(16-2)

§8 Epstein zeta function

Generalisation of Riemann ζ which has many applications

$(V, (\,,\,))$ real inner product space of dim $N < \infty$, $\|v\| = (v,v)^{1/2}$.

$\Lambda \subset V$ a lattice. (convenient initially not to fix a basis for either V or Λ)

$\Lambda' = \{ v \in V \mid \forall x \in \Lambda,\ (x,v) \in \mathbb{Z} \}$ - the dual lattice.

$(\,,\,)$ determines a canonical measure $d\mu_V$ on V; let $(v_1, \dots, v_N)$ be coordinates w.r.t.

ON basis of V, take $d\mu_V = dv_1 dv_2 \dots dv_N$

Notation: $m(\Lambda) = $ covolume of Λ $= \int_{V/\Lambda} d\mu_V = \mathrm{vol}(\mathcal{F}_\Lambda, d\mu_V)$ where $\mathcal{F}_\Lambda = \{ \sum x_i e_i \mid 0 \le x_i < 1 \}$ if $\Lambda = \bigoplus \mathbb{Z} e_i$.

Defn. $G(\Lambda, s) = \sum_{0 \neq x \in \Lambda} \frac{1}{\|x\|^{2s}}$ (warning: $\exists$ many different notations. Sometimes $2s$, s or Ns or ...)

Recall this is abs. cvgt. if $\mathrm{Re}(s) > N/2$.

Thm $Z(\Lambda, s) := \pi^{-s} \Gamma(s) G(\Lambda, s)$ has meromorphic ctn. to $\mathbb{C}$,

holomorphic except for simple poles at $s = \frac{N}{2}, 0$, residues $m(\Lambda)^{-1}, -1$ respectively.

It satisfies functional equation:-

$$Z(\Lambda, s) = m(\Lambda)^{-1} Z(\Lambda', \tfrac{N}{2} - s).$$

In particular, $\zeta(\Lambda, 0) = -1$.

For proof we need a generalisation of the Θ-function & Poisson summation.

Thm (Poisson summation for any lattice). Let $f \in \mathcal{S}(V)$, Fourier transform $\hat{f}(v) = \int e^{-2\pi i (u,v)} f(u)\, d\mu_V(u)$.

Then $\sum_{x \in \Lambda} f(x) = m(\Lambda)^{-1} \sum_{x \in \Lambda'} \hat{f}(x)$.

Proof: reduce to case of $(\mathbb{R}^N, \mathbb{Z}^N)$: let (e_i) be a $\mathbb{Z}$-basis for Λ, induces

$T: \mathbb{R}^N \xrightarrow{\sim} V$, $\underline{x} \mapsto \sum x_i e_i$; then $T: dx_1 \dots dx_N \mapsto \mu(\Lambda)^{-1} d\mu_V$ (consider volume of $\mathcal{F}_\Lambda$)

Let (e_i') be dual basis for Λ', so $(e_i, e_j') = \delta_{ij}$

and $T': \mathbb{R}^N \xrightarrow{\sim} V$, $\underline{y} \mapsto \sum y_i e_i'$. Let $f \in \mathcal{S}(V)$, $g = f \circ T \in \mathcal{S}(\mathbb{R}^N)$.

Then $\hat{g}(\underline{y}) = \int_{\mathbb{R}^N} e^{-2\pi i \underline{x} \cdot \underline{y}} f(T\underline{x})\, d\underline{x} = \int_V e^{-2\pi i (T^{-1}u) \cdot \underline{y}} f(u)\, \mu(\Lambda)^{-1} d\mu_V(u)$

$= \int_V e^{-2\pi i (u, T'\underline{y})} f(u) \mu(\Lambda)^{-1} d\mu_V(u) = \mu(\Lambda)^{-1} \hat{f}(T'\underline{y})$.

So $\sum_{u \in \Lambda} f(u) = \sum_{\underline{x} \in \mathbb{Z}^N} g(\underline{x}) = \sum_{\underline{y} \in \mathbb{Z}^N} \hat{g}(\underline{y}) = \mu(\Lambda)^{-1} \sum_{v \in \Lambda'} \hat{f}(v)$. $\square$

Thm Let $\Theta_\Lambda(t) = \sum_{x\in\Lambda} e^{-\pi\|x\|^2 t}$. Then

$$\Theta_\Lambda(t) = t^{-N/2}\, m(\Lambda)^{-1}\, \Theta_{\Lambda'}(1/t)$$

Proof. Let $f(u) = e^{-\pi(u,u)} = \prod_{i=1}^{N} e^{-\pi u_i^2}$ in ON coordinates $(u_1, \dots, u_N)$ $\Rightarrow \hat{f} = f$.

If $c \in \mathbb{R}_{>0}$, then $(c\Lambda)' = c^{-1}\Lambda'$ and $m(c\Lambda) = c^N m(\Lambda)$.

$\therefore\ \Theta_\Lambda(t) = \sum_{x\in t^{1/2}\Lambda} f(x) = t^{-N/2} m(\Lambda)^{-1} \sum_{x\in t^{-1/2}\Lambda'} f(x) = t^{-N/2} m(\Lambda)^{-1}\, \Theta_{\Lambda'}(1/t)$ $\square$

Epstein zeta. $\Theta_\Lambda(y) \to 1$ as $y \to \infty$ and $\Theta_\Lambda(y) - 1 = \sum_{0\neq x\in\Lambda} e^{-\pi\|x\|^2 y}$ rapidly decreasing at ∞.

$\therefore\ \Theta_\Lambda(y) \sim m(\Lambda)^{-1} y^{-N/2}$ as $y \to 0$, so slowly increasing with $m = N/2$.

and so $M(\Theta_\Lambda(y) - 1, s) = \sum_{0\neq x\in\Lambda} (\pi\|x\|^2)^{-s}\,\Gamma(s) = Z(\Lambda, s)$ for $\mathrm{Re}(s) > \frac{N}{2}$.

$$= \int_1^\infty + \int_0^1 (\Theta_\Lambda(y) - 1)\, y^s \frac{dy}{y}$$

and
$$\int_0^1 = \int_0^1 (\Theta_\Lambda(y) - m(\Lambda)^{-1} y^{-N/2})\, y^s \frac{dy}{y} + \int_0^1 m(\Lambda)^{-1} y^{s-N/2} - y^s \frac{dy}{y}$$
$$= \int_0^1 m(\Lambda)^{-1} (\Theta_{\Lambda'}(1/y) - y^{-N/2})\, y^s \frac{dy}{y} + \frac{m(\Lambda)^{-1}}{s - N/2} - \frac{1}{s}$$
$$= \int_1^\infty m(\Lambda)^{-1} (y^{N/2}\Theta_{\Lambda'}(y) - y^{N/2})\, y^{-s} \frac{dy}{y} + \frac{m(\Lambda)^{-1}}{s - N/2} - \frac{1}{s}$$

ie.
$$Z(\Lambda, s) = \frac{m(\Lambda)^{-1}}{s - \frac{N}{2}} - \frac{1}{s} + \underbrace{\int_1^\infty m(\Lambda)^{-1} (\Theta_{\Lambda'}(y) - 1)\, y^{\frac{N}{2} - s} + (\Theta_\Lambda(y) - 1)\, y^s \frac{dy}{y}}_{\text{entire function of } s \in \mathbb{C}}$$
$$= m(\Lambda)^{-1} Z(\Lambda', s)$$

using FE for $Z(\Lambda, s)$ and remark (i) below. $\square$

Remarks (i) $m(\Lambda') = m(\Lambda)^{-1}$! Either compute using bases for Λ, Λ' as in proof of Poisson summation, or observe
$$\Theta_\Lambda(1) = m(\Lambda)^{-1}\Theta_{\Lambda'}(1) = m(\Lambda)^{-1} m(\Lambda')^{-1}\Theta_\Lambda(1) \quad \text{as } (\Lambda')' = \Lambda.$$

(ii) If $V = \mathbb{R}$ with Euclidean norm, $\Lambda = \mathbb{Z}$ then $G(\Lambda, s) = 2\zeta(2s)$

(iii) Let $g \in O(V) = \{ g \in GL_{\mathbb{R}}(V) \mid \forall u, v \in V,\ (gu, gv) = (u, v) \}$.

Then $G(g\Lambda, s) = G(\Lambda, s)$ by defn.

(17-2) Suppose $V = \mathbb{R}^N$ with Euclidean norm. Then

$\{$lattices $\Lambda \subset \mathbb{R}^N$ with prescribed bases$\} \simeq GL_N(\mathbb{R})$

$$\Lambda = \bigoplus_{i=1}^{N} \mathbb{Z} e^{(i)} \longleftrightarrow \text{matrix} \begin{pmatrix} e_1^{(1)} & \cdots & e_N^{(1)} \\ \vdots & & \\ e_1^{(N)} & \cdots & e_N^{(N)} \end{pmatrix} = (e_i^{(j)})$$

$e^{(i)} = (e_1^{(i)}, \ldots, e_N^{(i)}) \in \mathbb{R}^N$

So $\{$lattices $\Lambda \subset \mathbb{R}^N\} \xleftrightarrow{\sim} GL_N(\mathbb{Z}) \backslash GL_N(\mathbb{R})$

If $g \in O(V)$ the matrix representing $g\Lambda$ is $(e_i^{(j)})\, {}^t g$

so (for fixed s) $G(\Lambda, s)$ is a function on

$$GL_N(\mathbb{Z}) \backslash GL_N(\mathbb{R}) / O(N).$$

Restricting to lattices with fixed $m(\Lambda)$, eg $m(\Lambda) = 1$: then for a suitable ordering of basis, $\det(e_i^{(j)}) = 1$, and so the space of lattices up to orthogonal transformations is

$$SL_N(\mathbb{Z}) \backslash SL_N(\mathbb{R}) / O(N)$$

If $N=2$ this is $SL_2(\mathbb{Z}) \backslash \mathfrak{H}$.

This is some group-theoretic explanation for what follows.

Special case ($N=2$): <u>real analytic Eisenstein series</u>.

<u>Defn.</u> $z \in \mathfrak{h}$, $s \in \mathbb{C}$, $\mathrm{Re}(s) > 1$. $\quad G(z,s) = \sum_{(m,n)\neq(0,0)} \frac{y^s}{|mz+n|^{2s}}$ (abs. cgce.)

<u>Theorem.</u> (i) $G(z,s) = G(\gamma(z), s)$ $\forall \gamma \in SL_2(\mathbb{Z})$

(ii) $\mathcal{E}(z,s) \stackrel{\text{def}}{=} \pi^{-s}\Gamma(s)\, G(z,s)$ has meromorphic ctn. to $s \in \mathbb{C}$, simple poles at $s = 1, 0$ residues $= +1, -1$.

<u>Proof</u> (i) $\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \gamma \in SL_2(\mathbb{Z})$. Then $\frac{\mathrm{Im}\,\gamma(z)^s}{|m\gamma(z)+n|^{2s}} = \frac{(\mathrm{Im}\, z)^s}{|m(az+b)+n(cz+d)|^{2s}} = \frac{y^s}{|m'z+n'|}$

with $(m' \ n') = (m \ n)\gamma$

(ii) Let $\Lambda_z = \frac{1}{y^{1/2}}(\mathbb{Z} + z\mathbb{Z}) \subset \mathbb{C}$, inner product $(x+iy, x'+iy') = xx'+yy'$ ($\Rightarrow \|z\| = |z|$).

Then $G(z,s) = G(\Lambda_z, s)$. Basis for Λ_z is $\left(\frac{1}{y^{1/2}}, \frac{x}{y^{1/2}} + iy^{1/2}\right)$, hence $m(\Lambda_z) = 1$.

and dual basis is $\left(y^{1/2} - \frac{ix}{y^{1/2}}, \frac{i}{y^{1/2}}\right)$, so $\Lambda'_z = i\Lambda_z$. As $i \in O(V)$ for $V = \mathbb{C}$,

$G(\Lambda'_z, s) = G(i\Lambda_z, s) = G(\Lambda_z, s)$. So follows from FE & AC of Epstein zeta. $\square$

(18-1) The (first) Kronecker limit formula.

Recall that $\zeta(s) = \zeta(0) + s\zeta'(0) + O(s^2)$
(sheet 1)
$= -1/2 - \frac{1}{2}\log(2\pi).s + O(s^2)$ as $s \to 0$.

Using functional equation, this is equivalent to knowing the first 2 terms of Laurent expansion at $s=1$, which is

$$\zeta(s) = \frac{1}{s-1} + \gamma + O(s-1) \quad \text{as } s \to 1.$$

We are going to prove the analogous formulae for $G(z,s)$:-

Theorem (KLF). $G'(z,0) = 4\zeta'(0) - \log y|\eta^4|$
$= -\log 4\pi^2 y|\eta^4|$

where

$$\eta = \eta(z) = q^{1/24}\prod_{n=1}^{\infty}(1-q^n). \qquad [q^{1/24} = e^{\pi i z/12}]$$

Corollary. (i) $\eta(\frac{-1}{z}) = (\frac{z}{i})^{1/2}\eta(z)$. (ii) $\Delta(z) = \eta(z)^{24} = q\prod_{n\geq 1}^{\infty}(1-q^n)^{24}$.

Proof of corollary (i) Since $G(\gamma(z),s) = G(z,s)$ $\forall \gamma \in SL_2(\mathbb{Z})$, the function $z \mapsto y|\eta(z)^4|$ is $SL_2(\mathbb{Z})$-invariant. Taking $\gamma = \begin{pmatrix} & -1 \\ 1 & \end{pmatrix}$ gives

$y|\eta(z)^4| = \mathrm{Im}(-1/z)|\eta(-1/z)^4| = \frac{y}{|z|^2}\cdot|\eta(-1/z)^4|$, and therefore
$\left|\sqrt{z/i}\,\eta(z).\,\eta(-1/z)^{-1}\right| = 1$. As any holomorphic function with constant modulus is constant, this implies

$$\eta(-1/z) = c\sqrt{z/i}.\eta(z), \quad c \in \mathbb{C}$$

and putting $z=i$ gives $c=1$.

(ii) Let $F(z) = \eta(z)^{24} = q\prod_{n\geq 1}(1-q^n)^{24}$. Then by (i), $F(-1/z) = z^{12}F(z)$, and clearly $F(z+1) = F(z)$. Since F is holomorphic and vanishing at ∞, this means $F \in S_{12} = \mathbb{C}.\Delta$, and comparing leading coefficient $\Rightarrow F = \Delta$. □

Proof of KLF. We'll compute Fourier series for the periodic function $x \mapsto G(x+iy, 0)$. (Can compute it for any s — involves Bessel functions) see eg. Bump §1.6)

Step 1. For $E_k(z)$, we first computed $\sum_{n\in\mathbb{Z}} \frac{1}{(z+n)^k}$.

For $z \in \mathbb{C}\setminus\mathbb{Z}$, define $H(z,s) = \pi^{-s}\Gamma(s)\sum_{n\in\mathbb{Z}}\frac{1}{|z+n|^{2s}}$ (Re $s > 1/2$)

18-2

Lemma (i) If $z \notin \mathbb{R}$, $H(z,s) = \pi^{1/2-s}|y|^{1-2s}\Gamma(s-\tfrac12) + H_1(z,s)$,

where $H_1(z,s)$ is an entire function of s.

(ii) $$H_1(z,0) = \begin{cases} -2\log|1-e^{2\pi i z}| & y>0 \\ -2\log|1-e^{-2\pi i z}| & y<0. \end{cases}$$

(iii) $x \in \mathbb{R}\setminus\mathbb{Z} \Rightarrow H(x,s)$ has A.C. to $\mathbb{C}\setminus\{1/2\}$, and

$$H(x,0) = -2\log|1-e^{-2\pi i x}|.$$

Proof (i). By Mellin transform, since $|n+z|^2 = (n+x)^2+y^2$,

$$H(z,s) = \int_0^\infty \sum_{n\in\mathbb{Z}} e^{-\pi t|n+z|^2} t^s \frac{dt}{t} = \int_0^\infty e^{-\pi y^2 t}\,\Theta(t;x)\,t^s\frac{dt}{t}$$

where (sheet 1 Q7)

$$\Theta(t,x) \overset{\text{def}}{=} \sum_{n\in\mathbb{Z}} e^{-\pi(n+x)^2 t} = t^{-1/2}\Theta^*(1/t;x) \qquad (1)$$

$$\Theta^*(t;x) \overset{\text{def}}{=} \sum_{n\in\mathbb{Z}} e^{-\pi n^2 t + 2\pi i n x}$$

and $t^{-1/2}\Theta^*(1/t;x) \sim t^{-1/2}$ as $t\to 0$. Taking out this term,

$$H(z,s) = H_1(z,s) + \int_0^\infty e^{-\pi y^2 t} t^{s-1/2}\frac{dt}{t} = H_1(z,s) + \pi^{1/2-s}|y|^{1-2s}\Gamma(s-\tfrac12)$$

where $$H_1(z,s) = \int_0^\infty e^{-\pi y^2 t}\left(\Theta(t;x) - t^{-1/2}\right)t^s\frac{dt}{t}.$$

$$= \int_0^\infty e^{-\pi y^2/t}\left(\Theta^*(t;x)-1\right)t^{1/2-s}\frac{dt}{t} \qquad (2)$$

substituting $t\mapsto \frac1t$ and using (1). As $\Theta^*(t;x)$ is rapidly decreasing at ∞, and $e^{-\pi y^2/t}$ is rapidly decreasing at 0, the integral defines an entire function of s.

(ii) By (2), $$H_1(z,0) = \sum_{0\neq n\in\mathbb{Z}} e^{2\pi i n x}\int_0^\infty e^{-\pi n^2 t - \pi y^2/t}\frac{dt}{t^{1/2}}$$ if abs. cvgt

Exercise (answer to be supplied in 1 week!): If $a>0$, $b\geq 0$ then
$$\int_0^\infty e^{-\pi(ax^2+b/x^2)}\,dx = \frac{1}{2\sqrt a}e^{-2\pi\sqrt{ab}}$$

(19-1) Recall from last time: if $z = x+iy \notin \mathbb{R}$

$$H(z,s) := \pi^{-s}\Gamma(s)\sum_{n\in\mathbb{Z}}\frac{1}{|z+n|} = \pi^{1/2-s}|y|^{1-2s}\Gamma(s-\tfrac{1}{2}) + H_1(z,s)$$

$$H_1(z,s) = \int_0^\infty e^{-\pi y^2/t}\left(\Theta^*(t;x)-1\right)t^{1/2-s}\,\frac{dt}{t} \qquad (2)$$

$$= \int_0^\infty \sum_{0\neq n\in\mathbb{Z}} e^{2\pi i n x}\, e^{-\pi n^2 t - \pi y^2/t}\, t^{-1/2-s}\,dt$$

Easy to check (see Remark [1] at end) that $\sum_{n\neq 0} e^{-\pi n^2 t-\pi y^2/t}\,|t^{-1/2-s}|$ is bounded by an integrable function on $(0,\infty)$, so can exchange $\int$ and $\sum$ by dominated convergence thm,

$s=0$: then $$\int_0^\infty e^{-\pi n^2 t - \pi y^2/t}\, t^{-1/2}\,dt = \frac{1}{|n|}e^{-2\pi|ny|} \qquad (n\neq 0)$$

ugly exercise, putting $t = x^2$.

So $$H_1(z,0) = \sum_{0\neq n\in\mathbb{Z}}\frac{1}{|n|}e^{-2\pi|ny|+2\pi i n x}$$

$$= \sum_{n\geq 1}\frac{1}{n}e^{2\pi i n(x+i|y|)} + \frac{1}{n}e^{2\pi i n(-x+i|y|)}$$

$$= -\log\left(\left(1-e^{2\pi i(x+i|y|)}\right)\left(1-e^{2\pi i(-x+i|y|)}\right)\right)$$

$$= \begin{cases} -2\log\left|1-e^{2\pi i z}\right| & y>0 \\ -2\log\left|1-e^{-2\pi i z}\right| & y<0. \end{cases}$$ — proves (ii)

(iii) $x\in\mathbb{R}\setminus\mathbb{Z} \Rightarrow H(x,s) = \int_0^\infty \Theta(t;x)\,t^s\,\frac{dt}{t} = \int_T^\infty + \int_0^T$

and $$\int_0^T = \int_0^T t^{s-1/2}\,\frac{dt}{t} + \int_0^T\left(\Theta(t;x)-t^{-1/2}\right)t^s\,\frac{dt}{t}$$

$$= \frac{T^{s-1/2}}{s-1/2} + \int_{T^{-1}}^\infty\left(\Theta^*(t;x)-1\right)t^{1/2-s}\,\frac{dt}{t}$$

and $\int_T^\infty, \int_{T^{-1}}^\infty$ are entire. Put $s=0$ and let $T\to\infty$: (see remark [2] to justify this)

$$H(x,0) = \int_0^\infty\left(\Theta^*(t,x)-1\right)t^{1/2}\,\frac{dt}{t} = \sum_{0\neq n\in\mathbb{Z}} e^{2\pi i n x}\int_0^\infty e^{-\pi n^2 t}\,t^{1/2}\,\frac{dt}{t}$$

$$= \sum_{0\neq n\in\mathbb{Z}} e^{2\pi i n x}\,\frac{\Gamma(1/2)}{(\pi n^2)^{1/2}} = \sum_{0\neq n\in\mathbb{Z}}\frac{1}{|n|}e^{2\pi i n x}. \qquad \square$$

Proof of KLF. Let $z = x+iy \in \mathfrak{h}$. Then for $\mathrm{Re}(s)>1$,

$$\mathcal{E}(z,s) = \pi^{-s}\Gamma(s)\sum_{(m,n)\neq(0,0)}\frac{y^s}{|mz+n|^{2s}} = 2\pi^{-s}\Gamma(s)\,y^s\,\zeta(2s) + y^s\sum_{m\neq 0}H(mz,s)$$

$$= 2\pi^{-s}\Gamma(s)\,y^s\,\zeta(2s) + \pi^{1/2-s}\Gamma(s-\tfrac{1}{2})\,y^s\sum_{m\neq 0}|my|^{1-2s} + y^s\sum_{m\neq 0}H_1(mz,s)$$

$$\Rightarrow \quad G(z,s) = 2y^s\zeta(2s) + 2y^{1-s}\pi^{1/2}\frac{\Gamma(s-1/2)}{\Gamma(s)}\zeta(2s-1) + (\pi y)^s\,\Gamma(s)^{-1}\sum_{m\neq 0} H_1(mz,s)$$

(see remark $\boxed{1}$ at end for justification of the next step).

$\boxed{s \to 0}$

$$2y^s\zeta(2s) = 2\zeta(0) + (4\zeta'(0) + 2\zeta(0).\log y)s + O(s^2)$$
$$= -1 + (4\zeta'(0) - \log y)s + O(s^2)$$

$$2y^{1-s}\pi^{1/2}\frac{\Gamma(s-1/2)}{\Gamma(s)}\zeta(2s-1) = \left(2y\pi^{1/2}\Gamma(-1/2)\zeta(-1)\right)s + O(s^2)$$
$$= \frac{1}{3}\pi y s + O(s^2)$$

$$\left[\begin{array}{c}\Gamma(1/2) = \sqrt{\pi} = (-1/2)\Gamma(-1/2)\\ \therefore \Gamma(-1/2) = -2\sqrt{\pi}\\ \zeta(-1) = -1/12\end{array}\right]$$

$$(\pi y)^s\,\Gamma(s)^{-1}\sum_{m\neq 0}H_1(mz,s) = \Big(\sum_{m\neq 0}H_1(mz,0)\Big)s + O(s^2)$$

and $$\sum_{m\neq 0}H_1(mz,0) = \sum_{m>0} -4\log|1-q^m| = -4\log\left|q^{-1/24}\eta(z)\right| = -\frac{\pi y}{3} - 4\log|\eta|. \quad \square$$

We'll use Lemma (iii) to prove the 2nd part of :

Thm. $\chi : (\mathbb{Z}/N\mathbb{Z})^\times \to \mathbb{C}^\times$ non-trivial Dirichlet character. Then :-

(i) $$L(\chi,0) = -\frac{1}{N}\sum_{\substack{0<n<N\\(n,N)=1}} n\,\chi(n)$$

(ii) If $\chi(-1) = 1$ (χ is even) then $L(\chi,0) = 0$ and
$$L'(\chi,0) = -\sum_{\substack{0<n<N/2\\(n,N)=1}}\chi(n)\log\left|1-e^{2\pi i n/N}\right|.$$

Proof (i). Show that if $\psi : \mathbb{Z}/N\mathbb{Z} \to \mathbb{C}$ is periodic, $L(\psi,0) = \sum_{n=1}^{N}\left(\frac{1}{2}-\frac{n}{N}\right)\psi(n)$.

As $\sum_{\substack{n=1\\(n,N)=1}}^{N-1}\chi(n) = 0$, this proves (i). Recall (§3, lecture 4) that

$$\Gamma(s)\,L(\psi,s) = \sum_{n=1}^{N}\psi(n)\,M(f_n(y),s),$$

$$f_n(y) = \frac{e^{(N-n)y}}{e^{Ny}-1} = \frac{1+(N-n)y+O(y^2)}{N(y+\frac{1}{2}Ny^2+O(y^3))} = \frac{1}{Ny} + \frac{1}{N}\left(\tfrac{1}{2}N-n\right) + O(y) \quad \text{in nbd of } 0$$

So $$M(f_n(y)) = \int_1^\infty f_n(y)\,y^s\frac{dy}{y} + \int_0^1 \frac{y^{s-1}}{N} + \left(\frac{1}{2}-\frac{n}{N}\right)y^s + y^{s+1}g(y)\,dy$$

$g : [0,\infty) \to \mathbb{R}$ cts.

$$= (\text{entire}) + \frac{1}{N(s-1)} + \left(\frac{1}{2}-\frac{n}{N}\right)\frac{1}{s} + (\text{holomorphic for } \mathrm{Re}(s) > -1)$$

giving $L(\psi,0) = \sum \psi(n)\left(\frac{1}{2}-\frac{n}{N}\right)$.

(20-1) (ii) If $\chi(-1)=1$, $2L(\chi,0) = -\frac{1}{N}\sum_{\substack{n=1\\(n,N)=1}}^{N-1} \chi(n)(n+(N-n)) = -\sum\chi(n) = 0$

$$L(\chi,s) = \sum_{\substack{0<n<N\\(n,N)=1}} \chi(n)\sum_{m\geq 0}^{\infty}\frac{1}{(Nm+n)^s} = \sum_{\substack{0<n<N\\(n,N)=1}} \chi(n)\sum_{m\geq 0}^{\infty}\frac{1}{(Nm+(N-n))^s} \quad \text{as } \chi(N-n)=\chi(n)$$

$$\therefore \quad 2\pi^{-s}\Gamma(s)L(\chi,2s) = \pi^{-s}\Gamma(s)\sum_{\substack{0<n<N\\(n,N)=1}}\chi(n)\sum_{m\in\mathbb{Z}}\frac{1}{|Nm+n|^{2s}}$$

$$= N^{-2s}\sum_{\substack{1\leq n<N\\(n,N)=1}}\chi(n)\,H(\tfrac{n}{N},s)$$

So at $s=0$, $4L'(\chi,0) = \sum\chi(n)H(\tfrac{n}{N},0) = -2\sum_{\substack{0<n<N\\(n,N)=1}}\chi(n)\log|1-e^{2\pi i n/N}|$.

$= -4\sum_{\substack{0<n<N/2\\(n,N)=1}}\chi(n)\log|1-e^{2\pi i n/N}|$ as terms for $n, N-n$ are equal. $\square$

Now use this to do some number theory!

Remarks on convergence. $\boxed{1}$ Let $A\leq \operatorname{Re} s = \sigma \leq B$ for some fixed A, B. Claim:–

(i) $\forall y\neq 0$, $\exists\, g(y)<\infty$ s.t.

$$\sum_{n\neq 0}\int_0^\infty e^{-\pi n^2 t-\pi y^2/t}\,|t^{-1/2-s}|\,dt \leq g(y)$$

[This implies by dominated convergence thm that $\forall z\in\mathbb{C}\setminus\mathbb{R}$, $s\in\mathbb{C}$,

$$H_1(z,s) = \sum_{n\neq 0}\int_0^\infty e^{-\pi n^2 t-\pi y^2/t+2\pi i n x}\,t^{-1/2-s}\,dt = \int_0^\infty\sum_{n\neq 0}e^{-\pi n^2 t-\pi y^2/t+2\pi i n x}\,t^{-1/2-s}\,dt,$$

as used in proof of Lemma (ii) above.]

(ii) $\sum_{m=1}^{\infty} g(my)$ converges $\forall y\neq 0$, with g as in (i).

[This implies that $\sum_{m\neq 0}H_1(mz,s)$ converges absolutely $\forall z\in\mathbb{C}\setminus\mathbb{R}$ to an entire function of s, used in proof of KLF above].

Proof Let $f_n = e^{-\pi n^2 t-\pi y^2/t}\,t^{-1/2-\sigma} \leq e^{-\pi|n|t-\pi y^2/t}\,t^{-1/2-\sigma}$

$\Rightarrow \sum_{n\neq 0} f_n \leq 2e^{-\pi y^2/t}\,t^{-1/2-\sigma}/(e^{\pi t}-1) = h(y,\sigma,t)$ say

As $h(y,\sigma,t) = \begin{cases} O(t^{-1/2-\sigma}e^{-\pi t}) & \text{as } t\to\infty \\ O(e^{-\pi y^2}t^{-3/2-\sigma}) & \text{as } t\to 0\end{cases}$, we get (i) with $g(y) = \sup_{A \le \sigma \le B} \int_0^\infty h(y, \sigma, t)\, dt$.

To get (ii), split integral into $\int_y^\infty + \int_0^y$. (Obviously may assume $y > 0$)

Then for $y \le t < \infty$, $h(y, \sigma, t) < t^{-1/2 - \sigma} / (e^{\pi t} - 1)$

$$\Rightarrow \int_y^\infty h(y, \sigma, t)\, dt < \int_y^\infty \frac{t^{-1/2-\sigma}}{e^{\pi t} - 1}\, dt < \int_y^\infty \frac{1}{1 - e^{-\pi y}} \cdot \frac{t^{-1/2-\sigma}}{e^{\pi t}}\, dt$$

$$= \frac{1}{1 - e^{-\pi y}}\, O(e^{-cy}) \quad \text{uniformly for } A \le \sigma \le B, \text{ some } c > 0.$$

and $$\int_0^y h(y, \sigma, t)\, dt < \int_0^y e^{-\pi y^2/t}\, t^{-1/2 - \sigma} \cdot \frac{1}{t}\, dt$$

$$= y^{1+2\sigma} \int_y^\infty e^{-\pi u}\, u^{-1/2 - \sigma}\, \frac{du}{u} \qquad (t = y^2/u)$$

$$= y^{1+2\sigma}\, O(e^{-cy}) \quad \text{uniformly for } A \le \sigma \le B, \text{ some } c > 0$$

So $\exists\, g(y)$ as in (i) with $\sum_{m=1}^\infty g(my) < \infty$. $\square$

Remark $\boxed{2}$. We have, for $x \in \mathbb{R} \setminus \mathbb{Z}$,

$$H(x, 0) = \int_T^\infty \Theta(t; x)\, \frac{dt}{t} - 2T^{-1/2} + \int_{T^{-1}}^\infty \sum_{0 \ne n \in \mathbb{Z}} e^{2\pi i n x - \pi n^2 t}\, t^{1/2}\, \frac{dt}{t} \qquad (*)$$

As $$\sum_{0 \ne n \in \mathbb{Z}} e^{2\pi i n x - \pi n^2 t} = \Theta^*(t; x) - 1 = -1 + t^{-1/2}\, \Theta(1/t; x) = -1 + O(e^{-c/t}) \text{ as } t \to 0$$
$$\text{and } = O(e^{-ct}) \text{ as } t \to 0,$$

we can put $T = 0$ to get

$$H(x, 0) = \int_0^\infty (\Theta^*(t; x) - 1)\, t^{-1/2}\, dt = \int_0^\infty \sum_{0 \ne n \in \mathbb{Z}} e^{2\pi i n x - \pi n^2 t}\, t^{-1/2}\, dt. \qquad (**)$$

It's not obvious that we can interchange $\int$ and $\sum$; as the series of integrals is $\sum_{n \ne 0} \frac{1}{|n|} e^{2\pi i n x}$, which is not absolutely convergent, some extra estimates on the tail of the series is required to apply dominated convergence. One way to get round is is to use some properties of the function $H(x, 0)$, namely:–

- from the integral representation $(*)$, $H(x, 0)$ is differentiable on $\mathbb{R} \setminus \mathbb{Z}$,
- also from $(*)$, for $0 < |x| < 1$ we have

$$H(x, 0) = \int_T^\infty e^{-\pi x^2 t}\, \frac{dt}{t} + \left(\text{differentiable on } (-1, 1)\right)$$

and $f(x) = \int_T^\infty e^{-\pi x^2 t} \frac{dt}{t}$ is integrable on $[-1/2, 1/2]$; indeed

$$\int_{-1/2}^{1/2} f(x)\, dx < \int_{-\infty}^{\infty} f(x)\, dy = \int_{-\infty}^{\infty} \int_T^\infty e^{-\pi x^2 t} \frac{dt}{t}\, dx = \int_T^\infty t^{-3/2}\, dt < \infty$$

so $H(x, 0) = H(x+1, 0)$ is in $L^1(\mathbb{R}/\mathbb{Z})$.

Therefore $\forall x \in \mathbb{R}/\mathbb{Z}$, $H(x, 0)$ is the limit of its Fourier series, whose coefficients are:

$$c_n = \int_0^1 e^{-2\pi i n x} H(x, 0)\, dx = \int_0^1 \int_0^\infty (\Theta^*(t; x) - 1) e^{-2\pi i n x} t^{-1/2}\, dt\, dx$$

and as the integrand is L^1 on $[0, 1] \times [0, \infty)$, can apply Fubini to get

$$c_n = \int_0^\infty e^{-\pi n^2 t} t^{-1/2}\, dt = \frac{1}{|n|}.$$

[I would be happy to learn of a simpler justification of the exchange of $\int$ and Σ in $(**)$!]