Modular forms and L-functions (Michaelmas 2019) — example sheet #1

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1. Prove that if $f(x) = e^{-\pi x^2}$ then $\widehat{f}(x) = f(x)$.

2. Let G be a finite abelian group, and χ, χ' characters of G. Show that

$$\sum_{g\in G} \overline{\chi(g)}\chi'(g) = \begin{cases} 0 & \text{if } \chi \neq \chi' \\ \#G & \text{if } \chi = \chi' \end{cases}$$

3. Determine all continuous homomorphisms from $\mathbb{R}$ to $\mathbb{C}$, and show that $\widehat{\mathbb{R}} = \{\chi_y \colon x \mapsto e^{2\pi i x y} \mid y \in \mathbb{R}\}$. Show also that every continuous homomorphism $\chi\colon \mathbb{R}^\times_{>0} \to \mathbb{C}^\times$ is of the form $\chi(x) = x^s$ for some $s \in \mathbb{C}$.

4. Use the duplication and reflection formulae for $\Gamma(s)$ to show that the functional equation of the ζ-function may be written as

$$\zeta(1-s) = 2(2\pi)^{-s} \cos\frac{\pi s}{2}\Gamma(s)\zeta(s).$$

5. Recall that $\Gamma(s)\zeta(s)$ is the Mellin transform of $1/(e^y - 1)$.

   (i) Let $Y > 0$. Show that at $s \to 1$,

$$\int_Y^\infty \frac{y^s}{e^y - 1}\frac{dy}{y} \to -\log(1 - e^{-Y}).$$

   (ii) Writing $1/(e^y - 1) = 1/y + g(y)$, show that as $s \to 1$,

$$\int_0^Y \frac{y^s}{e^y - 1}\frac{dy}{y} = \frac{1}{s-1} + \log Y + \int_0^Y g(y)\,dy + O(s-1).$$

   (iii) Letting $Y \to 0$, deduce that $\Gamma(s)\zeta(s) - 1/(s-1)$ vanishes at $s = 1$.

   (iv) Now use the result of question 4 to show that $\zeta'(0) = -(1/2)\log 2\pi$.

6. Evaluate $\zeta(2k)$ is terms of Bernoulli numbers. Deduce that $(-1)^{k-1}B_{2k} > 0$ for every $k \geq 1$. (It is not easy to prove this directly from the generating function definition!)

7. Define for $c \in \mathbb{R}$ the functions

$$\theta(t;c) = \sum_{n\in\mathbb{Z}} e^{-\pi(n+c)^2 t}, \qquad \theta^*(t;c) = \sum_{n\in\mathbb{Z}} e^{-\pi n^2 t + 2\pi i n c}.$$

   Use the Poisson summation formula to show that

$$\theta(t;c) = t^{-1/2}\theta^*(1/t;c).$$