$\boxed{\text{Q1}}$ i) By Chinese remainder theorem, $(\mathbb{Z}/N\mathbb{Z})^\times \xrightarrow{\sim} \prod_i (\mathbb{Z}/p_i^{r_i}\mathbb{Z})^\times$.

So $\chi(n) = \prod_i \chi_i(n \bmod p_i^{r_i})$ where χ_i is the composite

$$\chi_i : (\mathbb{Z}/p_i^{r_i}\mathbb{Z})^\times \hookrightarrow (\mathbb{Z}/N\mathbb{Z})^\times \xrightarrow[\chi]{} \mathbb{C}^\times.$$

ii) Let $H_s = \{ x \in (\mathbb{Z}/p_i^{r_i}\mathbb{Z})^\times \mid x \equiv 1 \pmod{p_i^s} \} \subset (\mathbb{Z}/p_i^{r_i}\mathbb{Z})^\times$

and $s_i = \min\{ s \geq 0 \mid \chi_i|_{H_s} = 1 \}$.

Then χ_i factors through the quotient $(\mathbb{Z}/p_i^{r_i}\mathbb{Z})^\times / H_{s_i} \cong (\mathbb{Z}/p_i^{s_i}\mathbb{Z})^\times$

and the resulting χ_i' has the required properties.

iii) Let $M = \prod p_i^{s_i}$. Then χ factors uniquely as:-

$$(\mathbb{Z}/N\mathbb{Z})^\times \xrightarrow[\bmod M]{} (\mathbb{Z}/M\mathbb{Z})^\times \xrightarrow[\chi' = \prod \chi_i']{} \mathbb{C}^\times$$

and clearly M is the minimal divisor of N with this property.

iv) $$L(\chi', s) = \prod_{p \nmid M} (1 - \chi'(p) p^{-s})^{-1}$$

$$L(\chi, s) = \prod_{p \nmid N} (1 - \chi(p) p^{-s})^{-1} = \prod_{p \nmid N} (1 - \chi'(p) p^{-s})^{-1}$$

v) <u>p odd</u>. As $\ker[(\mathbb{Z}/p^r\mathbb{Z})^\times \longrightarrow (\mathbb{Z}/p\mathbb{Z})^\times]$ has order p^{r-1}, every quadratic character mod p^r factors through a character mod p, so the only (nontrivial) quadratic character has conductor p and is the Legendre symbol $\chi(x) = \left(\frac{x}{p}\right) \equiv x^{(p-1)/2} \pmod p$.

<u>$p = 2$</u>. Then $(\mathbb{Z}/2^r\mathbb{Z})^\times$ is generated by -1 and 5, so

$$(\mathbb{Z}/2^r\mathbb{Z})^\times/(\text{squares}) \xrightarrow{\sim} \begin{cases} (\mathbb{Z}/8\mathbb{Z})^\times & \text{if } r \geq 3 \\ (\mathbb{Z}/2^r\mathbb{Z})^\times & \text{if } r < 3. \end{cases}$$

There is no primitive quadratic character mod 2.

Mod 4: just one: $\chi_4 : x \longmapsto \pm 1$ if $x \equiv \pm 1 \pmod 4$

Mod 8: just two: $\chi_8 : x \longmapsto \begin{cases} +1 & x \equiv \pm 1 \pmod 8 \\ -1 & \pm 3 \end{cases}$

and $\chi_4 \chi_8 : x \longmapsto \begin{cases} +1 & x \equiv 1, 3 \pmod 8 \\ -1 & -1, -3 \end{cases}$

Q1 (v) ctd

General conductor. Combining the above with (i – iii), see that if χ is a primitive quadratic character mod N, then $N = 2^r N_1 > 1$ where $r \in \{0, 2, 3\}$, N_1 is odd and squarefree, and

$$\chi = \chi_2 \prod_{p \mid N_1} \left(\frac{\cdot}{p}\right), \quad \chi_2 = \begin{cases} \text{trivial} & r=0 \\ \chi_4 & 2 \\ \chi_8 \text{ or } \chi_4\chi_8 & 3. \end{cases}$$

(vi) By (v), $2 \mid N \Rightarrow 4 \mid N$, and if $(x, N) = 1$ then x is odd, hence $x(1 + N/2) \equiv x + \frac{N}{2} \pmod N$.

χ primitive $\Rightarrow \chi(1+N/2) = -1$ as $\ker\left[(\mathbb{Z}/N\mathbb{Z})^\times \to (\mathbb{Z}/\tfrac{N}{2}\mathbb{Z})^\times\right] = \{1, 1+\frac{N}{2}\}$

$\Rightarrow \chi(x + N/2) = \chi(x)\chi(1+N/2) = -\chi(x)$.

(vii) $K = \mathbb{Q}(\sqrt{d})$, $d \neq 0, 1$ squarefree integer. Let $N = |d_K| = 2^r N_1$, N_1 odd, squarefree,

$$r = \begin{cases} 0 & \Leftrightarrow \quad d = \pm N_1 \equiv 1 \pmod 4 \\ 2 & \Leftrightarrow \quad d = \pm N_1 \equiv 3 \pmod 4 \\ 3 & \Leftrightarrow \quad d = \pm 2N_1 = 2 \pmod 4 \end{cases}$$

For $p \nmid 2d$, $\chi_K(p) = \left(\frac{d}{p}\right) = \left(\frac{d_K}{p}\right)$, so for any $a \geq 1$ with $(a, 2d) = 1$, $\chi_K(a) = \left(\frac{d}{a}\right) = \left(\frac{d_K}{a}\right)$ (Jacobi symbol)

χ_K is primitive $\Leftrightarrow$ $\forall$ prime $q \mid N$, $\exists a$ with $(a, N) = 1$, $a \equiv 1 \pmod{N/q}$ and $\chi_K(a) \neq 1$.

<u>q odd</u>. Choose (by Chinese Rem. Thm.) $a \in \mathbb{Z}$ with

$$a \equiv 1 \bmod 8\frac{N_1}{q}, \quad \left(\frac{a}{q}\right) = -1$$

Then $\chi_K(a) = \left(\frac{\pm 2^r N_1}{a}\right) = \left(\frac{q}{a}\right)\left(\frac{N_1/q}{a}\right)$ as $a \equiv 1 \pmod 8$

$= \left(\frac{a}{q}\right)\left(\frac{a}{N_1/q}\right)$ by quadratic reciprocity

$= -1$.

<u>$q = 2$</u> ($\Rightarrow r = 2$ or 3) If $r = 2$, $d \equiv 3 \pmod 4$.
Let $a \equiv 1 \pmod{2|d|}$, $a \equiv 3 \pmod 4$. Then

$\chi_K(a) = \left(\frac{d}{a}\right) = -\left(\frac{a}{d}\right) = -1$ as $a \equiv d \equiv 3 \pmod 4$

If $r = 3$, let $a \equiv 1 \bmod N_1$, $a \equiv 5 \pmod 8$. Then
$\chi_K(a) = \left(\frac{\pm 8N_1}{a}\right) = \left(\frac{2}{a}\right)\left(\frac{N_1}{a}\right) = (-1) \times \left(\frac{a}{N_1}\right) = -1$.

Q1 ctd

(viii) If χ is primitive quadratic of conductor N then by (v),

$N = 2^r N_1$, N_1 odd & squarefree, $r \in \{0, 2, 3\}$. Moreover:-

- if $r \in \{0, 2\}$ then there is only one χ. Let $\varepsilon \in \{\pm 1\}$ where $N_1 \equiv \varepsilon \pmod 4$. Then there is exactly one quadratic field K with $|d_K| = N$, namely:-
  - if $r = 0$, $K = \mathbb{Q}(\sqrt{\varepsilon N})$
  - if $r = 2$, $K = \mathbb{Q}(\sqrt{-\varepsilon N})$

  So in either case, $\chi_K = \chi$.

- if $r = 3$ there are two primitive quadratic characters mod N.

  There are also 2 quadratic K with $|d_K| = N$, namely

  $$K_+ = \mathbb{Q}(\sqrt{2N_1}), \quad K_- = \mathbb{Q}(\sqrt{-2N_1}).$$

  So enough to show that $\chi_{K_+} \neq \chi_{K_-}$.

  Let p be a prime $\equiv -1 \pmod 4$ with $(p, N) = 1$ (there is a prime factor of $4N - 1$ with this property).

  Then $\chi_{K_\pm}(p) = \left(\frac{\pm 2N_1}{p}\right) = \pm\left(\frac{2N_1}{p}\right)$

  so $\chi_{K_+}(p) = -\chi_{K_-}(p)$

Q2 From lectures, $$h_K = -\frac{1}{|d_K|} \sum_{\substack{0 < n < |d_K| \\ (n, d_K) = 1}} n \chi_K(n)$$

d_K even $\Rightarrow 4 \mid d_K$, hence

$$h_K = -\frac{1}{|d_K|} \sum_{\substack{0 < n < |d_K|/2 \\ (n, d_K) = 1}} \left(n \chi_K(n) + \left(n + \frac{|d_K|}{2}\right) \chi_K\left(n + \frac{|d_K|}{2}\right)\right)$$

$$= -\frac{1}{2} \sum_{\substack{0 < n < |d_K|/2 \\ (n, d_K) = 1}} \chi_K(n) \quad \text{by (vi–vii) above.}$$

Q3

(Easy case) K complex. From lectures,

$$\zeta_K(\mathcal{C}, s) = N(I_0)^s\, w_K\, G(I_0, s)$$

where $I_0 \subset K \subset \mathbb{C}$ with standard inner product ie. $\{1, i\}$ is an orthonormal basis, and $\mathcal{C}$ = class of inverse of I_0.

Dual lattice to I_0 is then

$$I_0' = \{ x' \in \mathbb{C} \mid \forall x \in I_0 \ (x, x') \in \mathbb{Z} \}$$

and $x = a + b\sqrt{-d}$ $(a, b \in \mathbb{Q})$ $\Rightarrow (x, x') = aa' + bb'd$
$x' = a' + b'\sqrt{-d}$ $(a', b' \in \mathbb{R})$ $= \mathrm{Re}(x\overline{x'}) = \frac{1}{2}\mathrm{tr}_{\mathbb{C}/\mathbb{R}}(x\overline{x'})$

In particular, $x' \in I_0' \Rightarrow a' = (1, x')$ and $b' = \frac{1}{d}(\sqrt{-d}, x') \in \mathbb{Q}$
ie. $x' \in K$, hence

$$I_0' = \{ x' \in K \mid \forall x \in I_0,\ \tfrac{1}{2}\mathrm{tr}_{K/\mathbb{Q}}(x\overline{x'}) \in \mathbb{Z} \}$$
$$= 2\,\overline{I_0^{-1}\mathcal{D}_K^{-1}}, \quad \mathcal{D}_K = \text{different ideal} = (\sqrt{d_K}).$$

As complex conjugation is an isometry,

$$G(I_0', s) = G(2I_0^{-1}\mathcal{D}_K^{-1}, s) = 2^{-2s}\, G(I_0^{-1}\mathcal{D}_K^{-1}, s) \quad \text{by def. of } G(\Lambda, s)$$

Also saw that $m(I_0) = N(I_0)\,|d_K|^{1/2}/2$, so

$$(2\pi)^{-s}\Gamma(s)\zeta_K(\mathcal{C}, s) = 2^{-s} w_K N(I_0)^s Z(I_0, s)$$
$$= 2^{-s} w_K N(I_0)^s\, m(I_0)^{-1} Z(I_0', 1-s) \quad \text{by F.E. for } Z(\Lambda, s)$$
$$= w_K N(I_0)^{s-1} \frac{2^{1-s}}{|d_K|^{1/2}} Z(2I_0^{-1}\mathcal{D}_K^{-1}, 1-s)$$
$$= w_K N(I_0^{-1}\mathcal{D}_K^{-1})^{1-s} |d_K|^{\frac{1}{2}-s}\, 2^{s-1} Z(I_0^{-1}\mathcal{D}_K^{-1}, 1-s) \quad \text{as } N(\mathcal{D}_K) = |d_K|$$
$$= |d_K|^{1/2 - s} (2\pi)^{s-1}\Gamma(1-s)\zeta_K(\mathcal{C}', 1-s)$$

where $\mathcal{C}'$ = class of $I_0\mathcal{D}_K$. Summing over ideal classes

$$\Rightarrow \quad Z_K(s) = |d_K|^{1/2 - s} Z_K(1-s)$$

$\boxed{\text{Q3 ctd}}$ Case K real is harder (deserves a *). Recall that we proved

$$\zeta_K(\mathcal{C}, s) = N(I_0)^s \frac{\Gamma(s)}{\Gamma(s/2)^2} \int_{-\varepsilon}^{\varepsilon} G(\Lambda_u, s) \frac{du}{u}$$

where Λ_u = lattice $I_0 \subset K \subset V_u = \mathbb{R}^2$ with inner product
$x \longmapsto (x, x')$

$((x_1, x_2), (y_1, y_2))_u = u x_1 y_1 + u^{-1} x_2 y_2$; and $m(\Lambda_u) = N(I_0)\, d_K^{1/2}$.

Therefore $\pi^{-s}\Gamma(s/2)^2 \zeta_K(\mathcal{C}, s) = N(I_0)^s \int_{\varepsilon^{-1}}^{\varepsilon} Z(\Lambda_u, s)\, du/u$

$$= N(I_0)^s \int_{\varepsilon^{-1}}^{\varepsilon} m(\Lambda_u)^{-1} Z((\Lambda_u)', 1-s) \frac{du}{u}$$

Now $(\Lambda_u)' = \{(y_1, y_2) \in \mathbb{R}^2 \mid \forall x \in I_0,\ u x y_1 + u^{-1} x' y_2 \in \mathbb{Z}\}$.

First do case $u = 1$. Then $\underline{y} \in \Lambda_1' \Rightarrow$ $y_1 + y_2 \in \mathbb{Q}$ $(x = 1)$
$\sqrt{d}\, y_1 - \sqrt{d}\, y_2 \in \mathbb{Q}$ $(x = \sqrt{d})$

$\Rightarrow (y_1, y_2) = (y, y')$ for some $y \in K$.

$\therefore (\Lambda_1)' = \{ y \in K \subset \mathbb{R}^2 \mid \forall x \in I_0,\ \underbrace{xy + x'y'}_{= Tr_{K/\mathbb{Q}}(xy)} \in \mathbb{Z} \}$

$= I_0^{-1} \mathfrak{D}_K^{-1} \subset \mathbb{R}^2$.

In general, $(\Lambda_u)' = \{(y_1, y_2) \in \mathbb{R}^2 \mid \forall x \in I_0,\ u x y_1 + u^{-1} x' y_2 \in \mathbb{Z}\}$.

$\simeq \downarrow \varphi_u : (y_1, y_2) \longmapsto (u^{-1} y_1, u y_2)$

$\{(z_1, z_2) \in \mathbb{R}^2 \mid \forall x \in I_0,\ x z_1 + x' z_2 \in \mathbb{Z}\}$

$\parallel$

$I_0^{-1}\mathfrak{D}_K^{-1}$ as for $u = 1$

and $\|\varphi_u(\underline{y})\|_{u^{-1}} = \|\underline{y}\|_u$, i.e. φ_u is an isometry between V_u and $V_{u^{-1}}$. So if $\Lambda^*_{u^{-1}} = I_0^{-1}\mathfrak{D}_K^{-1} \subset V_{u^{-1}}$,

we have
$$G((\Lambda_u)', s) = G(\Lambda^*_{u^{-1}}, s) \quad \text{and so}$$

$$\pi^{-s}\Gamma(s/2)^2 \zeta_K(\mathcal{C}, s) = N(I_0)^s \int_{\varepsilon^{-1}}^{\varepsilon} m(\Lambda_u)^{-1} Z(\Lambda^*_{u^{-1}}, 1-s) \frac{du}{u}$$

$$= N(I_0^{-1}\mathfrak{D}_K^{-1})^{1-s} |d_K|^{1/2 - s} \int_{\varepsilon^{-1}}^{\varepsilon} Z(\Lambda^*_{u^{-1}}, 1-s) \frac{du}{u}$$

$$= |d_K|^{1/2 - s}\, \pi^{-s}\Gamma(s/2)^2 \zeta_K(\mathcal{C}', 1-s)$$

with $\mathcal{C}'$ = class of $I_0^{-1}\mathfrak{D}_K^{-1}$. Then sum over ideal classes as before.

$\boxed{Q4}$

$$\Delta(z) = q\prod_{n\geq 1}(1-q^n)^{24} = z^{-12}\Delta(-1/z)$$

$$\therefore \quad \frac{\Delta'(z)}{\Delta(z)} = \frac{q'}{q} + \sum_{n\geq 1} 24\frac{(1-q^n)'}{1-q^n} = -\frac{12}{z} + z^{-2}\frac{\Delta'(-1/z)}{\Delta(-1/z)}$$

(where in the middle, $' = \frac{d}{dz}$)

$$= 2\pi i\left(1 - 24\sum_{n\geq 1}\frac{nq^n}{1-q^n}\right)$$

$$= 2\pi i\left(1 - 24\sum_{n\geq 1}\sum_{m\geq 1} nq^{mn}\right)$$

$$= 2\pi i\left(1 - 24\sum_{N\geq 1}\sigma_1(N)q^N\right) = 2\pi i E_2(z).$$

ie. $2\pi i E_2(z) = -\frac{12}{z} + 2\pi i z^{-2}E_2(-1/z)$, giving answer.

So $G_2(-1/z) = z^2G_2(z) - 2\pi i z$

Let $F(z) = \frac{\pi}{\operatorname{Im} z}$, Then as $\operatorname{Im}(-1/z) = \operatorname{Im} z/|z|^2$

$$F(-1/z) - z^2F(z) = \frac{\pi|z|^2}{\operatorname{Im} z} - \frac{\pi z^2}{\operatorname{Im} z} = \frac{\pi z}{\operatorname{Im} z}(\bar{z} - z) = -2\pi i z$$

hence if $G_2^*(z) = G_2(z) - \frac{\pi}{\operatorname{Im} z}$, $G_2^*(-1/z) = z^2G_2^*(z)$.

Clearly $G_2^*(z+1) = G_2^*(z)$, so G_2^* is modular of weight 2.

$\boxed{Q5}$

$$\Theta(t;a,b) = \sum_{n\in\mathbb{Z}} e^{-\pi(n+a)^2t + 2\pi i(n+\frac{a}{2})b} = \sum_{n\in\mathbb{Z}} f(n)$$

where $f(x) = f_{a,b,t}(x) = \exp\left[-\pi(x+a)^2t + 2\pi i(x+\frac{a}{2})b\right]$

$$= e^{2\pi i(x+\frac{a}{2})b}h(x) \qquad \in \mathcal{S}(\mathbb{R}).$$

with $h(x) = g(\sqrt{t}(x+a))$, $g(x) = e^{-\pi x^2}$. So

$\hat{h}(y) = e^{-2\pi i a y}t^{-1/2}\hat{g}(y/t^{1/2}) = e^{-2\pi i a y}t^{-1/2}e^{-\pi y^2/t}$ (as in soln to Q7, sheet 1)

and $\hat{f}(y) = \int_{-\infty}^{\infty} e^{-2\pi i x(y-b) + \pi i ab}h(x)\,dx = e^{\pi i ab}\hat{h}(y-b)$

$$= t^{-1/2}\exp\left[-\pi(y-b)^2/t - 2\pi i a(y+b/2)\right]$$

$$= t^{-1/2}f_{-b,a,1/t}(y)$$

So by Poisson summation,

$$\Theta(t;a,b) = t^{-1/2}\,\Theta(1/t;-b,a).$$