(1-1)

Modular forms & L-functions (M2019)

Intro. Big area of research in NT: Langlands programme relating:

"arithmetic" objects $\longleftrightarrow$ "analytic" objects.

eg: elliptic curves $\longleftrightarrow$ modular forms

$E: y^2 + y = x^3 - x \longleftrightarrow f(z) = q\prod_{n\geq 1}(1-q^n)^2(1-q^{11n})^2$ $\quad q = e^{2\pi i z}$, $\mathrm{Im}(z) > 0$

$= \sum_{n\geq 1} a_n q^n$

$p \neq 11$ prime: $\#E(\mathbb{F}_p) = 1 + p - a_p$ $\quad = \eta(z)^2\eta(11z)^2$

$\eta(z) = q^{1/24}\prod_{n\geq 1}(1-q^n)$ Dedekind eta function

$\eta(z+1) = e^{\pi i/12}\eta(z)$; $\eta(-1/z) = (z/i)^{1/2}\eta(z)$

Relation $E \longleftrightarrow f$ can also be expressed using L-series

$$L(E,s) = \sum_{n=1}^{\infty}\frac{a_n}{n^s}$$

- generalising Riemann zeta f. $\zeta(s) = \sum_{n=1}^{\infty}\frac{1}{n^s}$.

$\boxed{\text{TOOLS}}$

§1. Characters (fourier analysis) [NB no subtle analysis here].

G abelian topological group

A (unitary) character of G is a cts. hom^m $\chi: G \to U(1) = \{z \in \mathbb{C} \mid |z| = 1\}$

$\hat{G} := \{$characters of $G\}$ - group under multiplication.

Exs. $G = \mathbb{R}$; for $y \in \mathbb{R}$ let $\chi_y : \mathbb{R} \to U(1)$ be $\chi_y(x) = e^{2\pi i xy}$

$\chi_y \in \hat{G}$, $\chi_y \cdot \chi_{y'} = \chi_{y+y'}$

Fact: $\hat{G} = \{\chi_y \mid y \in \mathbb{R}\} \simeq \mathbb{R}$ (ex sheet)
$\chi_y \longleftrightarrow y$

$G = \mathbb{Z}$ (discrete topology): then $\hat{G} \xrightarrow{\sim} U(1)$
$\chi \longmapsto \chi(1)$.

$\mathbb{Z}/N\mathbb{Z}$ (" —): $\hat{G} \cong \mu_N = \{\zeta \in \mathbb{C}^\times \mid \zeta^N = 1\} \subset U(1)$
$\chi \mapsto \chi(1)$

$G = G_1 \times G_2 \Rightarrow \hat{G} \simeq \hat{G}_1 \times \hat{G}_2$ eg. $G = \mathbb{R}^n$; $\hat{G} = \{\chi_y : x \mapsto e^{2\pi i (x\cdot y)} \mid y \in \mathbb{R}^n\}$

$G = \mathbb{R}^\times = \{\pm 1\} \times \mathbb{R}^\times_{>0}$: $\hat{G} \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{R}$
$x \mapsto ((\mathrm{sgn}\, x)^\varepsilon \cdot |x|^{i\sigma}) \longleftarrow (\varepsilon, \sigma)$

[Rem: $\hat G$ has natural topology*, for which the evaluation maps $\hat G \xrightarrow{ev_g} U(1)$, $\chi \mapsto \chi(g)$ $(g \in G)$ are cts. This gives hom $G \longrightarrow \hat{\hat G}$, which is an isomorphism of top. gps. if G is locally compact: Pontryagin duality]

* compact-open.

Prop^n. G finite ab. gp. Then $\# \hat G = \# G$, and $G, \hat G$ are isomorphic (non-canonically).

Proof. $G \simeq \prod_{i=1}^{k} \mathbb{Z}/N_i\mathbb{Z} \Rightarrow \hat G \simeq \prod_{i=1}^{k} \mu_{N_i}$, and $\mu_N \simeq \mathbb{Z}/N\mathbb{Z}$ $\square$

Fourier transform. $f : \mathbb{R} \longrightarrow \mathbb{C}$ an L^1-function (ie. $\int_{-\infty}^{\infty} |f(x)|\,dx < \infty$)

$$\hat f(y) := \int_{-\infty}^{\infty} e^{-2\pi i xy} f(x)\,dx = \int \chi_y(x)^{-1} f(x)\,dx,$$ bded cts. fn. on $\mathbb{R} = \hat{\mathbb{R}}$.

Mainly interested in very special f: $f \in \mathcal{S}(\mathbb{R}) = \{ f \in C^\infty(\mathbb{R}) \mid \forall k, n,\ |x^n f^{(k)}(x)| \to 0$ as $|x| \to \infty \}$ (Schwartz space)

– then $\hat f \in \mathcal{S}(\mathbb{R})$ also, and have

Fourier inversion formula: $\hat{\hat f}(x) = f(-x)$ for $f \in \mathcal{S}(\mathbb{R})$.

Same holds for $\mathbb{R}^n$.

$G = \mathbb{R}/\mathbb{Z}$: Fourier transform is Fourier series: $\widehat{\mathbb{R}/\mathbb{Z}} = \{ \chi_n(x) = e^{2\pi i n x} \mid n \in \mathbb{Z} \} \simeq \mathbb{Z}$

$f : \mathbb{R}/\mathbb{Z} \to \mathbb{C}$; $c_n(f) = \int_{\mathbb{R}/\mathbb{Z}} \chi_n(x)^{-1} f(x)\,dx = \int_0^1 e^{-2\pi i n x} f(x)\,dx$

$$\hat f : \mathbb{Z} \longrightarrow \mathbb{C}, \quad n \longmapsto c_n(f)$$

Fourier inversion then is the statement: for suitable f (eg. $f \in C^\infty(\mathbb{R}/\mathbb{Z})$)

$$f(x) = \sum_{n \in \mathbb{Z}} c_n(f) \cdot \chi_n(x).$$

$G = \mathbb{Z}/N\mathbb{Z}$: $f : \mathbb{Z}/N\mathbb{Z} \longrightarrow \mathbb{C}$

$\hat f : \mu_N \longrightarrow \mathbb{C}$, $\hat f(\zeta) = \sum_{a \in \mathbb{Z}/N\mathbb{Z}} \zeta^{-a} f(a)$

Prop^n. $$f(x) = \frac{1}{N} \sum_{\zeta \in \mu_N} \zeta^x \hat f(\zeta).$$

Proof both sides linear in $f = \sum_{a \in \mathbb{Z}/N\mathbb{Z}} f(a)\delta_a$, $\delta_a(x) = \begin{cases} 1 & x = a \\ 0 & \neq \end{cases}$

$\therefore$ Enough to treat $f = \delta_a$.

$\hat f(\zeta) = \zeta^{-a}$, RHS $= 1/N \sum_{\zeta \in \mu_N} \zeta^{x-a}$.

(1-3) Result then follows from:-

<u>Lemma</u>. $k \in \mathbb{Z}$. Then $\displaystyle\sum_{\zeta \in \mu_N} \zeta^k = \begin{cases} N & k \equiv 0 \pmod N \\ 0 & \not\equiv \end{cases}$. $\square$

<u>Poisson summation formula</u>, on $\mathbb{R}^n$.

<u>Thm</u>. $f \in \mathcal{S}(\mathbb{R}^n)$. Then

$$\sum_{a \in \mathbb{Z}^n} f(a) = \sum_{b \in \mathbb{Z}^n} \hat{f}(b)$$

<u>Proof</u> Let $g(x) = \displaystyle\sum_{a \in \mathbb{Z}^n} f(x+a) \in C^\infty(\mathbb{R}^n/\mathbb{Z}^n)$ (as $f \in \mathcal{S}(\mathbb{R}^n)$).

Then $g(x) = \displaystyle\sum_{b \in \mathbb{Z}^n} c_b(g)\, e^{2\pi i b \cdot x}$, Fourier series.

$$c_b(g) = \int_{(\mathbb{R}/\mathbb{Z})^n} e^{-2\pi i b \cdot x} g(x)\, dx = \sum_{a \in \mathbb{Z}^n} \int_{[0,1]^n} e^{-2\pi i b \cdot x} f(x+a)\, dx$$

$$= \sum_{a \in \mathbb{Z}^n} \int_{a_1}^{a_1+1} \cdots \int_{a_n}^{a_n+1} e^{-2\pi i b \cdot x} f(x)\, dx = \int_{\mathbb{R}^n} e^{-2\pi i b \cdot x} f(x)\, dx = \hat{f}(b)$$

$$\therefore \sum_{a \in \mathbb{Z}^n} f(a) = g(0) = \sum_{b \in \mathbb{Z}^n} c_b(g) = \sum_{b \in \mathbb{Z}^n} \hat{f}(b).$$ $\square$

(2-1) First application: functional equation of Θ-function.

Thm. Let $\Theta(t) = \sum_{n\in\mathbb{Z}} e^{-\pi n^2 t}$ $(t>0)$

Then $\Theta(1/t) = t^{1/2}\,\Theta(t)$.

Proof. Let $g(x) = e^{-\pi x^2} \in \mathcal{S}(\mathbb{R})$.

Recall* that $\hat{g}(y) = g(y)$. Also:

- FT of $f(\alpha x)$ is $\alpha^{-1}\hat{f}(y/\alpha)$

So if $g_t(x) = e^{-\pi t x^2} = g(\sqrt{t}\,x)$, $\hat{g}_t(y) = t^{-1/2} g(y/\sqrt{t}) = t^{-1/2} e^{-\pi y^2/t}$.

Poisson summation $\Rightarrow$

$$\Theta(t) = \sum_{a\in\mathbb{Z}} g_t(a) = \sum_{b\in\mathbb{Z}} \hat{g}_t(b) = t^{-1/2}\,\Theta(1/t) \qquad \square$$

§2. Mellin transform, Γ-function.

Gamma function: $\Gamma(s) = \int_0^\infty e^{-y} y^s \frac{dy}{y}$, $s\in\mathbb{C}$, $\mathrm{Re}(s)>0$

– integral defines analytic fn. for $\mathrm{Re}(s)>0$.

Integration by parts $\Rightarrow$ $\Gamma(s+1) = s\Gamma(s)$

$\Gamma(1) = \int_0^\infty e^{-y}\,dy = 1$ $\Rightarrow$ $\Gamma(n) = (n-1)!$ $\forall n\in\mathbb{Z}_{>0}$.

Also $\Gamma(s) = \frac{1}{s(s+1)\cdots(s+N-1)}\Gamma(s+N)$ $\forall N\geq 1$, defining a meromorphic continuation to $\{\mathrm{Re}(s) > -N\}$ with simple pole at $s = 1-N$, residue $= \frac{(-1)^{N-1}}{(N-1)!}$

Other properties (see e.g. Ahlfors Ch. 2 §4):–

1. $\Gamma(s)^{-1} = e^{\gamma s} s \prod_{n\geq 1}\left(1-\frac{s}{n}\right) e^{s/n}$; in partic., $\Gamma(s)\neq 0$ $\forall s$
2. $\pi^{1/2}\Gamma(2s) = 2^{2s-1}\Gamma(s)\Gamma(s+1/2)$ ("duplication formula")

$\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin\pi s}$ ("reflection formula")

[$\gamma = \lim_{n\to\infty}\left(1+\frac{1}{2}+\cdots+\frac{1}{n} - \log n\right)$ Euler–Mascheroni constant]

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* Proof:– $\hat{f}(y) = \int_{-\infty}^{\infty} e^{-\pi x^2 - 2\pi i x y}\,dx = e^{-\pi y^2}\int_{-\infty}^{\infty} e^{-\pi(x+iy)^2}\,dx = f(y)\int_{-\infty+iy}^{\infty+iy} e^{-\pi z^2}\,dz$

– then shift path of integration to $[-\infty,\infty]$ and use $\int_{-\infty}^{\infty} e^{-\pi x^2}\,dx = 1$.

(2-2) Γ-fn. is special case of <u>Mellin transform</u> :-

$f : \mathbb{R}_{>0} \xrightarrow[\text{cts}]{} \mathbb{C}$, <u>rapidly decreasing</u> at ∞ ie. $y^n f(y) \to 0$ as $y \to \infty$ $\forall n$

<u>moderate growth</u> at 0 ie. for some $m \geq 0$, $|y^m f(y)|$ bounded as $y \to 0$.

$$M(f,s) := \int_0^\infty f(y)\, y^s \frac{dy}{y}$$

- converges to analytic fn. on $\{\mathrm{Re}(s) > m\}$.

<u>Proof.</u> $\int_r^R f(y)\, y^s \frac{dy}{y}$ is analytic $\forall s \in \mathbb{C}$; and conditions on f imply:-

$\int_R^\infty \to 0$ uniformly on compact subsets of $\mathbb{C}$

$\int_0^r \to 0$ — " — " — " — $\{\mathrm{Re}(s) > m\}$ $\square$

<u>Rem.</u> Suppose $\int_0^\infty |f| \frac{dy}{y} < \infty$. Then $M(f,s)$ converges for $s = i\sigma \in i\mathbb{R}$

and equals FT of $f \in L^1(\underset{G}{\mathbb{R}^\times_{>0}})$, $\hat{G} = \{y \mapsto y^{i\sigma}\} \simeq \mathbb{R}$.

- usually interested in f for which $\int |f| \frac{dy}{y} = \infty$.

Change of vars. $\Rightarrow$ $\boxed{M(f(\alpha y), s) = \alpha^{-s} M(f,s) \qquad (\alpha > 0)}$

<u>Ex. 0.</u> $f(y) = e^{-y}$ $(m = 0)$. Then $M(f,s) = \Gamma(s)$.

In general, Mellin transform converts <u>power series</u> into <u>Dirichlet series</u> :-

Let $f(z) = \sum_{n=1}^\infty a_n e^{2\pi i n z}$, $|a_n| < C \cdot n^K$; $f(iy) = \sum_{n=1}^\infty a_n e^{-2\pi n y}$

If $\mathrm{Re}(s) > \max(0, K+1)$ (so that first series below is covergent) then

$$(2\pi)^{-s}\Gamma(s) \sum_{n=1}^\infty \frac{a_n}{n^s} = \sum_n a_n (2\pi n)^{-s} M(e^{-y}, s) = \sum_n a_n M(e^{-2\pi n y}, s)$$

$$= M(f(iy), s)$$

<u>Ex 1:</u> (Riemann zeta function) $\zeta(s) = \sum_{n=1}^\infty \frac{1}{n^s}$ $(\mathrm{Re}(s) > 1)$

<u>Thm</u> $\Gamma(s)\zeta(s) = \int_0^\infty \frac{y^s}{e^y - 1} \frac{dy}{y} = M(f,s)$

$f(y) = 1/(e^y - 1)$.

(2-3)

__Prof__ $f(y) = \dfrac{e^{-y}}{1-e^{-y}} = \sum_{n\geq 1} e^{-ny}$ if $y > 0$

- rapidly decreasing at ∞
- $f(y) \sim \dfrac{1}{y}$ as $y \to 0$ $(m = 1)$

$$\therefore \quad M(f, s) = \sum_{n\geq 1} M(e^{-ny}, s) = \sum_{n\geq 1} n^{-s} M(e^{-y}, s) = \Gamma(s) \sum_{n\geq 1} \frac{1}{n^s} \qquad \square$$

(3-1) Recall from last time:

$$\Gamma(s)\zeta(s) = M(f,s), \qquad f(y) = \frac{1}{e^y - 1}.$$

Coroll 1. $\zeta(s)$ has mero. ctn. to $\mathbb{C}$ with simple pole at $s=1$, residue $=1$, as only singularity.

Proof

$$M(f,s) = \int_0^\infty \frac{y^s}{e^y-1}\frac{dy}{y} = \int_1^\infty + \int_0^1 = M_\infty + M_0 \quad \text{say.}$$

$$M_\infty = \int_1^\infty \frac{y^s}{e^y-1}\frac{dy}{y}$$ is entire (holomorphic $\forall s \in \mathbb{C}$).

For any $N > 0$, $f(y) = c_{-1}y^{-1} + c_0 + \dots + c_{N-1}y^{N-1} + y^N g_N(y)$, (*)

$g_N \in C^\infty(\mathbb{R})$.

$$\Rightarrow M_0 = \underbrace{\sum_{n=-1}^{N-1} c_n \int_0^1 y^{s+n-1}\,dy}_{\overset{\shortparallel}{\sum_{n=0}^{N} \frac{c_{n-1}}{s-(1-n)}}} + \underbrace{\int_0^1 g_N(y)\,y^{s+N-1}\,dy}_{\text{holomorphic for } Re(s) > -N}$$

$\therefore$ $(2\pi)^{-s}\Gamma(s)\zeta(s)$ has AC to $\{Re(s) > -N\}$ with at worst simple poles at $s = 1-n \in \{1-N, 2-N, \dots, 0, 1\}$, residue $= c_{n-1}$ (any $N > 0$).

$\Gamma(s)$ has simple pole at $s \in \{0,-1,-2,\dots\}$ $\Rightarrow$ $\zeta(s)$ has AC with only singularity at $s=1$.

(*) rem. The infinite series only converges for $|y| < 1$.

The Bernoulli numbers $B_n \in \mathbb{Q}$ $(n \geq 0)$ are given by

$$\sum_{n=0}^\infty B_n \frac{t^n}{n!} = \frac{t}{e^t-1} = \left(1 + \frac{t}{2!} + \frac{t^2}{3!} + \dots\right)^{-1} = 1 - \frac{1}{2}t + \frac{1}{6}\cdot\frac{t^2}{2!} \dots .$$

So $B_0 = 1$, $B_1 = -1/2$, $B_2 = 1/6$, ..., $B_{12} = -691/2730 = -691/2\cdot3\cdot5\cdot7\cdot13$

3-2

Prop^n $B_n = 0$ if n is odd and ≥ 3.

Proof. $f(t) = \dfrac{t}{e^t - 1} + \dfrac{t}{2} = \displaystyle\sum_{\substack{n\geq 0\\ n\neq 1}} B_n \frac{t^n}{n!} = \frac{t}{2}\,\frac{e^t+1}{e^t-1} = f(-t)$. $\square$

Corollary 2 $\zeta(0) = -\tfrac12 = B_1$, and $\forall n\geq 1$, $\zeta(1-n) = -\dfrac{B_n}{n}$

In particular, $\zeta(1-n)\in\mathbb{Q}$ $\forall n\geq 1$ and $=0$ if n is odd >1.

Proof $\operatorname{Res}_{s=1-n} \Gamma(s)\zeta(s) = c_{n-1}$, $\displaystyle\sum_{n\geq -1} c_n y^n = \frac{1}{e^y - 1} \Rightarrow c_{n-1} = \frac{B_n}{n!}$

$\boxed{s=1}$ $\Gamma(1) = 1 \Rightarrow \zeta$ simple pole, residue $= c_{-1} = 1$

$\boxed{s = 1-n \leq 0}$ $\operatorname{Res}_{s=1-n}\Gamma(s) = \dfrac{(-1)^{n-1}}{(n-1)!}$

$$\therefore\ \zeta(1-n) = \left(\operatorname{Res}_{s=1-n}\Gamma(s)\right)^{-1}\cdot c_{n-1} = (-1)^{n-1}\frac{B_n}{n} = \begin{cases} B_1 & n=1\\ -B_n/n & n \text{ even} >0 \\ 0 & n \text{ odd} > 0\end{cases}$$ $\square$

The functional equation.

Thm Let $Z(s) = \pi^{-s/2}\Gamma(s/2)\zeta(s)$. Then $Z(s)$ has a simple pole at $s=1, 0$, residues $1, -1$, and $Z(s) = Z(1-s)$.

Proof Recall $\Theta(y) = \displaystyle\sum_{n\in\mathbb{Z}} e^{-\pi n^2 y} = 1 + 2\sum_{n\geq 1} e^{-\pi n^2 y} = y^{-1/2}\Theta(1/y)$

So $\Theta(y) \to 1$ as $y\to\infty$ and $\Theta(y) - 1$ is rapidly decreasing there.

and $\Theta(y) \sim y^{-1/2}$ as $y\to 0$ $(\alpha = 1/2)$.

Then $M\left(\dfrac{\Theta(y)-1}{2}, \dfrac{s}{2}\right) = M\left(\displaystyle\sum_{n\geq 1} e^{-\pi n^2 y}, s/2\right) = \sum_{n\geq 1}(\pi n^2)^{-s/2}\Gamma(s/2) = Z(s)$

So $2Z(s) = \displaystyle\int_0^\infty [\Theta(y) - 1]\, y^{s/2}\frac{dy}{y} = \int_0^1 + \int_1^\infty$ $(\operatorname{Re}(s)>1)$

$$\int_0^1[\Theta(y)-1]\,y^{s/2}\frac{dy}{y} = \int_0^1[\Theta(y) - y^{-1/2}]\,y^{s/2}\frac{dy}{y} + \int_0^1 y^{(s-1)/2} - y^{s/2}\,\frac{dy}{y}$$

$$= \int_0^1 (y^{-1/2}\Theta(1/y) - y^{-1/2})\,y^{s/2}\frac{dy}{y} + 2\left[\frac{y^{(s-1)/2}}{s-1} - \frac{y^{s/2}}{s}\right]_0^1$$

$$= \int_1^\infty [\Theta(y) - 1]\,y^{(1-s)/2}\frac{dy}{y} + \frac{2}{s-1} - \frac{2}{s}$$

$$\therefore\ Z(s) = \underbrace{\frac12\int_1^\infty (\Theta(y)-1)(y^{s/2} + y^{(1-s)/2})\frac{dy}{y}}_{\text{entire fn. of } s} + \frac{1}{s-1} - \frac1s$$

– visibly $Z(1-s) = Z(s)$. $\square$

4-1

§3. Dirichlet L-functions.

Terminology: Dirichlet series : series of the form $\sum_{n\geq 1} \frac{a_n}{n^s}$ (+ growth condition on a_n)

Fix some modulus $N \geq 1$

Dirichlet character (mod N): any character $\chi : (\mathbb{Z}/N\mathbb{Z})^\times \longrightarrow \mathbb{C}^\times$

[If $N=1$, $\mathbb{Z}/N\mathbb{Z} = \{0\} = (\mathbb{Z}/N\mathbb{Z})^\times$, so just one character, namely $\chi(0) = 1$]
$\qquad\qquad\qquad = \{1\}$

Def. $L(\chi, s) = \sum_{\substack{n\geq 1 \\ (n,N)=1}} \chi(n) n^{-s}$, Dirichlet L-function (converges absolutely if $\mathrm{Re}(s) > 1$)
[or $L(s,\chi)$]

Basic properties:

- Euler product $L(\chi, s) = \prod_{p\nmid N} \frac{1}{1-\chi(p)p^{-s}} = \prod_{p \nmid N} (1 + \chi(p)p^{-s} + \chi(p^2)p^{-2s} + \dots)$

(same as $\zeta(s)$ since $\chi(mn) = \chi(m)\chi(n)$).

- analytic continuation. More generally, let $\psi : \mathbb{Z}/N\mathbb{Z} \longrightarrow \mathbb{C}$ be any periodic function on $\mathbb{Z}$. Put $L(\psi, s) = \sum_{n\geq 1} \psi(n) n^{-s}$, cvgs for $\mathrm{Re}(s) > 1$ (as ψ bounded)

Then $\Gamma(s) L(\psi, s) = \sum_{n\geq 1} \psi(n) M(e^{-ny}, s) = M(f, s)$ with

$$f(y) = \sum_{n\geq 1} \psi(n) e^{-ny} = \sum_{n=1}^{N} \sum_{r=0}^{\infty} \psi(n) e^{-(n+Nr)y}$$

$$= \sum_{n=1}^{N} \psi(n)\, e^{-ny}/1-e^{-Ny} = \sum_{n=1}^{N} \psi(n) \frac{e^{(N-n)y}}{e^{Ny}-1}$$

$y \to \infty$: $f(y) = O(e^{-y})$ so rapidly decreasing.

$y \to 0$: $f(y) = \sum_{n=1}^{N} \frac{\psi(n)}{Ny} + g(y)$, g cts at $y=0$.

$\therefore$ $\Gamma(s) L(\psi, s) = M_\infty(s) + M_0(s)$ (just as for $\zeta(s)$)

$M_\infty(s) = \int_1^\infty f(y) y^s \frac{dy}{y}$ analytic $\forall s$

$$M_0(s) = \int_0^1 f(y) y^s \frac{dy}{y} = \underbrace{\sum_{n=1}^{N} \int_0^1 \frac{\psi(n)}{N} y^{s-2} dy}_{= \left(\frac{1}{N}\sum_{n\in\mathbb{Z}/N\mathbb{Z}} \psi(n)\right)\cdot \frac{1}{s-1}} + \underbrace{\int_0^1 g(y)\, y^{s-1}\, dy}_{\text{analytic for } \mathrm{Re}(s) > 0}$$

Since Γ analytic and $\neq 0$ for $\mathrm{Re}(s) > 0$, deduce part (i) of :-

4-2

Thm (i) For any $\psi : \mathbb{Z}/N\mathbb{Z} \to \mathbb{C}$, $L(\psi, s)$ has a meromorphic continuation to $\{s \mid Re(s) > 0\}$, holomorphic for $s \neq 1$. At $s=1$ it has at worst a simple pole with residue $= \frac{1}{N} \sum_{n \in \mathbb{Z}/N\mathbb{Z}} \psi(n)$.

(ii) Let $\chi : (\mathbb{Z}/N\mathbb{Z})^\times \to \mathbb{C}^\times$ be a Dirichlet character mod N. Then:-

- if $\chi \neq \chi_0$ (where χ_0 is the trivial character $\chi_0(n) = 1$), $L(\chi, s)$ is analytic at $s=1$
- $\operatorname{Res}_{s=1} L(\chi_0, s) = \frac{\varphi(N)}{N} = \prod_{p \mid N} (1 - \frac{1}{p})$

__Proof of (ii)__ Let $H = \ker \chi \subset (\mathbb{Z}/N\mathbb{Z})^\times$, so $\operatorname{im}(\chi) = \mu_d$, $d = \varphi(N)/\# H$.

$$\operatorname{Res}_{s=1} L(\chi, s) = \frac{1}{N} \sum_{n \in (\mathbb{Z}/N\mathbb{Z})^\times} \chi(n) = \frac{\# H}{N} \sum_{n \in (\mathbb{Z}/N\mathbb{Z})^\times / H} \chi(n)$$

$$= \frac{\varphi(N)}{N d} \sum_{\zeta \in \mu_d} \zeta = \begin{cases} 0 & \text{if } \chi \neq \chi_0 \quad (\Leftrightarrow d > 1) \\ \frac{\varphi(N)}{N} & \text{if } \chi = \chi_0 . \end{cases}$$ □

__Remarks__ i) just as for $\zeta(s)$ can get AC to entire complex plane.

ii) Note that $L(\chi_0, s) = \prod_{p \nmid N} \frac{1}{1 - p^{-s}} = \prod_{p \mid N} (1 - \frac{1}{p^s}) . \zeta(s)$

__Dirichlet's Thm on primes in AP__ $1 \leq a < N$, $(a, N) = 1$

Then $\exists \infty$ many primes p with $p \equiv a \pmod N$.

__Remarks:__ i) there are some cases (eg $N=4$) where can do this by elementary methods. But even in these cases, L-functions give a much stronger result:-

$$\frac{\sum_{p \equiv a \bmod N} p^{-x}}{\sum_{\text{all } p} p^{-x}} \longrightarrow \frac{1}{\varphi(N)} \quad \text{as } x \to 1+.$$

So in some sense, $\frac{1}{\varphi(N)}$ of all primes are $\equiv a \pmod N$. ("Analytic density")

In fact we can prove (somewhat harder) the 'naive' statement

$$\frac{\# \{p \leq X, \ p \equiv a \pmod N\}}{\# \{p \leq X\}} \longrightarrow \frac{1}{\varphi(N)} \quad \text{as } X \to \infty$$

4-3 Key fact (prove later): If $\chi \neq \chi_0$ then $L(\chi, 1) \neq 0$.

__Lemma.__ (i) $\sum_p p^{-x} \sim -\log(x-1)$ as $x \to 1+$.

(ii) $\chi \neq \chi_0$. Then $\sum_p \chi(p) p^{-x}$ is bounded as $x \to 1+$

__Proof__ For any χ and N, $x>1$, and for some choice of logarithm on LHS,

$$\log L(\chi, s) = \sum_{p \nmid N} -\log(1-\chi(p)p^{-x})$$

$$= \sum_{p\nmid N} \sum_{r \geq 1} \frac{\chi(p)^r p^{-rx}}{r}$$

$$\Rightarrow \left| \log L(\chi,s) - \sum_{p\nmid N} \frac{\chi(p)}{p^x} \right| \leq \sum_{p\nmid N} \sum_{r\geq 2} p^{-rx}$$

$$= \sum_{p\nmid N} p^{-2x}/1-p^{-x} = \sum_{p\nmid N} \frac{1}{p^x(p^x-1)} \leq \sum_{n \geq 2} \frac{1}{n(n-1)} = 1.$$

i.e. $\sum_{p\nmid N} \frac{\chi(p)}{p^x} = \log L(\chi,s) + O(1)$.

i) $\chi = \chi_0$. $L(\chi_0, x) = \prod_{p \mid N} (1-p^{-x})\zeta(x)$, $\left|\zeta(x) - \frac{1}{x-1}\right|$ bounded as $x \to 1+$

$\Rightarrow \log L(\chi_0, x) \sim \log \zeta(s) \sim -\log(x-1)$ as $x \to 1+$.

ii) $\chi \neq \chi_0$. $L(\chi, s) \neq 0 \Rightarrow \operatorname{Log} L(\chi, s)$ bounded as $x \to 1+$. □

__Proof of Dirichlet's Thm.__ For finite G, $a \in G$, the delta function

$$\delta_a(x) = \begin{cases} 1 & x = a \\ 0 & x \neq a \end{cases}$$

has Fourier transform $\chi(a)^{-1}$ (obviously), so Fourier inversion gives

$$\delta_a(x) = \frac{1}{\# G} \sum_{\chi \in \hat{G}} \chi(a)^{-1} \chi(x).$$

Take $G = (\mathbb{Z}/N\mathbb{Z})^\times$:

$$\sum_{p \equiv a \pmod N} p^{-x} = \frac{1}{\varphi(N)} \sum_{\chi \in \widehat{(\mathbb{Z}/N\mathbb{Z})^\times}} \left(\sum_{p\nmid N} \chi(a)^{-1}\chi(p)\, p^{-x} \right)$$

Lemma $\Rightarrow$ for $\chi = \chi_0$, inner $\sum_{p\nmid N} = \sum_{p \nmid N} p^{-x} \sim -\log(1-x)$ as $x \to 1+$.

and remaining terms $\chi \neq \chi_0$ are bounded as $x \to 1+$.

$\therefore$ LHS $\sim \frac{1}{\varphi(N)} \cdot -\log(1-x)$ as $x \to 1+$, and $\therefore$ $\{p \equiv a \pmod N\}$ is infinite □