11-3

§6. Hecke operators.

Aim: construct action on M_k of a certain algebra associated to $GL_2(\mathbb{Q})$.

In this section, $\Gamma = SL_2(\mathbb{Z})$.

f modular of wt. k $\iff$ $f(\gamma(z)) = j(\gamma, z)^k f(z)$ $\forall \gamma \in \Gamma$.

Convenient to express this as invariance for a group action.

Def^n. f a function on $\mathfrak{H}$, $\gamma \in GL_2(\mathbb{R})^+$, $k \in \mathbb{Z}$.

("slash operator") $$f|_k \gamma \underset{\text{defn}}{=} \frac{(\det \gamma)^{k/2}}{j(\gamma, z)^k} f(\gamma(z)) = \frac{(ad-bc)^{k/2}}{(cz+d)^k} f\left(\frac{az+b}{cz+d}\right).$$

Prop^n: This defines an action on the right of $PGL_2(\mathbb{R})^+ = GL_2(\mathbb{R})^+/\mathbb{R}^\times . I$ on functions

Proof: enough to check $(f|_k \gamma)|_k \delta = f|_k(\gamma\delta)$ and $f|_k \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} = f$,

which are easy & trivial, respectively. $\square$

So f modular of wt. k $\iff$ $f|_k \gamma = f$ $\forall \gamma \in \Gamma$.

1st try: Let $\gamma \in GL_2(\mathbb{R})^+$, f modular of wt. k. What about $g = f|_k \gamma$? Well,

if $\delta \in \Gamma$, $g|_k \delta = f|_k \gamma\delta = (f|_k \gamma\delta\gamma^{-1})|_k \gamma$; so $g|_k \delta = g \iff f|_k \gamma\delta\gamma^{-1} = f$.

Now $SL_2(\mathbb{Z})$ is not normal in $GL_2(\mathbb{R})^+$ [easy to show that normaliser $= \langle SL_2(\mathbb{Z}), I\mathbb{R}^\times \rangle$]

so g is unlikely to be modular of wt k.

2nd try: Lemma: Suppose $\gamma_1, \dots, \gamma_n \in GL_2(\mathbb{R})^+$, such that the family of cosets $\Gamma\gamma_1, \dots, \Gamma\gamma_n$ is invariant under right translation by Γ.

Then if f is modular of wt. k, so is $\sum_{i=1}^n f|_k \gamma_i = g$.

Proof. Let $\gamma \in \Gamma$; by hypothesis $\exists$ permutation $\sigma \in S_n$ st. $\Gamma\gamma_i\gamma = \Gamma\gamma_{\sigma(i)}$

so $\gamma_i\gamma = \delta_i\gamma_{\sigma(i)}$, $\delta_i \in \Gamma$. Then

$$g|_k \gamma = \sum_{i=1}^n f|_k \gamma_i\gamma = \sum_{i=1}^n f|_k \delta_i\gamma_{\sigma(i)} = \sum_{i=1}^n f|\gamma_{\sigma(i)} = \sum_{j=1}^n f|\gamma_j = g. \quad \square$$

(12-1) Recall: Lemma: $(\Gamma\gamma_i)_{1\le i\le r}$ invariant under right multiplication by Γ. Then

f modular of wt k $\Rightarrow$ $\sum_i f|_k \gamma_i$ modular of wt k. $\square$

Remark: this is just algebra, and has nothing to do with modular forms:

G group, $\Gamma \subset G$ subgroup, V v.space with (right) action of G.
$(\Gamma g_i)_{i=1,\dots,r}$ family of cosets invariant under Γ;
then if $v \in V^\Gamma = \{v \in V \mid v\gamma = v \ \forall \gamma \in \Gamma\}$, $\sum_{i=1}^r v g_i \in V^\Gamma$

Taking linear combinations $\Rightarrow$ linear map $\mathbb{Z}[\Gamma\backslash G]^\Gamma \longrightarrow \mathrm{End}(V^\Gamma)$
Lemma is case: $G = GL_2(\mathbb{R})^+ \supset \Gamma = SL_2(\mathbb{Z})$, $V = \{$cx. functions on $\mathfrak{H}\}$ with right action
$G \ni \gamma : f \mapsto f|_k \gamma$.

Propn. Let $\Delta_m = \{\delta \in \mathrm{Mat}_2(\mathbb{Z}) \mid \det \delta = m\}$ $(m \ge 1)$

Then $\Delta_m = \coprod_{\delta \in \Pi_m} \Gamma\delta$, $\Pi_m = \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \middle| \begin{array}{l} a, d \ge 1,\ ad = m \\ b \in R_d \end{array} \right\}$

where R_d is any complete set of residues mod d. (eg. $\{0,1,\dots,d-1\}$)

In particular, the cosets $\{\Gamma\delta \mid \delta \in \Pi_m\}$ satisfy the condition of the Lemma.

Proof. Let $\delta \in \Delta_m$. Then the rows of δ generate a sgp. $\Lambda \subset \mathbb{Z}^2$ (row vectors)

of index $= m$. Every such sgp. is of this form, and δ, δ' determine the same

subgroup $\Leftrightarrow$ $\delta' = \gamma\delta$ for some $\gamma \in GL_2(\mathbb{Z})$ (necessarily $\in SL_2(\mathbb{Z})$, as $\det\delta = \det\delta'$)

Consider $\Lambda \cap \mathbb{Z}.e_2$; it is generated by unique de_2, $d \ge 1$, and then

$\Lambda = \langle ae_1 + be_2, de_2 \rangle$ for some $a, b \in \mathbb{Z}$ with $|ad| = (\mathbb{Z}^2 : \Lambda) = m$.

WLOG $a \ge 1$. Adding multiple to de_2 to 1^{st} generator

$\Rightarrow \Lambda = \langle ae_1 + be_2, de_2 \rangle$ for $b \in R_d$. Then a, b, d unique, $ad = m$, ie.

$\delta' = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \in \Pi_m$ and so $\Delta_m = \coprod_{\delta \in \Pi_m} \Gamma\delta$. $\square$

Coroll. Define $f|_k T_m = m^{\frac{k}{2}-1} \sum_{\delta \in \Pi_m} f|_k \delta$. Then

f modular of wt k $\Rightarrow$ $f|_k T_m$ modular of weight k.

Proof: Lemma + Propn. (The factor $m^{k/2-1}$ is inserted to make certain formula nicer)

The $\{T_m\}$ are Hecke operators. They satisfy nice relations:-

12–2 **Thm.** (i) If $(m,m')=1$, $f\big|_k T_m \big|_k T_{m'} = f\big|_k T_{mm'}$; $f\big|_k T_1 = f$

(ii) p prime, $r \geq 1$. Then
$$f\Big|_k T_{p^r}\Big|_k T_p = f\Big|_k T_{p^{r+1}} + p^{k-1} f\Big|_p T_{p^{r-1}}.$$

(iii) For every m, m' the operators $T_m, T_{m'}$ commute.

__Proof__ : $f\big|_k T_m \big|_k T_{m'} = (mm')^{\frac{k}{2}-1} \sum_{\delta\in\Pi_m} \sum_{\delta'\in\Pi_{m'}} f\big|_k \delta\delta'$. Two ways to proceed:–

__Method A__. (i) $\delta = \begin{pmatrix} a & b \\ & d\end{pmatrix}$, $\delta' = \begin{pmatrix} a' & b' \\ & d'\end{pmatrix}$, $ad = m$, $a'd' = m'$, $b \bmod d$, $b' \bmod d'$.

$\Rightarrow \delta\delta' = \begin{pmatrix} A & B \\ & D\end{pmatrix}$, $A = aa'$, $D = dd'$, $B = ab' + d'b$.

$(m,m')=1$. Then $AD = mm' \Rightarrow \exists!\ a, d, a', d'$ with $ad = m$, $a'd' = m'$, $aa' = A$, $dd' = D$ (take $a = (A, m)$ etc).

and $R_d \times R_{d'} \longrightarrow \mathbb{Z}/dd'\mathbb{Z}$ is bijective ; so $\Pi_m \times \Pi_{m'} \xrightarrow{\sim} \Pi_{mm'}$
$(b, b') \longmapsto ab' + d'b$. $\qquad (\delta, \delta') \longmapsto \delta\delta'$

(ii) $\delta = \begin{pmatrix} p^{r-i} & b \\ & p^i\end{pmatrix}$ $b \bmod p^i$; $\delta' = \begin{cases} \begin{pmatrix} p & 0 \\ & 1\end{pmatrix} & \text{or} \\ \begin{pmatrix} 1 & b' \\ & p\end{pmatrix} & b' \bmod p.\end{cases}$

1st case :– $\delta\delta' = \begin{pmatrix} p^{r+1-i} & b \\ 0 & p^i\end{pmatrix}$; as i, b vary, this gives all $\begin{pmatrix} a & * \\ & *\end{pmatrix} \in \Pi_{p^{r+1}}$ with $p | a$.

2nd case :– $\delta\delta' = \begin{pmatrix} p^{r-i} & p^{r-i}b' + pb \\ & p^{i+1}\end{pmatrix}$; for $i = r$, as b', b vary this gives all $\begin{pmatrix} 1 & * \\ & *\end{pmatrix} \in \Pi_{p^{r+1}}$

remaining elts :– $0 \leq i \leq r-1$: $\begin{pmatrix} p & \\ & p\end{pmatrix}\begin{pmatrix} p^{r-1-i} & b + p^{r-1-i}b' \\ & p^i\end{pmatrix}$; for each b', as b, i varies this runs over all elts. of $\begin{pmatrix} p & \\ & p\end{pmatrix}\Pi_{p^{r-1}}$.

and $f\big|\begin{pmatrix} p & \\ & p\end{pmatrix} = f$, so
$$f\big| T_{p^r} \big| T_p = p^{(r+1)(k/2 - 1)}\Big[\sum_{\delta\in\Pi_{p^{r+1}}} f\big|_k \delta + p \sum_{\delta\in\Pi_{p^{r-1}}} f\big|\delta\Big]$$
$$= f\Big|_k T_{p^{r+1}} + p^{k-1} f\Big|_k T_{p^{r-1}}. \qquad \square$$

(iii) By (i), $T_m, T_{m'}$ commute if $(m, m') = 1$.

So if $m = \prod p_i^{r_i}$, $T_m = \prod_i T_{p_i^{r_i}}$ (commuting product)

By applying (ii) successively, T_{p^r} is expressed as a polynomial in T_p.

So $\{T_{p^r}\}_{r\geq 1}$ commute. $\square$

(B – 1)

Method B. (More conceptual)

$(\delta, \delta') \in \Pi_m \times \Pi_{m'} \iff \underset{\text{index } m}{\Lambda} \subset \underset{\text{index } m'}{\Lambda'} \subset \mathbb{Z}^2$ Λ = row span of $\delta\delta'$, Λ' = — " — δ'.

(i) If $(m, m') = 1$, $\Lambda \underset{mm'}{\subset} \mathbb{Z}^2$, $\exists!$ intermediate Λ' of index m' in $\mathbb{Z}^2$.

(ii) If $m = p^r, m' = p$, then $\Lambda \not\subset p\mathbb{Z}^2 \Rightarrow \exists!\ \Lambda'$ of index p
$\Lambda \subset p\mathbb{Z}^2 \Rightarrow \exists\ (p+1)$ such Λ'

$$\Rightarrow \sum_{\delta \in \Pi_{p^r}, \delta' \in \Pi_p} f|\delta\delta' = \sum_{\varepsilon \in \Pi_{p^{r+1}} \setminus \left(\begin{smallmatrix} p & 0 \\ 0 & p \end{smallmatrix}\right)\Pi_{p^{r-1}}} f|\varepsilon + (p+1) \sum_{\varepsilon \in \left(\begin{smallmatrix} p & 0 \\ 0 & p \end{smallmatrix}\right)\Pi_{p^{r-1}}} f|\varepsilon = f|T(p^{r+1}) + pf|T(p^{r-1}).$$ □

Now compute action of $\{T_m\}$ on modular forms. As they commute, we'll write action on left instead:

Defn: f modular of wt k: $T_m f \overset{\text{def}}{=} f|_k T_m$ (or T_m^k to avoid any ambiguity).

Thm. i) $f \in \begin{cases} M_k \\ S_k \end{cases} \Rightarrow T_m^k f \in \begin{cases} M_k \\ S_k. \end{cases}$

ii) $f = \sum_{n \in \mathbb{Z}} a_n q^n$ wk. modular of weight k

$$\Rightarrow T_m^k f = \sum_{n \in \mathbb{Z}} b_n q^n, \qquad b_n = \sum_{1 \le d | (m,n)} d^{k-1} a_{mn/d^2}.$$

Proof (i) f holo. on $\mathfrak{H}$ $\Rightarrow T_m^k f$ so. So (i) follows from (ii).

(ii)
$$q^n|_k T_m = m^{k/2-1} \sum_{\delta = \left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right) \in \Pi_m} \frac{m^{k/2}}{d^k} e^{2\pi i n(az+b)/d}$$

$$= m^{k-1} \sum_{1 \le d | m} d^{-k} \left[\sum_{b \bmod d} e^{2\pi i nb/d}\right] q^{mn/d^2} \qquad (a = m/d)$$

Inner sum $= \begin{cases} 0 & \text{if } d \nmid n \\ d & \text{if } d | n. \end{cases}$ $\therefore\ q^n|_k T_m = m^{k-1} \sum_{1 \le d | (m,n)} d^{1-k} q^{mn/d^2}$

Write $m = de$; then $d | n \iff n = \frac{m}{e} l,\ l \in \mathbb{Z}$, and $\frac{mn}{d^2} = el$

$$\therefore\ f|_k T_m = \sum_{l \in \mathbb{Z}} \sum_{1 \le e | m} e^{k-1} a_{ml/e} q^{el}$$

$$= \sum_{n \in \mathbb{Z}} \left[\sum_{1 \le e | (m,n)} e^{k-1} a_{mn/e}\right] q^n = \sum_{n \in \mathbb{Z}} b_n q^n$$ □

Coroll. Suppose $T_m f = \lambda f$, some $m \geq 1$, $\lambda \in \mathbb{C}$, $f = \sum_{n \geq 0} a_n q^n \in M_k$

i) $a_m = \lambda a_1$, and more generally if $(m, n) = 1$, $a_{mn} = \lambda a_n$.

ii) If $a_0 \neq 0$, $\lambda = \sigma_{k-1}(m)$.

Proof (i) $\lambda a_1 = b_1 = a_m$ by Thm (ii); (ii) $\lambda a_0 = b_0 = \sum_{1 \le d | m} d^{k-1} a_0$. □
$\lambda a_n = b_n = a_{mn}$ as $(m, n) = 1$.

(13-2) Of particular interest is the case when $T_m f = \lambda_m f$ for all $m \geq 1$, some $\lambda_m \in \mathbb{C}$.
– f is said to be a <u>Hecke eigenform</u>.

<u>Thm</u> Let $f = \sum_{n\geq 0} a_n q^n \in M_k$, $k>0$. Suppose $a_0 \neq 0$. Then TFAE:–

(a) $f = a_0 E_k$

(b) $\forall m \geq 1$, $T_m f = \lambda_m f$ $\lambda_m \in \mathbb{C}$

and if so, $\lambda_m = \sigma_{k-1}(m)$.

<u>Proof (i)</u>: (a) $\Rightarrow$ (b). $E_k = 1 - \frac{2k}{B_k} \sum_{n\geq 1} \sigma_{k-1}(n) q^n$

Since every T_m is a polynomial in $\{T_p \mid p \text{ prime}\}$, enough to show that $T_p E_k = \lambda_p E_k$, some λ_p. By Thm (iii), $T_p E_k = \sum_{n\geq 1} b_n q^n$ where:

$b_0 = \sigma_{k-1}(p) \quad [= 1+p^{k-1}]$

$b_n = \frac{-2k}{B_k} \sum_{d \mid (p,n)} d^{k-1} \sigma_{k-1}\left(\frac{mn}{d^2}\right)$ [recall: $\sigma_r(mn) = \sigma_r(m)\sigma_r(n)$ if $(m,n)=1$]

$p \nmid n$: $b_n = -\frac{2k}{B_k} \sigma_{k-1}(np) = -\frac{2k}{B_k} \sigma_{k-1}(p)\sigma_{k-1}(n)$

$p \mid n$: $b_n = -\frac{2k}{B_k}\left[\sigma_{k-1}(np) + p^{k-1}\sigma_{k-1}(n/p)\right]$

and if $p \mid n$, $\sigma_r(n)\sigma_r(p) = \sigma_r(np) + p^r \sigma_r(n/p)$

since $\{\text{divisors of } np\} = \{d \mid d \mid n\} \sqcup \{dp \mid d \mid n,\ d \nmid \tfrac{n}{p}\}$

so $\sigma_r(np) = \sigma_r(n) + p^r[\sigma_r(n) - \sigma_r(\tfrac{n}{p})] = \sigma_r(n)\sigma_r(p) - p^r\sigma_r(n/p)$.

$\therefore$ $T_p E_k = \sigma_{k-1}(p) E_k \quad \forall p$.

(b) $\Rightarrow$ (a). By Coroll (ii), $\lambda_m = \sigma_{k-1}(m)$ $\forall m$.

$\therefore$ $a_n = \sigma_{k-1}(n) a_1$ $\forall n \geq 1$ and so $f = A + BE_k$, $A, B \in \mathbb{C}$

$\Rightarrow A = f - BE_k \in M_k \Rightarrow A = 0$ as $k > 0$. $\square$

(14-1) What about cusp forms?

The Petersson inner product. Recall: $\operatorname{Im}\gamma(z) = \frac{\operatorname{Im} z}{|j(\gamma,z)|^2}$; $\frac{dx\,dy}{y^2}$ is Γ-invariant.

Defⁿ. $f, g \in M_k$, at least one $\in S_k$.

$$\langle f, g\rangle \overset{\text{def}}{=} \int_{\mathcal{D}} \Omega(f,g) \quad \text{where} \quad \Omega(f,g) = f(z)\overline{g(z)}\, y^k \, \frac{dx\,dy}{y^2}.$$

Remark. (i) as $|f\bar{g}| = O(q)$ as $\operatorname{Im} z \to \infty$, this is convergent.

(ii) Ω is Γ-invariant, since $f(\gamma(z))\overline{g(\gamma(z))}\,(\operatorname{Im}\gamma(z))^k = j(\gamma,z)^k\,\overline{j(\gamma,z)}^k f(z)\overline{g(z)} \frac{y^k}{|j(\gamma,z)|^{2k}}$

$= f(z)\overline{g(z)}\, y^k$

So can replace $\mathcal{D}$ by any (measurable) fundamental set for Γ. & makes sense

to write $\langle f, g\rangle = \int_{\Gamma\backslash \mathfrak{H}} \Omega(f,g)$.

Thm $\langle T_m f, g\rangle = \langle f, T_m g\rangle$.

Coroll. $\exists$ basis of S_k composed of Hecke eigenforms, and the eigenvalues of the T_n are real.

Thm (multiplicity one) Let $f, g \in M_k \setminus 0$. Suppose that for all primes p,

$$T_p f = \lambda_p f, \quad T_p g = \lambda_p g.$$

Then $f = cg$.

Proof. WLOG, $k>0$; so f, g nonconstant

Since T_m are polys. in T_p, $\exists \lambda_m$ s.t. $T_m f = \lambda_m f$, $T_m g = \lambda_m g$

$\forall m \geq 1$. Let $f = \sum_{n\geq 0} a_n q^n$, $g = \sum_{n \geq 0} b_n q^n$. Then

$$a_n = \lambda_n a_1, \quad b_n = \lambda_n b_1.$$

$\Rightarrow$ $a_1, b_1 \neq 0$ and $\exists c$ s.t. $a_n = c b_n$ $\forall n \geq 0$.

Then $f - cg = a_0 - cb_0$ is constant and $k \neq 0$ $\Rightarrow$ $f = cg$. $\square$

A cusp form which is a Hecke eigenform and has $a_1 = 1$ is said to be normalised. In this case $T_m f = a_m f$ $\forall m \geq 1$.

Ex. $\Delta = \sum_{n\geq 1} \tau(n) q^n \in S_{12}$ is a normalised eigenform ($\tau(1) = 1$; eigenform since $\dim S_{12} = 1$)

So $T_n \Delta = \tau(n)\Delta$ $\forall n \geq 1$.

Now recall: $T_m T_{m'} = T_{mm'}$ if $(m,m') = 1$

$$T_p^k T_{p^r}^k = T_{p^{r+1}}^k + p^{k-1} T_{p^{r-1}}^k.$$

So deduce (conj. of Ramanujan, proved by Mordell) :-

<u>Coroll.</u> If $(m,n)=1$, $\tau(mn) = \tau(m)\tau(n)$

and $\tau(p^{r+1}) = \tau(p)\tau(p^r) - p^{11}\tau(p^{r-1})$ (p prime, $r \geq 1$)

Same thing holds for S_k, $k \in \{16, 18, 20, 22, 26\}$ (the remaining k for which $\dim S_k = 1$).

<u>Remark.</u> Can prove: "<u>Strong multiplicity one</u>" theorem.

Let $f, g \in S_k$ be normalized eigenforms. Assume $T_p f = \lambda_p f$, $T_p g = \lambda_p g$ for all but finitely many p. Then $f = g$.

$\boxed{k = 24}$ $\dim S_{24} = 2$, spanned by $g_1 = E_4^3\Delta$ and $g_2 = \Delta^2$.

$$\Delta = q - 24q^2 + 252q^3 - \dots$$
$$E_4^3 = 1 + 720q + 179280q^2 + 16954560q^3 - \dots$$
$$\Rightarrow \quad g_2 = q^2 - 48q^3 + 1080q^4$$
$$g_1 = q + 696q^2 + 162252q^3 + 12831808q^4 + \dots$$

$$T_2^k f = \sum_1^\infty b_n q^n \quad \text{with} \quad b_n = \begin{cases} a_{2n} + 2^{k-1}a_{n/2} & n \text{ even} \\ a_{2n} & n \text{ odd.} \end{cases} \quad \text{if } f = \sum_1^\infty a_n q^n$$

$$\therefore \quad T_2(g_1) = 696q + 21220416q^2 + \dots = 696g_1 + 20736000g_2$$
$$T_2(g_2) = q + 1080q^2 + \dots = g_1 + 384g_2$$

$\therefore$ matrix of T_2 is $\begin{pmatrix} 696 & 1 \\ 20736000 & 384 \end{pmatrix}$

– char. poly is $x^2 - 1080x - 20468736 = (x - 540 + 12\sqrt{144169}) \times (x - 540 - 12\sqrt{144169})$

$\therefore$ action of T_2 is sufficient to decompose S_{24} into 2 eigenspaces.

& coeffs. of the eigenforms lie in $\mathbb{Q}(\sqrt{\underbrace{144169}_{\text{prime!}}})$.

This seems to be a universal phenomenon:-

<u>Conjecture</u> (Maeda) : char poly of T_2 acting on S_k is always irreducible.

(so the only Hecke eigenforms in S_k with <u>rational</u> q-exp. coeffs. are those when $\dim S_k = 1$).

Proof (sketch) of self-adjointness of T_n.

- enough to prove for T_p, p prime.
- recall $T_p f = p^{\frac{k-1}{2}} \sum_{\delta \in \Pi_p} f|_k \delta = p^{\frac{k-1}{2}} \left[f|_k \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} + \sum_{b \bmod p} f|_k \begin{pmatrix} 1 & b \\ 0 & p \end{pmatrix} \right]$.

The cosets $\Gamma\delta$ ($\delta \in \Pi_p$) are permuted by right action of δ.

- Γ acts transitively on $\{\Gamma\delta \mid \delta \in \Pi_p\}$.

Proof: $\Gamma\begin{pmatrix} 1 & b \\ 0 & p \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \Gamma\begin{pmatrix} 1 & b+1 \\ 0 & p \end{pmatrix}$, $\Gamma\begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} & -1 \\ 1 & \end{pmatrix} = \Gamma\begin{pmatrix} & -1 \\ 1 & \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix} = \Gamma\begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix}$

- stabiliser of $\Gamma\begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}$ is $\begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}^{-1}\Gamma\begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} \cap \Gamma = \left\{\begin{pmatrix} a & p^{-1}b \\ pc & d \end{pmatrix}\right\} \cap \Gamma = \Gamma_0(p)$

$= \left\{\begin{pmatrix} a & b \\ pc & d \end{pmatrix} \in \Gamma\right\}$,

and of $\Gamma\begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix}$ is $\begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix}^{-1}\Gamma\begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix} \cap \Gamma$

$= \Gamma^0(p) = \left\{\begin{pmatrix} a & pb \\ c & d \end{pmatrix} \in \Gamma\right\} = \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix}^{-1}\Gamma_0(p)\begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix}$

- so $\sum_{\delta \in \Pi_p} f|\delta = \sum_i f|\begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}|\gamma_i$, where $\Gamma = \coprod \Gamma_0(p)\gamma_i$.

$\sum_{\delta \in \Pi_p} g|\delta = \sum_i g|\begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix}|\gamma_i'$, $\Gamma = \coprod \Gamma^0(p)\gamma_i'$.

$$\therefore \langle T_p f, g\rangle = p^{(k-1)/2}\sum_i \int_{\mathcal{D}} \underbrace{f|\begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}|\gamma_i \cdot \bar{g}\; y^k \frac{dx\,dy}{y^2}}_{= f|\begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} \cdot \bar{g}\; y^k \frac{dx\,dy}{y^2}\Big|_{z \mapsto \gamma_i(z)} \text{ as } g = g|_k \gamma_i}$$

$$= p^{(k-1)/2}\sum_i \int_{\gamma_i \mathcal{D}} f|\begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} \cdot \bar{g}\; y^k \frac{dx\,dy}{y^2}$$

$$= p^{(k-1)/2}\int_{\Gamma_0(p)\backslash\mathfrak{H}} f|\begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} \bar{g}\; y^k \frac{dx\,dy}{y^2}$$

$$= p^{(k-1)/2}\int_{\Gamma^0(p)\backslash\mathfrak{H}} f \cdot \underbrace{\bar{g}|_k\begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}^{-1}}_{= \bar{g}|_k\begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix}} y^k \frac{dx\,dy}{y^2}$$

$= $ (reversing argument) $\langle f, T_p g\rangle$ $\square$

Estimates for eigenvalues. For Eis series E_k, eigenvalue of T_n = $\sigma_{k-1}(n) = O(n^{k-1}\log n)$

What about cusp forms?

Thm $f = \sum_{n=1}^{\infty} a_n q^n \in S_k$. Then $\exists C$ s.t. $|a_n| \le C n^{k/2}$

15-2

Remark. In fact it's known that if f is a normalised eigenfn, $|a_p| \le 2p^{\frac{k-1}{2}}$.
Proof uses hard alg. geometry – Deligne 1972 (Fields medal!).
In particular, $|\tau(p)| \le 2p^{11/2}$ ("Ramanujan conjecture").

Deligne estimate $\Rightarrow$ for any $f \in S_k$, $|a_n| = O(n^{\frac{k-1}{2}} \log n)$.

Proof of Thm. Recall (Ex Sheet 2): $y^{k/2}|f|$ bounded on $\mathfrak{H}$ by some $M \ge 0$.

Now $f = \sum_{n\ge1}^{\infty} a_n q^n \Rightarrow a_n = \int_0^1 e^{-2\pi i n(x+iy)} f(x+iy)\,dx$ (any $y>0$)

(using $\int_0^1 e^{2\pi i m x}\,dx = \begin{cases} 1 & m=0 \\ 0 & m\ne 0\end{cases}$).

$$\therefore \quad |a_n| \le e^{2\pi n y} \sup_{x\in[0,1]} |f(x+iy)| \le e^{2\pi n y} M y^{-k/2} \quad \forall y \in (0,\infty)$$

Take $y = 1/n \Rightarrow |a_n| \le M e^{2\pi} n^{k/2}$. $\square$

§7. L-functions of modular fns.

$0 \ne f = \sum_1^{\infty} a_n q^n \in S_k$. Then $a_n = O(n^{k/2})$ implies that the Dirichlet series

$$L(f,s) = \sum_{n\ge1} a_n n^{-s}$$ is abs. cgt. for $\mathrm{Re}(s) > 1 + \frac{k}{2}$

Thm. $L(f,s)$ is entire and satisfies the functional equation

$$\Lambda(f,s) \overset{\text{def}}{=} 2(2\pi)^{-s}\Gamma(s)L(f,s) = (-1)^{k/2}\Lambda(f,k-s).$$

Proof. Look at Mellin transform of $f(iy) = \sum_{n\ge1} a_n e^{-2\pi n y}$:-

$\boxed{\infty}$ $f(iy) = O(e^{-2\pi y})$ as $y \to \infty$ so rapidly decreasing at ∞;

$\boxed{0}$ $f(iy) = (iy)^{-k} f(i/y)$ so rapidly decreasing at 0 : $y^M f(iy) \to 0$ $\forall M \in \mathbb{Z}$.

$$\therefore \quad M(f(iy),s) = \int_0^\infty \sum_{n=1}^\infty a_n e^{-2\pi n y} y^s \frac{dy}{y}$$ is entire.

and if $\mathrm{Re}\, s > \frac{k}{2}+1$, $M(f(iy),s) = \sum_{n\ge1} a_n M(e^{-2\pi n y}, y) = (2\pi)^{-s}\Gamma(s)L(f,s)$

$\Rightarrow L(f,s)$ entire, as $\Gamma(s) \ne 0$.

$$M(f(iy),s) = \int_1^\infty + \int_0^1 f(iy)\, y^s \frac{dy}{y} = \int_1^\infty f(iy)\, y^s \frac{dy}{y} + \int_0^1 (iy)^{-k} f(i/y)\, y^s \frac{dy}{y}$$

$$= \int_1^\infty f(iy)\left[y^s + (-1)^{k/2} y^{k-s}\right]\frac{dy}{y},$$ invariant under $s \leftrightarrow k-s$. $\square$

(16-1) Particular case of interest:

<u>Thm</u> (Euler product) $f = \sum_1^\infty a_n q^n \in S_k$ normalized Hecke eigenform. Then

$$L(f,s) = \prod_p \frac{1}{1 - a_p p^{-s} + p^{k-1-2s}} \qquad \left(\text{if } Re(s) > \tfrac{k}{2}+1.\right)$$

<u>Proof.</u> $|a_p p^{-s} + p^{k-1-2s}| = O(p^{k/2 - \sigma}) + O(p^{k-1-2\sigma}) \qquad \sigma = Re(s)$

$= O(p^{-1-\delta}) \qquad$ if $\sigma \geq \frac{k}{2}+1+\delta$ and $2\sigma \geq k+\delta$

So $\prod_p$ converges $\Leftarrow$ $\sum_p p^{-1-\delta}$ converges $\Leftarrow$ $\delta > 0$.

<u>Claim</u>: $\dfrac{1}{1-a_p p^{-s} + p^{k-1-2s}} = 1 + a_p p^{-s} + a_{p^2} p^{-2s} + \cdots$

(multiply through by $1 - a_p p^{-s} + p^{k-1-2s}$ and use identity

$a_{p^{r+1}} = a_p a_{p^r} - p^{k-1} a_{p^{r-1}}$, since a_n = eigenvalue of T_n).

$\therefore \prod_p = \prod_p (1 + a_p p^{-s} + a_{p^2} p^{-2s} + \cdots) = \sum_{n \geq 1} a_n n^{-s}$

since $a_{mn} = a_m a_n$ for $(m,n) = 1$. ◻

$\boxed{\text{Case of } E_k}$ T_n eigenvalue $= \sigma_{k-1}(n)$

<u>Prop</u>: $\sum_{n \geq 1}^{\infty} \sigma_{k-1}(n) n^{-s} = \zeta(s)\zeta(s-k+1) \qquad (Re(s) > k)$

<u>Proof.</u> Same Euler product calculation gives

$$LHS = \prod_p \frac{1}{1 - \sigma_{k-1}(p)p^{-s} + p^{k-1-2s}} = \prod_p \frac{1}{1 - (1+p^{k-1})p^{-s} + p^{k-1-2s}}$$

$$= \prod_p \frac{1}{(1-p^{-s})(1-p^{k-1-s})} = \zeta(s)\zeta(s-k+1)$$ ◻