sum_{\substack{0 < n < d/2 \\ (n, d) = 1}} \chi_K(n)$

<u>Corollary</u> Let $K = \mathbb{Q}(\sqrt{-q})$ where $q \equiv 3 \pmod 4$ is <u>prime</u> > 3. Then:

$$h_K = \begin{cases} \frac{1}{3}(R - N) & \text{if } q \equiv 3 \pmod 8 \\ R - N & \phantom{\text{if } q \equiv\ } 7 \pmod 8 \end{cases}$$

where $\begin{cases} R = \#\text{ of quadratic residues mod } q \in (0, \frac{q}{2}) \\ N = \#\text{ of non-residues.} \end{cases}$

(In particular, h_K is <u>odd</u>, and $R > N$.)

<u>Proof</u>. $q \equiv 3 \ (4) \Rightarrow d_K = -q$.

p odd: then $\chi_K(p) = \left(\frac{-q}{p}\right) = \left(\frac{-1}{p}\right) \cdot \left(\frac{q}{p}\right) = \left(\frac{p}{q}\right)$ by quadratic reciprocity

$\Rightarrow$ $\forall n$ with $(n, q) = 1$, $\chi_K(n) = \left(\frac{n}{q}\right)$ (since n or $n+q$ is a product of odd primes)

Now $\chi_K(2) = \left(\frac{2}{q}\right) = \begin{cases} +1 \\ -1 \end{cases}$ if $q \equiv \begin{cases} \pm 1 \\ \pm 3 \end{cases} \pmod 8$, hence formula. $\square$

Real quadratic fields

Thm. K real quadratic. Then $\zeta_K(s)$ has a meromorphic continuation to $\mathbb{C}$, simple pole at $s=1$, simple zero at $s=0$, and

$$\zeta_K'(0) = -\frac{h_K}{w_K}\log\varepsilon = -\tfrac{1}{2}h_K\log\varepsilon$$

$$\operatorname{Res}_{s=1}\zeta_K(s) = \frac{4}{d_K^{1/2}}\,\frac{h_K\log\varepsilon}{w_K} = \frac{2}{d_K^{1/2}}h_K\log\varepsilon.$$

Here $\varepsilon > 1$ is the fundamental unit of $K \subset \mathbb{R}$, i.e. the smallest unit > 1 (so $\mathcal{O}_K^\times = \{\pm\varepsilon^n \mid n\in\mathbb{Z}\}$).

Proof. As above, we have

$$\zeta_K(s) = \sum_{\mathcal{C}\in Cl(K)}\zeta_K(\mathcal{C},s), \quad \zeta_K(\mathcal{C},s) = (NI_0)^s\sum_{\in I_0}^{*}\frac{1}{|N_{K/\mathbb{Q}}(x)|^s}$$

But now there are 2 problems:–

- $\sum^*_{I_0} \neq (\text{number})\times\sum_{0\neq x\in I_0}$ since $\mathcal{O}_K^\times$ is infinite.
- $N_{K/\mathbb{Q}}(x) = a^2-b^2d$ $(x=a+b\sqrt{d})$ is an indefinite quadratic form.

[For a general # field, $N_{K/\mathbb{Q}}(x)$ will not even be quadratic.]

Device to solve both problems:–

$\boxed{\text{Hecke transform.}}$ (in 2 dimensions).

Lemma $x_1, x_2 \in \mathbb{R}^\times$. Then

$$\frac{1}{|x_1x_2|^s} = \frac{2\Gamma(s)}{\Gamma(s/2)^2}\int_0^\infty \frac{1}{(ux_1^2+u^{-1}x_2^2)^s}\cdot\frac{du}{u}$$

Proof.

$$\Gamma(s)\int_0^\infty\frac{1}{(ux_1^2+u^{-1}x_2^2)^s}\frac{du}{u} = \int_0^\infty\int_0^\infty e^{-t}\cdot\left(\frac{t}{ux_1^2+u^{-1}x_2^2}\right)^s\frac{du}{u}\frac{dt}{t}$$

$$= \int_0^\infty\int_0^\infty e^{-(ux_1^2+u^{-1}x_2^2)t}\,t^s\,\frac{du}{u}\,\frac{dt}{t}$$

Put $(y_1, y_2) = (tu, tu^{-1})$; then $\quad dy_1/y_1 = dt/t + du/u$ $\quad$ Jacobian $= 2$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad dy_2/y_2 = dt/t - du/u$

$$\therefore \int = \int_0^\infty\int_0^\infty \exp(-x_1^2 y_1 - x_2^2 y_2)\,(y_1y_2)^{s/2}\,\frac{1}{2}\,\frac{dy_1}{y_1}\frac{dy_2}{y_2}$$

$$= \frac{1}{2}\prod_{i=1}^{2}\int_0^\infty e^{-x_i^2 y}\,y^{s/2}\,\frac{dy}{y} = \frac{1}{2}|x_1x_2|^{-s}\,\Gamma(s/2)^2. \qquad \square$$

Back to $\zeta_K(\mathcal{C}, s)$

Let $x \in K \setminus 0$. Then Hecke transform $\Rightarrow$ $\quad$ writing $u = \varepsilon^{2t}$, $t \in \mathbb{R}$

$$\frac{1}{|N_{K/\mathbb{Q}}(x)|^s} = \frac{1}{|xx'|^s} = \frac{2\Gamma(s)}{\Gamma(s/2)^2}\int_0^\infty \frac{du/u}{(ux^2 + u'x'^2)^s}$$

$$= \frac{2\Gamma(s)}{\Gamma(s/2)^2}\sum_{n\in\mathbb{Z}}\int_{\varepsilon^{2n-1}}^{\varepsilon^{2n+1}} \frac{du/u}{(ux^2 + u'x'^2)^s}$$

$$= \frac{2\Gamma(s)}{\Gamma(s/2)^2}\sum_{n\in\mathbb{Z}}\int_{\varepsilon^{-1}}^{\varepsilon} \frac{du/u}{(u(\varepsilon^n x)^2 + u'(\varepsilon'^n x')^2)^s}$$

So consider $\displaystyle\sum_{0\neq x\in I_0}\frac{1}{(ux^2 + u^{-1}x'^2)^s} = G(\Lambda_u, s)$

where the lattice $\Lambda_u \subset V_u = \mathbb{R}^2$ is the image of $I_0 \hookrightarrow \mathbb{R}^2$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad x \longmapsto (x, x')$

and the inner product on V_u is

$$((x_1, x_2), (y_1, y_2)) = u x_1 y_1 + u^{-1} x_2 y_2.$$

As $x, y \in I_0$ are associate $\Leftrightarrow$ $y = \pm\,\varepsilon^n x$, $n \in \mathbb{Z}$,

$$G(\Lambda_u, s) = 2\sum_{n\in\mathbb{Z}}\ \sum_{x\in I_0}{}^{*}\ \frac{1}{(u(\varepsilon^n x)^2 + u^{-1}(\varepsilon'^n x')^2)^s}$$

and therefore

$$\boxed{\begin{aligned}\zeta_K(\mathcal{C}, s) &= (NI_0)^s \sum_{x\in I_0}{}^{*}\frac{1}{|N_{K/\mathbb{Q}}x|^s} \\ &= (NI_0)^s\,\frac{\Gamma(s)}{\Gamma(s/2)^2}\int_{\varepsilon^{-1}}^{\varepsilon} G(\Lambda_u, s)\,\frac{du}{u}\,.\end{aligned}}$$

(22-3)

i.e. we have expressed $\zeta_K(\mathcal{C}, s)$ as an <u>integral</u> of a family of Epstein zeta functions. In particular this gives the meromorphic continuation, since $G(\Lambda_u, s)$ has A.C. to $\mathbb{C} - \{1\}$, simple pole at $s = 1$.

$\boxed{s=0}$ $N(I_0)^s \dfrac{\Gamma(s)}{\Gamma(s/2)^2} \sim 1 \cdot \dfrac{1/s}{(2/s)^2} \sim \dfrac{s}{4}$; $G(\Lambda_u, 0) = -1 \quad \forall u$

$\int_{-\varepsilon}^{\varepsilon} du/u = 2 \log \varepsilon$.

so $\zeta_K(\mathcal{C}, s) = \frac{1}{2}(\log \varepsilon) s + O(s^2)$

$\Rightarrow$ formula for $\zeta'_K(0)$.

$\boxed{s=1}$ $V_u = \mathbb{R}^2$ with inner product $((x_1, x_2), (y_1, y_2)) = u x_1 y_1 + u^{-1} x_2 y_2$.

O.N. basis is $\{(u^{-1/2}, 0), (0, u^{1/2})\}$, so the associated measure is $dx_1\, dx_2$ $\forall u$.

If $I_0 = \mathcal{O}_K$ the basis for $\Lambda_u \hookrightarrow \mathbb{R}^2$, $x \mapsto (x, x')$ is $(1, 1), (\theta, \theta')$.

So $m(\Lambda_u) = \left|\det\begin{pmatrix} 1 & 1 \\ \theta & \theta' \end{pmatrix}\right| = d_K^{1/2}$ in this case; for general I_0, $m(\Lambda_u) = N(I_0)\, d_K^{1/2}$.

so $\mathrm{Res}_{s=1}\, G(\Lambda_u, s) = \pi \cdot \mathrm{Res}_{s=1}\, \pi^{-s} \Gamma(s) G(\Lambda_u, s) = \pi\, m(\Lambda_u)^{-1} = \pi N(I_0)^{-1} d_K^{-1/2}$

is independent of u.

$\Rightarrow$ $\mathrm{Res}_{s=1} \zeta_K(\mathcal{C}, s) = N(I_0) \cdot \dfrac{\Gamma(1)}{\Gamma(1/2)^2} \cdot \pi N(I_0)^{-1} d_K^{-1/2} \cdot 2 \log \varepsilon = \dfrac{2}{d_K^{1/2}} \log \varepsilon$ as $\Gamma(\frac{1}{2}) = \sqrt{\pi}$

giving formula for $\mathrm{Res}_{s=1} \zeta_K(s)$. □

[The general case of a number field of degree $n = r_1 + 2r_2$:

Write $r = r_1 + r_2 - 1 = \mathrm{rk}\, \mathcal{O}_K^\times$, and let $\varepsilon_1, \ldots, \varepsilon_r \in \mathcal{O}_K^\times$ such that

$$\mathcal{O}_K^\times = (\text{finite}) \times \langle \varepsilon_1, \ldots, \varepsilon_r \rangle$$

The regulator R_K of K is the abs. value of any $(r \times r)$-minor of the matrix

$$\begin{pmatrix} \log|\sigma_1(\varepsilon_1)| & \cdots & \log|\sigma_{r_1+r_2}(\varepsilon_1)| \\ & & \\ \log|\sigma_1(\varepsilon_r)| & \cdots & \log|\sigma_{r_1+r_2}(\varepsilon_r)| \end{pmatrix}$$

where $\sigma_i : K \hookrightarrow \mathbb{R}$, $\sigma_{r_1+i} = \overline{\sigma_{r_1+r_2+i}} : K \hookrightarrow \mathbb{C}$ are the n complex embeddings of K.
$(i = 1, \ldots r_1)$

If $\mathcal{O}_K = \mathbb{Z}\theta_1 \oplus \cdots \oplus \mathbb{Z}\theta_n$, then the discriminant of K is

$$d_K = \det\left(\sigma_i(\theta_j)\right)^2 .$$

Thm The Dedekind ζ-function $\zeta_K(s)$ has a meromorphic continuation to $\mathbb{C}$, holomorphic except for a simple pole at $s=1$. It has a zero of order $r = r_1 + r_2 - 1$ at $s = 0$, and

$$\lim_{s \to 0} \zeta_K(s)/s^r = -\frac{h_K R_K}{w_K} .$$

Moreover

$$\mathrm{Res}_{s=1}\, \zeta_K(s) = \frac{2^{r_1}(2\pi)^{r_2}}{|d_K|^{1/2}} \cdot \frac{h_K R_K}{w_K}$$

See supplement on course webpage for proof.]

23-1

I. Prove that for any D.C. $\chi \neq \chi_0$ mod N, $L(\chi, 1) \neq 0$.

Case 1 (hard): χ quadratic. Then one knows $\exists$ quadr. field K st. $\chi = \chi_K$.

So $L(\chi, s) = \zeta_K(s)/\zeta(s)$.

But as we've seen, $\forall K$, $\zeta_K(s)$ has simple pole at $s=1$ $\therefore$ $\zeta_K(s)\zeta(s)^{-1}$ is analytic and $\neq 0$.

Case 2 (easy): $\chi^2 \neq 1$, so $\chi \neq \chi^{-1} = \bar{\chi}$. Then $\overline{L(\chi, \bar{s})} = L(\bar{\chi}, s)$ (obvious in range of abs. cvgce, so true $\forall s$ by identity theorem)

So $L(\chi, 1) = 0 \Rightarrow L(\chi^{-1}, 1) = 0$.

Let $m =$ order of χ $(\chi^m = 1)$

For $p \nmid N$ let $\chi(p) = \omega_p$, root of unity of order $m_p \mid m$.

Consider
$$F(s) = \prod_{j=0}^{m-1} L(\chi^j, s) = \prod_{p \nmid m} \prod_{j=0}^{m-1} (1 - \omega_p^j p^{-s})^{-1}$$
$$= \prod_{p \nmid m} \prod_{j=0}^{m_p - 1} (1 - \omega_p^j p^{-s})^{-m/m_p} = \prod_{p \nmid m} (1 - p^{-m_p s})^{-m/m_p}$$
$$= \prod_{p \nmid m} (1 + p^{-m_p s} + p^{-2m_p s} + \dots)^{m/m_p} = \sum_{n \geq 1} a_n n^{-s}$$

where all $a_n \geq 0$. (and ∞ many are > 0)

So $F(s)$ is monotone decreasing and positive for $s \in (1, \infty)$.

$\Rightarrow F(s) \not\to 0$ as $s \to 1+$.

But for $0 < j < m$, $L(\chi^j, s)$ is analytic at $s=1$

$L(\chi, 1) = L(\chi^{-1}, 1) = 0$, and $L(\chi^0, s)$ has simple pole at $s=1$

$\Rightarrow F(s)$ has a zero at $s=1$, contradiction. $\Box$

II. Class no. formulae for real quadratic fields.

Thm K real quadratic, fundamental unit $\varepsilon > 1$. Then

(i) $$h_K = \frac{1}{\log \varepsilon} \sum_{\substack{0 < n < d_K/2 \\ (n, d_K) = 1}} \chi_K(n) \log \sin \frac{\pi n}{d_K}$$

(ii) Let $$\eta = \prod_{\substack{0 < n < d_K/2 \\ (n, d_K) = 1}} \left(\sin \frac{\pi n}{d_K}\right)^{-\chi(n)}.$$ Then $\eta \in \mathcal{O}_K^\times$, and $\eta = \varepsilon^h$.

23-2

Proof. Write $N = d_K$.

(i) $$-\tfrac{1}{2} h_K \log \varepsilon = \zeta_K'(0) = \zeta(0) \cdot L'(\chi_K, 0)$$
$$= -\tfrac{1}{2} \cdot -\sum_{\substack{0<n<N/2 \\ (n,N)=1}} \chi(n) \log |1 - e^{2\pi i n/N}|$$

and $|1 - e^{2\pi i n/N}| = |e^{\pi i n/N} - e^{-\pi i n/N}| = 2 \left| \sin \frac{\pi n}{N} \right|$

$$\therefore -h_K \log \varepsilon = \sum_{\substack{0<n<N/2 \\ (n,N)=1}} \chi(n) \left(\log 2 + \log \left| \sin \frac{\pi n}{N} \right|\right) = \sum_{\substack{0<n<N/2 \\ (n,N)=1}} \chi(n) \log \sin \frac{\pi n}{N}$$

(as $\sin \frac{\pi n}{N} > 0$)

(ii) Exponentiate (i).

$\square$

<u>Remark</u>. There are other examples of a similar nature:- $\square$

1) any totally real ($r_2 = 0$) <u>abelian</u> number field
 - eg. $K = \mathbb{Q}(\cos \frac{2\pi}{p})$; there is an "obvious" subgroup of $\mathcal{O}_K^\times$, generated by -1 and . Its index in $\mathcal{O}_K^\times$ equals h_K.

2) $E/\mathbb{Q}$ an elliptic curve. Suppose $E(\mathbb{Q})$ has rank 1. Then in many (conjecturally all) cases, it is known how to construct an "explicit" (ie. given by a formula) point P of infinite order on E, such that the index $(E(\mathbb{Q}) : \langle P \rangle)$ is given in terms of the order of $Ш_{E/\mathbb{Q}}$, the <u>Tate - Shafarevich group</u> of E. These are called <u>Heegner points</u>. The proof of this result goes via the computation of the derivative of an L-function (the <u>Gross - Zagier formula</u>)

A lot of contemporary number theory is concerned with relating values/derivatives of L-functions with invariants of arithmetic objects (number fields, elliptic curves....). There are many more conjectures than theorems.