Digression: lattices. V f.d. real vector space. A *lattice* in V is a subgroup $\Lambda \subset V$ of the form $\Lambda = \sum_{i=1}^{d} \mathbb{Z} e_i$, (e_i) a $\mathbb{R}$-basis for V.

eg. $\mathbb{Z}^n \subset \mathbb{R}^n$ and obviously, given $\Lambda \subset V$, $\exists\, V \simeq \mathbb{R}^n$ mapping Λ to $\mathbb{Z}^n$.

If Λ is a lattice then
- Λ is a discrete subgroup (i.e. induced topology on Λ is discrete)
- Λ is cocompact (i.e. V/Λ with quotient topology)

[topology on V coming say from any norm].

Prop. $\dim V = d$, $\Lambda \subset V$ lattice, $\|\cdot\| : V \to \mathbb{R}$ a norm. Then if $\alpha \in \mathbb{R}$,
$$\sum_{0 \neq x \in \Lambda} \frac{1}{\|x\|^\alpha} \text{ converges} \iff \alpha > d.$$

Proof. Recall that any two norms $\|\cdot\|, \|\cdot\|'$ on V are *equivalent*: $\exists\, 0 < A < B$ st. $\forall v \in V$, $A\|v\| \le \|v\|' \le B\|v\|$. So series converges for one norm $\iff$ it converges for any norm. So enough to take $V = \mathbb{R}^n$, $\Lambda = \mathbb{Z}^n$,
$$\|x\| = \|x\|_\infty = \max\{|x_i|\}.$$
Then
$$\sum_{0 \neq x \in \mathbb{Z}^d} \frac{1}{\|x\|^\alpha} = \sum_{n \ge 1} \frac{N(d,n)}{n^\alpha}.$$
$$N(d,n) = \#\{x \in \mathbb{Z}^d \mid \|x\| = n\} = \#\{\|x\| \le n\} - \#\{\|x\| \le n-1\}$$
$$= (2n+1)^d - (2n-1)^d = 2^{d+1} d\, n^{d-1} + O(n^{d-3}).$$
$\therefore$ series cvges $\iff \sum_{n \ge 1} n^{d-1-\alpha}$ cvges $\iff \alpha > d$. $\square$

§4. The modular group

Have seen examples of functions on $\mathfrak{H} = \{z \in \mathbb{C} \mid \operatorname{Im}(z) > 0\}$ satisfying transformation laws under Möbius maps:- eg.
$$\vartheta(z) = \sum_{n \in \mathbb{Z}} e^{\pi i n^2 z} = \underset{\begin{pmatrix}1&2\\&1\end{pmatrix}}{\vartheta(z+2)} = \underset{\begin{pmatrix}&-1\\1&\end{pmatrix}}{(z/i)^{-1/2}\, \vartheta(-1/z)}$$
Systematically study these.

$GL_2(\mathbb{R})$ acts on $\mathbb{C} \setminus \mathbb{R}$ on left by $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} : z \mapsto \frac{az+b}{cz+d}$
and on $\mathbb{R} \cup \{\infty\}$

$$\operatorname{Im} \gamma(z) = \frac{1}{2i}\left(\frac{az+b}{cz+d} - \frac{a\bar z + b}{c\bar z + d}\right) = \frac{1}{2i}\frac{(ad-bc)(z-\bar z)}{|cz+d|^2} = \det(\gamma)\frac{\operatorname{Im} z}{|cz+d|^2}.$$

So $\operatorname{Im} z$, $\operatorname{Im}\gamma(z)$ have same sign $\iff \det(\gamma) > 0$

(5-2) $\therefore$ $GL_2(\mathbb{R})^+ = \{\gamma \in GL_2(\mathbb{R}) \mid \det \gamma > 0\}$ maps $\mathfrak{H}$ to itself;

centre $\mathbb{R}^\times \cdot \begin{pmatrix} 1 & \\ & 1 \end{pmatrix}$ acts trivially $\Rightarrow$ action on $\mathfrak{H}$.

$$PGL_2(\mathbb{R})^+ = GL_2(\mathbb{R})^+ / \mathbb{R}^\times \cdot \begin{pmatrix} 1 & \\ & 1 \end{pmatrix}$$

Now $GL_2(\mathbb{R})^+ = \langle \mathbb{R}^\times \begin{pmatrix} 1 & \\ & 1 \end{pmatrix}, SL_2(\mathbb{R}) \rangle$

$\Rightarrow$ $PGL_2(\mathbb{R})^+ \xrightarrow{\sim} SL_2(\mathbb{R}) / \{\pm I\}$ (often written $PSL_2(\mathbb{R})$ although this notation can be confusing)

Fact $PGL_2(\mathbb{R})^+$ is the group of holomorphic automorphisms of $\mathfrak{H}$.

[later will need GL as well as SL]

Action is transitive, stabiliser of $z = i$ is $SO(2)/\{\pm I\}$ (a maximal compact subgroup of $PGL_2(\mathbb{R})^+$). This follows from Iwasawa decomposition

$$SL_2(\mathbb{R}) = KAN = NAK, \quad K = SO(2) = \left\{ \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \right\}$$

$$A = \begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix}, \quad N = \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}$$

(= Gram-Schmidt orthonormalisation)

since $z = x + iy = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \sqrt{y} & 0 \\ 0 & 1/\sqrt{y} \end{pmatrix} (i)$.

Mainly interested in $\Gamma = SL_2(\mathbb{Z})$ and its sgps. of finite index.

$\bar{\Gamma} = \Gamma / \{\pm I\}$ is the modular group. Describe it, and action on $\mathfrak{H}$, completely.

6-1 Interested in $SL_2(\mathbb{Z})$ and its subgroups of finite index acting on $\mathfrak{h}$.

$\Gamma \subset SL_2(\mathbb{Z})$; define $\bar\Gamma = \Gamma/\Gamma \cap \{\pm 1\}$ = image of Γ in $PGL_2(\mathbb{R})^+$.

If $\Gamma = SL_2(\mathbb{Z})$, $\bar\Gamma$ $(= PSL_2(\mathbb{Z}))$ is the <u>modular group</u>.

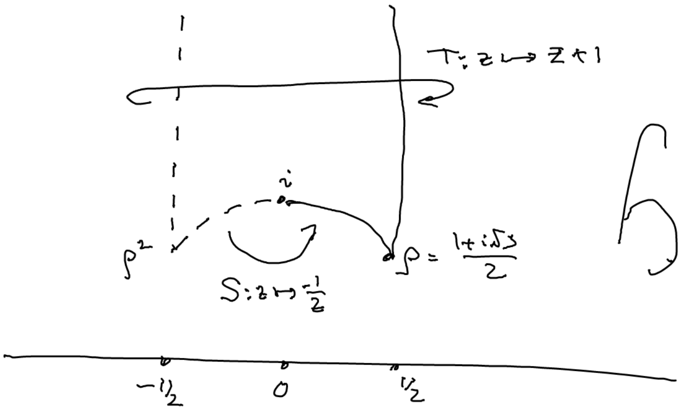
Let $S = \pm\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, $T = \pm\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ $\qquad (z \mapsto -\frac{1}{z},\ z \mapsto z+1)$.

<u>Thm.</u> Let $\Gamma = SL_2(\mathbb{Z})$. Then:

(i) $\bar\Gamma = \langle S, T \rangle$

(ii) The set $\mathcal{D} = \left\{ z \in \mathfrak{h} \;\middle|\; \begin{array}{c} -\frac12 < \operatorname{Re} z \le \frac12 \\ |z| \ge 1 \\ |z| = 1 \Rightarrow \operatorname{Re} z \ge 0 \end{array} \right\}$ is a

<u>fundamental set</u> for action of $\bar\Gamma$ on $\mathfrak{h}$ i.e. $\forall z \in \mathfrak{h}$, $\exists!\, z_0 \in \mathcal{D}$ and some $\gamma \in \bar\Gamma$ with $z = \gamma(z_0)$ (so as sets, $\mathcal{D} \simeq \bar\Gamma \backslash \mathfrak{h}$).



(iii) <u>Stabilisers:</u> $z \in \mathcal{D}$, then

$z \notin \{i, \rho = e^{\pi i/3}\} \Rightarrow \bar\Gamma_z = \{1\}$.

$\bar\Gamma_i = \langle S \rangle \simeq \mathbb{Z}/2\mathbb{Z}$

$\bar\Gamma_\rho = \langle TS \rangle \simeq \mathbb{Z}/3\mathbb{Z}$

$TS = \pm\begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}$

<u>Proof.</u> (same as Gauss reduction for pos def. binary QF).

<u>Step 1.</u> Let $\bar\Gamma^* = \langle S, T \rangle \subset \bar\Gamma$.

If $z \in \mathfrak{h}$, construct $\gamma \in \bar\Gamma^*$ st. $\gamma(z) \in \mathfrak{h}$.

<u>Step 2.</u> Let $z, z' \in \mathcal{D}$. Show that if $\gamma \in \bar\Gamma$, $\gamma(z) = z'$ then

$z' = z$ and either $\gamma = 1$

$z = i$, $\gamma = S$

$z = \rho$, $\gamma = TS$ or $(TS)^2$.

Granted these: then (ii), (iii) follow at once. As for (i), let $\gamma \in \bar\Gamma$.

Then $\exists \gamma' \in \bar\Gamma^*$ st. $\gamma'(\gamma^{-1}(2i)) \in \mathfrak{h}$ by step 1. So $\gamma'\gamma^{-1} \in \bar\Gamma_{2i}$.

By step 2, this implies $\gamma = \gamma' \in \bar\Gamma^*$.

(6-2) <u>Proof of step 1</u>. Let $z \in \mathfrak{H}$. Then $\{cz+d \mid c,d \in \mathbb{Z}\} \subset \mathbb{C}$ is a lattice

so $\{ |cz+d| , c,d \in \mathbb{Z}\}$ is a discrete subset of $\mathbb{R}$.

$$\therefore \quad \left\{ \operatorname{Im} \gamma(z) = \frac{\operatorname{Im} z}{|cz+d|^2} \;\middle|\; \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \bar\Gamma^* \right\} = \operatorname{Im} \bar\Gamma^* z$$

is a discrete subset of $\mathbb{R}_{>0}$, bounded above.

Let $z' = \gamma(z)$, $\gamma \in \bar\Gamma^*$ with $\operatorname{Im} z'$ <u>maximal</u>. WLOG (replacing z' with $T^n z' = z+n$ for suitable $n \in \mathbb{Z}$), $-\frac12 < \operatorname{Re} z' \le \frac12$. If $|z'|<1$ then

$$|-1/z'| > 1 \quad \text{and} \quad \operatorname{Im}(-1/z') = \frac{\operatorname{Im} z'}{|z'|^2} > \operatorname{Im} z';$$

contradicting maximality. So either:

- $|z'|>1$, $z' \in \mathcal{D}$, or
- $|z'|=1$, $z' = e^{i\theta}$ say, so one of $z', -1/z' = e^{i(\pi-\theta)} \in \mathcal{D}$.

<u>Proof of step 2.</u> Assume $z, z' = \gamma(z) \in \mathcal{D}$, $\gamma = \pm\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \bar\Gamma$.

WLOG $\operatorname{Im} z \le \operatorname{Im} z' = \dfrac{\operatorname{Im} z}{|cz+d|^2}$, so

$1 \ge |cz+d| \ge c \operatorname{Im} z \ge \frac{\sqrt3}{2} c$ as $z \in \mathcal{D}$. $\therefore$ $c \in \{0, \pm 1\}$

$\boxed{c=0}$ $\gamma = \pm\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$, $z' = z+b \Rightarrow b = 0$, $z = z'$, $\gamma \in \{\pm I\}$.

$\boxed{c=1}$ $|z+d| \le 1$, $|z| \ge 1 \Rightarrow d \in \{0, \pm 1\}$ ($d = \pm 2$ impossible as $z \notin \mathbb{R}$)

- $d=0 \Rightarrow |z|=1$, $\gamma = \pm\begin{pmatrix} a & -1 \\ 1 & 0 \end{pmatrix}$, $z' = a - 1/z = a + e^{i(\pi-\theta)}$ ($z = e^{i\theta}$, $\frac{\pi}{3} \le \theta \le \frac{\pi}{2}$)
  $\Rightarrow a = 0$, $z = z' = i$, $\gamma = S$
  or $a = 1$, $z = z' = \rho$, $\gamma = TS = \pm\begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}$
- $d = 1$ impossible as $\rho^2 \notin \mathcal{D}$.
- $d = -1 \Rightarrow z = \rho$; $|cz+d| = |\rho - 1| = 1 \Rightarrow \operatorname{Im} z' = \operatorname{Im} z = \frac{\sqrt3}{2} \Rightarrow z' = \rho$.

$$\frac{a\rho+b}{\rho-1} = \rho \Rightarrow a\rho + b = \rho^2 - \rho = -1 \Rightarrow \gamma = \pm\begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix} = (TS)^2. \qquad \square$$

<u>Prop$^{\underline{n}}$</u>. (i) The measure $d\mu = \dfrac{dx\,dy}{y^2}$ ($z = x+iy$) on $\mathfrak{H}$ is invariant under $SL_2(\mathbb{R})$

(ii) If $\Gamma \subset SL_2(\mathbb{Z})$ is a subgroup of finite index, then $\mu(\Gamma \backslash \mathfrak{H}) < \infty$.

<u>Proof</u> (i). <u>Naive</u>: check $d\mu$ invariant under $\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}$ (easy) and under $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ (fiddly)
— these generate the whole group.

Easier: enough to show the associated 2-form

$$\eta = \frac{dx \wedge dy}{y^2} = \frac{i\, dz \wedge d\bar z}{2(\operatorname{Im} z)^2}$$

is invariant.

(6-3) But if $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{R})$, $d\gamma(z) = \gamma'(z)\,dz = \frac{a(cz+d)-(az+b)c}{(cz+d)^2}\,dz = \frac{dz}{(cz+d)^2}$

and $\operatorname{Im}\gamma(z) = \frac{\operatorname{Im} z}{(cz+d)\overline{(cz+d)}}$ $\Rightarrow$ η is invariant.

(ii) Let $PSL_2(\mathbb{Z}) = \coprod_{i=1}^{M} \overline{\Gamma}\,\gamma_i$.

<u>Claim</u>: (a) $\mathcal{D}' \stackrel{\text{def}}{=} \bigcup_{i=1}^{M} \gamma_i \mathcal{D}$ is, up to a finite set, a fundamental set for $\overline{\Gamma}$.

(b) $i \neq j \Rightarrow \gamma_i \mathcal{D} \cap \gamma_j \mathcal{D}$ is finite.

Assuming this, then

$$\mu(\overline{\Gamma}\backslash \mathfrak{H}) \stackrel{(a)}{=} \mu(\mathcal{D}') = \mu\Big(\bigcup_{i=1}^{M} \gamma_i \mathcal{D}\Big) \stackrel{(b)}{=} \sum_{i=1}^{M} \mu(\gamma_i \mathcal{D})$$

$$= M\,\mu(\mathcal{D}) \qquad \text{as } \mu \text{ is invariant}$$

$$< M \int_{\substack{y \geq \sqrt{3}/2 \\ |x| \leq 2}} \frac{dx\,dy}{y^2} < \infty. \qquad \square$$

(Easy exercise : $\mu(\mathcal{D}) = \pi/3$).

<u>Proof of claim</u>: If $z \in \mathfrak{H}$ $\exists\, z_0$ (unique) $\in \mathcal{D}$ and $\gamma \in PSL_2(\mathbb{Z})$ st. $z = \gamma(z_0)$. Then $\gamma = \delta\gamma_i$ for some i, $\delta \in \overline{\Gamma}$ $\Rightarrow$ $z = \delta(\gamma_i(z_0))$. So every z is $\overline{\Gamma}$-equivalent to an element of $\mathcal{D}'$. If $\gamma_j(z_0')$ is equivalent to $\gamma_i(z_0)$ by $\overline{\Gamma}$ then $z_0' = z_0$ (as z_0', z_0 are $PSL_2(\mathbb{Z})$-equivalent), and so

$\gamma_j(z_0) = \delta'\gamma_i(z_0)$, $\delta' \in \overline{\Gamma}$. ie

$\gamma_j^{-1}\delta'\gamma_i \in$ (stabiliser of z_0 in $PSL_2(\mathbb{Z})$)

So if $z_0 \notin \{i, \rho\}$, $\gamma_j = \delta'\gamma_i \Rightarrow i = j$, $\delta' = 1$. So, apart from maybe the finite set of translates of i, ρ, no 2 elements of $\mathcal{D}'$ are $\overline{\Gamma}$-equivalent.

For (b), if $\gamma_i(z_0) = \gamma_j(z_1) \in \gamma_i\mathcal{D} \cap \gamma_j\mathcal{D}$ then $\gamma_j^{-1}\gamma_i(z_0) = z_1$. So $z_1 = z_0$ and either $\gamma_j = \gamma_i$ or $z_0 \in \{i, \rho\}$.

Useful finite index sgps of $SL_2(\mathbb{Z})$:-

$N \geq 1$ $\quad \Gamma(N) = \{\gamma \in SL_2(\mathbb{Z}) \mid \gamma \equiv I \bmod N\}$ "principal congruence subgroup of level N"

$\Gamma_0(N) = \{\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \mid c \equiv 0 \bmod N\}$

$\Gamma_1(N) = \{\gamma \in \Gamma_0(N) \mid d \equiv 1 \bmod N\}$

$\Gamma(1) = SL_2(\mathbb{Z})$. If $\Gamma(N) \subset \Gamma \subset SL_2(\mathbb{Z})$ for some N, say Γ is a congruence subgroup. These are the most interesting arithmetically (but most subgroups of f.i. aren't congruence sgps: in fact

$$\frac{\#\{\text{congr. subgps of index } M\}}{\#\{\text{all} \quad - \text{"} - \quad - \text{"} - \quad - \}} \longrightarrow 0 \quad \text{as} \quad M \to \infty. \quad).$$

Defn $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\mathbb{R})^+$, $z \in \mathfrak{H}$

$j(\gamma, z) := cz + d$ $\quad \left[\text{So} \quad \gamma \begin{pmatrix} z \\ 1 \end{pmatrix} = j(\gamma, z) \begin{pmatrix} \gamma(z) \\ 1 \end{pmatrix} \quad (*) \right]$

Propn (i) $j(\gamma\delta, z) = j(\gamma, \delta(z))\, j(\delta, z)$ $\quad (\Leftarrow (*))$.

[j is a 1-cocycle $GL_2(\mathbb{R}) \longrightarrow \{$invertible hol. fns. on $\mathfrak{H}\}$]

(ii) $j(\gamma^{-1}, z) = j(\gamma, \gamma^{-1}(z))^{-1}$ $\quad (\delta = \gamma^{-1}$ in (i)).

§5. Modular fns of level 1.

Defn. A ($\mathbb{C}$-valued) function f on $\mathfrak{H}$ is modular of weight $k \in \mathbb{Z}$

$\forall \gamma \in SL_2(\mathbb{Z})$, $\quad f(\gamma(z)) = (cz+d)^k f(z). = j(\gamma, z)^k f(z)$ $\quad (M)$

Remarks (i) take $\gamma = -I$. Then $f(z) = (-1)^k f(z)$. So if $k \notin 2\mathbb{Z}$, f is identically 0

Henceforth assume k even; (M) only depends on image of γ in $PSL_2(\mathbb{Z})$.

(ii). If $f(\gamma(z)) = j(\gamma, z)^k f(z)$ and $f(\delta(z)) = j(\delta, z)^k f(z)$

then $f((\gamma\delta)(z)) = f(\gamma(\delta(z))) = j(\gamma, \delta(z))^k f(\delta(z)) = j(\gamma, \delta(z))^k j(\delta, z)^k f(z)$

$= j(\gamma\delta, z)^k f(z)$.

So to check modularity, enough to check (M) for $\gamma = S, T$.

(7-2) 1^{st} example: <u>Eisenstein series</u>. $k \geq 4$ even.

$$G_k(z) = \sum_{c,d}{}' \frac{1}{(cz+d)^k} \overset{\text{def}}{=} \sum_{(c,d)\in\mathbb{Z}^2\setminus\{0,0\}} \frac{1}{(cz+d)^k}.$$

<u>Prop.</u> (i) $G_k : \mathfrak{h} \to \mathbb{C}$ is holomorphic & modular of wt k $\forall k \geq 4$.

(ii) $$G_k(z) = 2\zeta(k) + \frac{2(2\pi i)^k}{(k-1)!} \sum_{n\geq 1} \sigma_{k-1}(n) q^n \qquad q = e^{2\pi i z}$$

$$\sigma_{k-1}(n) = \sum_{1\leq d | n} d^{k-1}$$

<u>Proof</u> (i) Since $\{cz+d \mid c,d\in\mathbb{Z}\} \subset \mathbb{C}$ is a lattice, $\sum_{c,d}' \frac{1}{|cz+d|^\alpha}$ converges for $\alpha > 2$. So series for G_k is abs. cgt.

Let $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Then $$\frac{1}{(m\gamma(z)+n)^k} = \frac{1}{\left(m\frac{az+b}{cz+d}+n\right)^k} = \frac{(cz+d)^k}{(m'z+n')^k}, \quad (m' \ n') = (m \ n)\gamma$$

So $G_k(\gamma(z)) = (cz+d)^k G_k(z)$. Holomorphy follows from the expansion (ii):-

(ii) $$G_k(z) = 2\sum_{d=1}^{\infty} \frac{1}{d^k} + 2\sum_{c=1}^{\infty}\sum_{d\in\mathbb{Z}} \frac{1}{(cz+d)^k} \quad \text{as } k \text{ even.}$$

<u>Lemma</u>. $$\sum_{n\in\mathbb{Z}} \frac{1}{(z+n)^k} = \frac{(2\pi i)^k}{(k-1)!} \sum_{m=1}^{\infty} m^{k-1} q^m, \quad q = e^{2\pi i z} \quad \forall k \geq 2 \text{ if } \operatorname{Im}(z) > 0.$$

<u>Proof</u>. Recall $$\frac{1}{z} + \sum_{n=1}^{\infty}\left(\frac{1}{z-n} + \frac{1}{z+n}\right) = \pi\cot\pi z = \pi i \frac{e^{\pi i z}+e^{-\pi i z}}{e^{\pi i z}-e^{-\pi i z}}$$

$$= -\pi i\left(\frac{1+q}{1-q}\right) = -\pi i - 2\pi i \frac{q}{1-q} = -\pi i - 2\pi i \sum_{m=1}^{\infty} q^m.$$ Differentiate $(k-1)$ times. $\square$

$$\therefore G_k(z) = 2\zeta(k) + 2\sum_{c=1}^{\infty} \frac{(2\pi i)^k}{(k-1)!} \sum_{m=1}^{\infty} m^{k-1} q^{cm}$$

$$= 2\zeta(k) + 2\frac{(2\pi i)^k}{(k-1)!}\sum_{n=1}^{\infty} \underbrace{\left(\sum_{1\leq m|n} m^{k-1}\right)}_{=\sigma_{k-1}(n)} q^n$$ [writing $n = cm$] $\square$

Remarks. 1) Origin of G_k: elliptic functions.

Let $\Lambda = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2 \subset \mathbb{C}$ lattice ($\Leftrightarrow$ ω_1, ω_2 lin. indep. over $\mathbb{R}$)

$$\wp(u) = \frac{1}{u^2} + \sum_{0 \neq \omega \in \Lambda} \left[\frac{1}{(u-\omega)^2} - \frac{1}{\omega^2}\right]$$ is the Weierstrass $\wp$-function

$$\wp(u+\omega) = \wp(\omega) \quad \forall \omega \in \Lambda.$$

Laurent expansion of $\wp$ about $u=0$:

$$\wp(u) = \frac{1}{u^2} + \sum_{k \text{ even} \geq 4} (k-1)\, G_k(\Lambda)\, z^k$$

where $G_k(\Lambda) := \sum_{0 \neq \omega \in \Lambda} \frac{1}{\omega^k} = \frac{1}{\omega_2^k} \cdot G_k\left(\frac{\omega_1}{\omega_2}\right)$.

2) Recall from Ex. sheet:— $\zeta(k) = -\frac{(2\pi i)^k}{k!} B_k$ if $k \geq 2$ even.

So define the normalised Eisenstein series

$$E_k(z) = \frac{1}{2\zeta(k)} G_k(z) = 1 - \frac{2k}{B_k} \sum_{n=1}^{\infty} \sigma_{k-1}(n)\, q^n \quad ; \text{ rational coefficients.}$$

E.g.

$B_2 =$

$B_4 =$ $\qquad E_4 = 1 + 240 \sum \sigma_3(n) q^n$

$B_6 =$ $\qquad E_6 = 1 - 504 \sum \sigma_5(n) q^n$

$B_8 =$ $\qquad E_8 = 1 + 480 \sum \sigma_7(n) q^n$

$B_{10} =$ $\qquad E_{10} = 1 - 264 \sum \sigma_9(n) q^n$

$B_{12} =$ $\qquad E_{12} = 1 + \frac{65520}{691} \sum \sigma_{11}(n) q^n$

$B_{14} =$ $\qquad E_{14} = 1 - 24 \sum \sigma_{13}(n) q^n.$

Now define modular forms.

Defn. Let f be a complex function on $\mathfrak{H}$, such that $f(z+N) = f(z)$ for some $N > 0$. Say that f is

$\begin{cases} \text{(a) meromorphic at infinity} \\ \text{(b) holomorphic at infinity} \end{cases}$, respectively

if (i) f is holomorphic on $\{\operatorname{Im} z > Y\}$ for some $Y \geq 0$; and

(ii) the function on $\{0 < |q_N| < e^{-\pi Y/N}\}$

$$\tilde{f}(q_N) = f(z), \quad q_N = e^{2\pi i z/N}$$

(which is well-defined as $f(z+N) = f(z)$, and is holomorphic by (i)) is

$\begin{cases} \text{(a) meromorphic at } q_N = 0 \\ \text{(b) holomorphic at } q_N = 0 \end{cases}$, respectively.

Remark More concretely, (i), (ii) are equivalent to:-

$$f(z) = \sum_{n=-\infty}^{\infty} c_n e^{2\pi i n z/N} \quad \forall z \text{ with } \operatorname{Im}(z) > Y$$

and $\begin{cases} (a) & \exists N_0 \text{ st. } c_n = 0 \text{ for } n \leq -N_0, \text{ respectively} \\ (b) & c_n = 0 \ \forall n < 0. \end{cases}$

This series is called the q-expansion (= Fourier expansion) of f.

Def. A modular form of weight $k \in \mathbb{Z}$ (and level 1) is a holomorphic function $f : \mathfrak{H} \to \mathbb{C}$ s.t.

i) f is modular of weight k; and
ii) f is holomorphic at ∞

If ii) is replaced by ii') f is meromorphic at ∞, then say that f is a weak(ly) modular form of weight k.

Eg. Eisenstein series: G_k (∴ also E_k) is a modular form of weight k $\forall$ even $k \geq 4$.

If k is odd the only (weak) modular form of weight k is 0.

$\{$Modular forms of weight $k\} \overset{\text{defn}}{=\!=} M_k$ is a $\mathbb{C}$-vector space (obvious).

Next task: show $\dim_{\mathbb{C}} M_k < \infty$ and compute it.

Thm. Let f be a weak modular form of weight k, not identically 0

Then have degree formula:-

$$\sum_{z_0 \in \mathcal{D} \setminus \{i, \rho\}} \operatorname{ord}_{z=z_0} f + \frac{1}{2} \operatorname{ord}_{z=i} f + \frac{1}{3} \operatorname{ord}_{z=\rho} f + \operatorname{ord}_{\infty} f = \frac{k}{12}.$$

Here $\mathcal{D}$ = fundamental set for action of $SL_2(\mathbb{Z})$ on $\mathfrak{H}$ defined earlier.

$\operatorname{ord}_{z=z_0} f$ = order of vanishing of f at $z = z_0$.

$\operatorname{ord}_{\infty} f$ = least n for which $c_n \neq 0$ in q-expansion $f(z) = \sum_{n=-\infty}^{\infty} c_n q^n$.

Note: if $f \not\equiv 0$ then $\exists \varepsilon > 0$ st. $\tilde{f}(q) \neq 0$ for $0 < |q| < \varepsilon$ (by principal of isolated zeros).

∴ $\exists Y$ st. $f(z) \neq 0$ for $\operatorname{Im} z > Y$ ($Y = -\log \varepsilon / 2\pi$ in fact)

$\Rightarrow$ # of zeros of f in $\mathcal{D}$ is finite (so formula makes sense!).

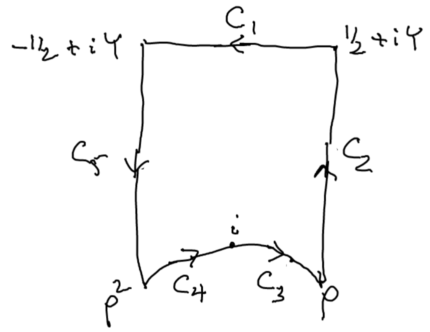
(9-1) Recall what we are proving.

Proof. Assume $f \neq 0$ on the boundary of $\mathcal{D}$ (i.e. on $\{|\operatorname{Re} z| = \frac{1}{2}, \operatorname{Im} z \geq \frac{\sqrt{3}}{2}\}$
$\cup \{|z|=1, |\operatorname{Re} z| \leq 1/2\}$).

and $f \neq 0$ for $\operatorname{Im} z \geq Y$, some $Y > 1$.

(So in particular, $\operatorname{ord}_{z=i} f = 0 = \operatorname{ord}_{z=\rho} f$).

Consider contour integral



$$\frac{1}{2\pi i}\int_C \frac{f'(z)}{f(z)}\,dz = \sum_{z_0 \in \mathcal{D}} \operatorname{ord}_{z=z_0} f$$

(by argument principle).

i) as $f(z+1) = f(z)$, $\int_{C_5} = -\int_{C_2}$.

ii) as $f(-1/z) = z^k f(z)$, $\dfrac{f'(-\frac{1}{z})}{f(-\frac{1}{z})}\cdot\dfrac{d(-1/z)}{dz} = \dfrac{d/dz\, f(-1/z)}{f(-1/z)} = \dfrac{k}{z} + \dfrac{f'(z)}{f(z)}$

and as S maps C_3 to C_4 (with opposite orientation),

$$\int_{C_4} \frac{f'(z)}{f(z)}\,dz = -\int_{C_3}\frac{f'(-1/z)}{f(-1/z)}\,d(-1/z) = -\int_{C_3}\left(\frac{k}{z} + \frac{f'(z)}{f(z)}\right)dz.$$

and so $\displaystyle\int_{C_3} + \int_{C_4}\frac{f'(z)}{f(z)}\,dz = -\int_{C_3}\frac{k}{z}\,dz = -\int_{\pi/2}^{\pi/3} k\, i\,d\theta = \frac{2\pi i\,k}{12}$ $\quad(z = e^{i\theta})$

iii) $f(z) = \tilde{f}(q)$ on $\{0 < |q| \leq e^{-2\pi Y}\}$, $q = e^{2\pi i z}$

and $\quad f'(z) = \tilde{f}'(q)\dfrac{dq}{dz}$, so

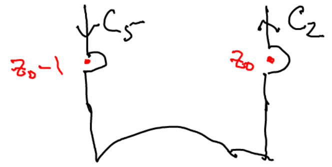
$$\int_{C_1}\frac{f'(z)}{f(z)}\,dz = -\int_{|q|=e^{-2\pi Y}}\frac{\tilde{f}'(q)}{f(q)}\cdot dq \qquad \text{(sign from orientations of paths)}$$

$$= -2\pi i \operatorname{ord}_\infty f.$$

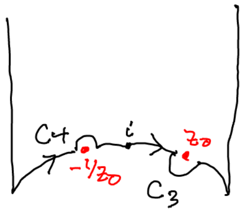
Combining, $\displaystyle\sum_{z \in \mathcal{D}} \operatorname{ord}_{z=z_0} f + \operatorname{ord}_\infty f = k/12.$

So have to consider the case of zeros along the boundary.

If we have a 0 at $z_0 \neq i$ or ρ, for consistently indent contour twice:-



(if $\operatorname{Re} z_0 = -1/2$) $\quad C_2$ = translate of C_5 by $z \mapsto z+1$
(with opposite orientation)



(of $|z| = 1$)

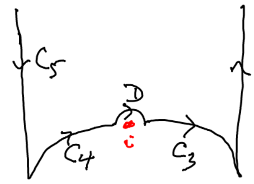
C_4 = transform of C_3 by $z \mapsto -1/z$ (with opposite orientation).

Then $\int_C f'/f \, dz = 2\pi i \sum_{z_0 \in \mathcal{D}} \mathrm{ord}_{z=z_0} f$

and $\int_{C_2} + \int_{C_5} = \int_{C_3} + \int_{C_4} = 0$ as before.

Remains to treat case of zero at $z_0 = i$ or ρ.

$\boxed{z_0 = i}$ Add indentation:

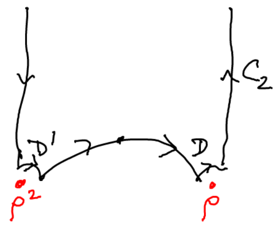


$\frac{f'}{f}$ has (at worst) simple pole at $z = i$; residue = $\mathrm{ord}_{z=i} f$.

Standard complex analysis $\Rightarrow$ $\int_D \frac{f'}{f} dz \longrightarrow -\pi i \, \mathrm{Res}_{z=i} \frac{f'}{f}$ as radius(D) $\to 0$

$$= -\frac{\mathrm{ord}_{z=i} f}{2} \times 2\pi i.$$

$\boxed{z_0 = \rho}$ Add 2 indentations:



(D, D' 1/6 of circle)

$f(z) = f(z+1) \Rightarrow \mathrm{ord}_\rho f = \mathrm{ord}_{\rho^2} f$, and

$\int_D \frac{f'}{f} dz \longrightarrow -\frac{\pi}{3} i \, \mathrm{Res}_{z=\rho} f'/f$ as radius $\to 0$

$\int_{D'} \longrightarrow -\frac{\pi}{3} i \, \mathrm{Res}_{z=\rho^2} f'/f$

ie. $\int_D + \int_{D'} f'/f \, dz \longrightarrow -\frac{\mathrm{ord}_{z=\rho} f}{3} \times 2\pi i.$ □

First applications. (i) $M_k = 0$ if $k < 0$.

Proof:- if $0 \neq f \in M_k$, then LHS of formula is ≥ 0, RHS $= k/12 < 0$.

(ii) $M_0 = \mathbb{C}$

Proof:- let $f \in M_0$, $f(2i) = c \in \mathbb{C}$. Then $g = f - c \in M_0$, $g(2i) = 0$. So LHS > 0 but RHS $= 0$. $\therefore$ $g \equiv 0$ ie. $f \equiv c$.

(iii) $M_2 = 0$. Proof: formula gives $1/6 = \frac{a}{2} + \frac{b}{3} + c$ for $a, b, c \geq 0$ integers; impossible.

Prop: $\forall k \geq 0$, $\dim_{\mathbb{C}} M_k \leq 1 + \frac{k}{12}$ (in particular, M_k is finite-dimensional)

Proof. Suppose (for some $d \geq 0$) $f_0, \dots, f_d \in M_k$. Let $z_1, \dots, z_d \in \mathcal{D} \setminus \{i, \rho\}$ be distinct. Then $\exists\, a_0, \dots, a_d \in \mathbb{C}$, not all zero, such that $g = \sum_{j=0}^{d} a_j f_j$ satisfies $g(z_1) = \dots = g(z_d) = 0$.

(9-3) If $d > k/12$ then applying the degree formula, see that $g \equiv 0$. So $f_0 \dots f_d$ are lin. dependent ie. $\dim_{\mathbb{C}} M_k \le d$. $\square$

<u>Coroll</u>. $M_k = \mathbb{C}.E_k$ for $k = 4, 6, 8, 10$.

<u>Proof</u>: for these k, $\dim M_k \le 1$; but $E_k \in M_k \setminus \{0\}$. $\square$

<u>Coroll</u>. $E_4^2 = E_8$, $E_4 E_6 = E_{10}$.

<u>Proof</u> $f \in M_k$, $g \in M_\ell \Rightarrow fg \in M_{k+\ell}$ (from definition).

So $E_4^2 \in M_8 = \mathbb{C} E_8$; comparing constant coeff. of q-expansion $\Rightarrow E_4 = E_8$.
Same for $k = 10$. $\square$

$\boxed{k=12}$ From the above, $\dim_{\mathbb{C}} M_{12} \le 2$.

E_4^3, $E_6^2 \in M_{12}$. And $E_4^3 - E_6^2 = (1 + 240q + \dots)^3 - (1 - 504q - \dots)^2$
$= 0 + 1728q + \dots$

So E_4^3, E_6^2 are linearly independent, so $\dim_{\mathbb{C}} M_{12} = 2$

Define $\Delta = \dfrac{E_4^3 - E_6^2}{1728} = 0 + q - 24q^2 \dots = \sum_{n \ge 1} \tau(n) q^n$ $\tau(n) \in \mathbb{Q}$, $\tau(1) = 1$

τ is <u>Ramanujan's τ-function</u>.

Recall description of M_k ($k \in 12$) & defn of Δ.

Have $\mathrm{ord}_\infty \Delta = 1$ so $\mathrm{ord}_{z=z_0} \Delta = 0$ $\forall z_0 \in \mathcal{D}$ (by divisor formula)

i.e. $\Delta(z) \neq 0$ $\forall z \in \mathcal{D}$ ($\therefore$ also $\forall z \in \mathfrak{H}$)

<u>Defn.</u> $S_k = \{ f \in M_k \mid \mathrm{ord}_\infty f > 0 \}$ - the space of <u>cusp forms</u>.
(so Δ is a cusp form of wt 12)

<u>Propn.</u> (i) $M_k = S_k \oplus \mathbb{C} E_k$ $\forall k > 2$.

(ii) $f \mapsto f\Delta$ is an <u>isomorphism</u> $M_k \xrightarrow{\sim} S_{k+12}$.

<u>Proof.</u> (i) obviously $E_k \notin S_k$, and if $f \in M_k$, q-expansion $\sum_{n=0}^{\infty} a_n q^n$,
$g = f - a_0 . E_k$ has zero const. coeff. so $\mathrm{ord}_\infty g > 0$, $g \in S_k$.

(ii) f modular of wt k $\Leftrightarrow$ $f\Delta$ modular of wt $k+12$, and these defines map $M_k \hookrightarrow S_{k+12}$.

If $g \in S_{k+12}$, then consider $f = g/\Delta$. As $\Delta(z) \neq 0$ $\forall z \in \mathfrak{H}$, f is holomorphic on $\mathfrak{H}$ and is modular of wt. k. Finally

$\mathrm{ord}_\infty f = \mathrm{ord}_\infty g - \mathrm{ord}_\infty \Delta = \mathrm{ord}_\infty g - 1 \geq 0$ as $g \in S_{k+12}$ $\Rightarrow f \in M_k$. $\square$

<u>Coroll</u> (i) $k \geq 0$, even. Then

$$\dim M_k = \begin{cases} \lfloor k/12 \rfloor & \text{if } k \equiv 2 \pmod{12} \\ 1 + \lfloor k/12 \rfloor & \text{otherwise} \end{cases}$$

(ii) A basis for M_k is

$$\{ E_4^a E_6^b \Delta^c \mid a, c \geq 0,\ b \in \{0,1\},\ 4a + 6b + 12c = k \} \quad (*_k)$$

In particular, every modular form may be written as a polynomial in E_4, E_6.

<u>Proof</u> (i) Let d_k = RHS of desired formula.

Clearly $\forall k \geq 0$, $d_{k+12} = d_k + 1$.

Also, by Propn, $\dim M_{k+12} = 1 + \dim S_{k+12} = 1 + \dim M_k$.

So enough to check formula for $0 \leq k < 12$, which we have already done.

(ii) True for $k \leq 10$.

Let $k \geq 12$. $\exists!\ a_1 \geq 0$, $b_1 \in \{0,1\}$ st. $2a_1 + 3b_1 = k/2$.

So $(*_k) = \left\{ E_4^a E_6^b \Delta^c \;\middle|\; \begin{array}{l} a \geq 0,\ b \in \{0,1\},\ c \geq 1 \\ 2a + 3b + 6c = k/2 \end{array} \right\} \cup \{ E_4^{a_1} E_6^{b_1} \}$

$$= (*_{k-12}).\Delta \cup \{E_4^{a_1} E_6^{b_1}\}$$

By induction on k, $(*_{k-12})$ is a basis for M_{k-12}, hence by proposition, $(*_{k-12})\Delta$ is a basis for S_k; and $M_k = S_k \oplus \mathbb{C}\, E_4^{a_1} E_6^{b_1}$. $\square$

More about Δ. Propn. (i) $\tau(n) \in \mathbb{Z}$ $\forall n \geq 1$.

(ii) $\tau(n) \equiv \sigma_{11}(n) \pmod{691}$

Proof (i) $1728\Delta = (1+240A_3(q))^3 - (1-504A_5(q))^2$

$= 720A_3(q) + 1008A_5(q) \pmod{1728}$ $\quad A_k(q) = \sum_{n\geq 1} \sigma_k(n) q^n$

$\therefore\ 12\Delta \equiv 5A_3(q) + 7A_5(q) \pmod{12}$ $\quad 1728 = 12^3$, $504 = 21.24$

$\Rightarrow 12\tau(n) \equiv 5(\sigma_3(n) - \sigma_5(n)) \pmod{12}$

$\equiv 5 \sum_{d|n} \underbrace{d^3(1-d^2)}_{\equiv 0 \bmod 12\ \forall d \in \mathbb{Z}} \pmod{12}$

(ii) $E_{12} = 1 + \frac{65520}{691} \sum_{n\geq 1}^{\infty} \sigma_{11}(n) q^n \in M_{12}$, so

$E_{12} = E_4^3 + \lambda\Delta$, some λ. Put $E_4^3 = 1 + \sum_{n\geq 1}^{\infty} b_n q^n$, $b_n \in \mathbb{Z}$

$\Rightarrow 65520\,\sigma_{11}(n) = 691 b_n + \mu\tau(n)$, $\mu = 691\lambda$.

$n=1$: $\sigma_{11}(n) = 1 = \tau(n) \Rightarrow \mu \in \mathbb{Z}$, $\mu \equiv 65520 \pmod{691}$.

$\Rightarrow \sigma_{11}(n) \equiv \tau(n) \pmod{691}$. $\square$

The modular function j.

Define $j(z) = \frac{E_4^3}{\Delta}$. Then: - j is holomorphic on $\mathfrak{H}$ (as $\Delta \neq 0$ on $\mathfrak{H}$)
- j is modular of weight 0 (i.e. invariant under $SL_2(\mathbb{Z})$)
- j is meromorphic at ∞, $\operatorname{ord}_\infty j = -1$ $(= \operatorname{ord}_\infty E_4^3 - \operatorname{ord}_\infty \Delta)$

ie. j is a weak modular fn of wt. 0

Defn. A modular function (of level 1) is a meromorphic function on $\mathfrak{H}$ which is modular of wt. 0 & meromorphic at ∞

The set of modular functions is a field

Thm (i) $j : SL_2(\mathbb{Z})\backslash\mathfrak{H} \longrightarrow \mathbb{C}$ is bijective.

(ii) every modular function is a rational function of j.

10-3

Proof (i) Enough to show: $\forall c \in \mathbb{C}$, $g(z) = j(z) - c$ has a unique zero in $\mathcal{D}$ (ignoring multiplicities).

Degree formula $\Rightarrow$ $\displaystyle\sum_{z_0 \in \mathcal{D} \setminus \{i, \rho\}} \mathrm{ord}_{z_0} g + \tfrac{1}{2}\mathrm{ord}_i g + \tfrac{1}{3}\mathrm{ord}_\rho g = -\mathrm{ord}_\infty g = 1$

So either: g has a simple zero at some $z_0 \in \mathcal{D} \setminus \{i, \rho\}$, no others in $\mathcal{D}$
or — " — double — " — $z = i$ — " —
— " — triple — " — $z = \rho$ — " — .

11-1 Recall statement of thm. Remark: q-exp. of j begins $\frac{1}{q}$.

Proof (ii): let f be a modular function. Suppose its poles in $\mathcal{D}$ are at $\{a_i\}$

Then for suitable $n_i \geq 0$, $g = \prod_i (j(z) - j(a_i))^{n_i} \cdot f$ is holomorphic on $\mathfrak{H}$; ie. is a weak MF of weight 0. If $\mathrm{ord}_\infty g \geq 0$ then $g \in M_0 = \mathbb{C}$ ie. $f \in \mathbb{C}(j)$. If not, let $\mathrm{ord}_\infty g = -m < 0$, and let

$$g = b_{-m} q^{-m} + \dots$$

be its q-expansion. Then $g - b_{-m} j^m$ has $\mathrm{ord}_\infty g \geq -m+1$, so repeating, $\exists$ polynomial $P(j)$ s.t. $g - P(j) \in M_0 = \mathbb{C}$ $\Rightarrow$ $f \in \mathbb{C}(j)$ $\square$.

Remarks (i) j has triple zero at $z = \rho$ since $E_4(\rho) = 0$ (Q4 of sheet #2)

$j - 1728 = \dfrac{E_6^2}{\Delta}$ has double zero at $z = i$ since $E_6(i) = 0$

(ii) $j = \dfrac{1}{q} + 744 + 196884q + \dots$

– the coefficients are related to representation of the monster group (= largest sporadic finite simple group).

Application to C analysis.

Picard's Little Thm: If $f : \mathbb{C} \to \mathbb{C}$ is a nonconstant entire function, then $\exists$ at most one $c \in \mathbb{C}$ such that $\forall z$, $f(z) \neq c$.

(e.g. $f(z) = \exp(z) + c$).

Proof (uses some topology).

Let $\mathcal{U} = \mathfrak{H} - \{\gamma(i), \gamma(\rho)\}$. Then $j : \mathcal{U} \twoheadrightarrow \mathbb{C} - \{0, 1728\}$ and as the zeroes of j' in $\mathcal{U}$ are simple, has a local inverse at each point of $\mathcal{U}$.

Since $PSL_2(\mathbb{Z})$ acts freely on $\mathcal{U}$ (stabilisers are trivial), algebraic topology $\Rightarrow$ $j : \mathcal{U} \to \mathbb{C} \setminus \{0, 1728\}$ is a covering space, 11-2 Now let $f : \mathbb{C} \longrightarrow \mathbb{C} \setminus \{a, b\}$ be holomorphic. Replacing f by $\lambda f + \mu$ may assume $\{a, b\} = \{0, 1728\}$. Then as $\mathbb{C}$ is simply-connected, $\exists$ $\tilde{f} : \mathbb{C} \longrightarrow \mathcal{U}$ s.t. $j \circ \tilde{f} = f$:

$$\begin{tikzcd} & \mathcal{U} \arrow[d, "j"] \\ \mathbb{C} \arrow[ur, dashed, "\tilde{f}"] \arrow[r, "f"'] & \mathbb{C} \setminus \{0, 1728\} \end{tikzcd}$$

Because j has a local inverse at each pt. of $\mathcal{U}$ (i.e. $j' \neq 0$), $\tilde{f}$ is holomorphic. But then the composite:

$$\mathbb{C} \xrightarrow{\tilde{f}} \mathcal{U} \subset \mathfrak{h} \xrightarrow{\sim} \{|w| < 1\}$$
$$z \longmapsto \frac{z - i}{z + i}$$

is a bounded entire function, so by Liouville is constant. Hence $\tilde{f}$, and so f, constant. $\square$