Modular forms and L-functions (Michaelmas 2019) — example sheet #4

Comments/corrections to a.j.scholl@dpmms.cam.ac.uk

1. (More on Dirichlet characters). Let $\chi\colon (\mathbb{Z}/N\mathbb{Z})^\times \to \mathbb{C}^\times$ be a Dirichlet character, where $N > 1$ with prime factorisation $N = \prod_{1\le i\le k} p_i^{r_i}$, $r_i \ge 1$.

   (i) Show that there are unique characters $\chi_i\colon (\mathbb{Z}/p_i^{r_i}\mathbb{Z})^\times \to \mathbb{C}^\times$ such that $\chi(n) = \prod_i \chi_i(n \bmod p_i^{r_i})$.

   (ii) Show that for each i there exists a minimal $0 \le s_i \le r_i$ and a unique $\chi_i'\colon (\mathbb{Z}/p_i^{s_i}\mathbb{Z})^\times \to \mathbb{C}^\times$ such that χ_i factors as $\chi_i(n) = \chi_i'(n \bmod p_i^{s_i})$.

   (iii) Deduce that there exists a unique least positive divisor M of N such that χ factors as $\chi(n) = \chi'(n \bmod M)$, for some (necessarily unique) character χ' mod M. If $M = N$ we say that χ is *primitive of conductor* N. In general, we say that χ' is the primitive character attached to χ, and that M is the conductor of χ.

   (iv) With the notation of (iii), show that

$$L(\chi, s) = \prod_{p\mid N,\ p\nmid M} (1 - \chi'(p)p^{-s})L(\chi', s).$$

   (v) Determine all primitive Dirichlet characters that are quadratic (i.e. $\chi^2 = 1$), and their conductors. [First determine all quadratic characters of $(\mathbb{Z}/p^r\mathbb{Z})^\times$.]

   (vi) Show that if χ is a primitive quadratic Dirichlet character mod N and N is even, then for all $x \in (\mathbb{Z}/N\mathbb{Z})^\times$, $\chi(x + N/2) = -\chi(x)$.

   (vii) Recall that if K is a quadratic field, then χ_K is the unique quadratic Dirichlet character mod $|d_K|$ such that, if $(p, d_K) = 1$ then p splits in K iff $\chi_K(p) = +1$. Show that χ_K is primitive.

   (viii) Show that if $\chi \ne 1$ is a primitive quadratic Dirichlet character of conductor N, then there is a unique quadratic field K, with $d_K = \pm N$, such that $\chi = \chi_K$.

2. Use parts (vi) and (vii) of the previous question to show that if K is an imaginary quadratic field with even discriminant $d_K \ne -4$, then

$$h_K = \frac{1}{2} \sum_{\substack{0<n<|d_K|/2 \\ (n,d_K)=1}} \chi_K(n).$$

3. Use the functional equation of the Epstein zeta function to show that if, for a quadratic field K, we define

$$Z_K(s) = \begin{cases} \pi^{-s}\Gamma(s/2)^2\zeta_K(s) & (K \text{ real}) \\ 2(2\pi)^{-s}\Gamma(s)\zeta_K(s) & (K \text{ complex}), \end{cases}$$

   then $Z_K(s) = |d_K|^{1/2-s}\, Z_K(1 - s)$.

4. Let $E_2(z) = 1 - 24\sum_{n=1}^\infty \sigma_1(n)q^n = (3/\pi^2)G_2(z)$ as in sheet 2, Q8. Use the infinite product for $\Delta(z)$ to prove that

$$E_2(-1/z) = z^2E_2(z) + \frac{12z}{2\pi i}.$$

   Deduce that the function $G_2^*(z) = G_2(z) - \pi/\mathrm{Im}(z)$ is modular of weight 2.

5. Define, for $a, b \in \mathbb{R}$ and $t > 0$

$$\theta(t; a, b) = \sum_{n\in\mathbb{Z}} e^{-\pi(n+a)^2t+2\pi i(n+a/2)b}.$$

   Use the Poisson summation formula to compute $\theta(1/t; a, b)$.