Q1. Straightforward induction on n.

If $n = k-1$ and $f \in M^!_{2-k}$ then

$$f^{(k-1)}(-1/z) = z^{(2-k)+2(k-1)} + \sum_{j=0}^{n-1} c_{k-1,j}(\cdots\cdots)$$

with $$c_{k-1,j} = \binom{k-1}{j}(2-k+(k-1)-1)\cdots\cdots(2-k+j) \qquad (0 \le j \le k-2)$$

$$= 0$$

ie $f^{(k-1)}(-1/z) = z^k f^{(k-1)}(z)$. If $f = \sum_{n \ge -N} a_n q^n \in M^!_{2-k}$

then $f^{(k-1)} = (2\pi i)^{k-1} \sum_{n \ge -N} n^{k-1} a_n q^n$ as $dq^n/dz = (2\pi i n) q^n$

So $f^{(k-1)} \in M^!_k$.

Q2. (i) Recall that a basis for S_k is given by

$$\left\{ E_4^a E_6^b \Delta^c \;\middle|\; \begin{array}{l} a \ge 0,\ c \ge 1,\ 4a+6b+12c = k \\ b = 0 \text{ or } 1 \text{ for } k \equiv 0 \text{ or } 2 \pmod 4 \end{array} \right\}$$

Fix k with $k = 12$ or $k \ge 16$ ($\Leftrightarrow m = \dim_{\mathbb{C}} S_k \ge 1$). Then for each $1 \le c \le m$, there is a unique (a, b) as above. So define :-

$$f_i = E_4^a E_6^b \Delta^i \quad \text{as above}, \quad 4a + 6b + 12i = k, \qquad 1 \le i \le m.$$

Then $f_i = q^i + \sum_{n=i+1}^{\infty} b_{i,n} q^n$ with $b_{i,n} \in \mathbb{Z}$.

So $\exists!$ constants $A_{ij} \in \mathbb{Z}$, $1 \le i < j \le m$ such that if

$$g_i = f_i + A_{i,i+1} f_{i+1} + \cdots + A_{i,m} f_m$$

then $g_i = q^i + \sum_{n \ge m+1} c_{i,n} q^n$, $c_{i,n} \in \mathbb{Z}$, so $g_i \in S_k(\mathbb{Z})$

Then if $f = \sum_{n=1}^{\infty} b_n q^n \in S_k(\mathbb{Z})$, $f = \sum_{i=1}^{n} b_i g_i$, so $\{g_i\}$ is a $\mathbb{Z}$-basis for $S_k(\mathbb{Z})$ which is so free of rank m.

(ii) $f \in S_k(\mathbb{Z})$, $n \ge 1 \Rightarrow T_n f \in S_k(\mathbb{Z})$ by formula for q-expansion of $T_n f$.

$\therefore S_k(\mathbb{Z})$ is stable under $\mathbb{T}$. So in the basis $\{g_i\}$, the matrix of T_n has integer coefficients. So its char. poly. has integer coefficients, and so the eigenvalues of T_n are algebraic integers. As T_n is self-adjoint they are totally real.

(iii) Consider the basis $g_1, \ldots, g_m$ and the endomorphisms $T_1 = 1, \ldots, T_m$. Recall that if $f = \sum_{n \ge 1} a_n(f) q^n$, then

$$a_1(T_n f) = \sum_{d \mid (n,1)} d^{k-1} a_{1 \cdot n/d^2}(f) = a_n(f) \qquad (*)$$

In particular, $$a_1(T_n g_i) = a_n(g_i) = \delta_{i,n} \quad \text{if } 1 \le n \le m.$$

So if $\sum_{n=1}^{m} c_n T_n = 0$ then $0 = \sum_{n=1}^{m} c_n a_1(T_n g_i) = c_i$ for every i. So the $\{T_n \mid 1 \le n \le m\}$ are lin. independent over $\mathbb{C}$.

Let $\mathbb{T}' \subset \mathbb{T}$ be the $\mathbb{Z}$-submodule spanned by $T_1 = 1, T_2, \dots, T_m$. Then $\mathbb{T}' \simeq \mathbb{Z}^m$, and by the above, the restriction of b to $\mathbb{T}'$ is a duality

$$b' : S_k(\mathbb{Z}) \times \mathbb{T}' \longrightarrow \mathbb{Z}$$

i.e. induces isomorphisms

$$\beta' : S_k(\mathbb{Z}) \xrightarrow{\sim} \mathrm{Hom}_{\mathbb{Z}}(\mathbb{T}', \mathbb{Z})$$
$$\alpha' : \mathbb{T}' \xrightarrow{\sim} \mathrm{Hom}_{\mathbb{Z}}(S_k(\mathbb{Z}), \mathbb{Z}).$$

– indeed, under b', the basis $(g_i)_{1 \le i \le m}$ for $S_k(\mathbb{Z})$ and $(T_n)_{1 \le n \le m}$ for $\mathbb{T}'$ are dual. Remains to show that $\mathbb{T}' = \mathbb{T}$, so enough to show that the map

$$\alpha : \mathbb{T} \longrightarrow \mathrm{Hom}_{\mathbb{Z}}(S_k(\mathbb{Z}), \mathbb{Z}) \text{ is } \underline{\text{injective}}$$
$$T \longmapsto (f \mapsto a_1(Tf)).$$

Suppose $\alpha(T) = 0$, so $\forall f \in S_k(\mathbb{Z})$, $a_1(Tf) = 0$.
Then $\forall n \in \{1, \dots, m\}$, $\forall f \in S_k(\mathbb{Z})$,

$$0 = a_1(TT_n f) = a_1(T_n T f) = a_n(Tf).$$

So $a_1(Tf) = \dots = a_m(Tf) = 0$, implying (by (i)) that $Tf = 0$.
So $T = 0$.

For the last part, just have to check that β is $\mathbb{T}$-linear. But if $f \in S_k(\mathbb{Z})$, $T \in \mathbb{T}$, then

$$\beta(Tf) : Y \mapsto a_1(Y(Tf)) = a_1((TY)f) = \beta((TY)f) = \beta(f)(TY)$$

so β is $\mathbb{T}$-linear.

<u>Q3</u> (i) Let $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$, and map it to the tuple (c, d).

Then $\gamma, \gamma' = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}$ determine the same tuple up to equivalence

$\Leftrightarrow (c', d') = (\pm c, \pm d) \Leftrightarrow c'd = cd'$ (as $(c,d) = 1 = (c', d')$)

$\Leftrightarrow \gamma' \gamma^{-1} = \begin{pmatrix} a'd - b'c & ab' - a'b \\ c'd - d'c & ad' - bc' \end{pmatrix} \in \Gamma_\infty \Leftrightarrow \Gamma_\infty \gamma = \Gamma_\infty \gamma'$

Finally, if $(c, d) = 1$, $\exists a, b \in \mathbb{Z}$ st. $ad - bc = 1$, so we have a bijection

$$\Gamma_\infty \backslash \Gamma = \{\text{cosets } \Gamma_\infty \gamma\} \xrightarrow{\sim} \{\text{pairs } (c,d) \text{ up to } \sim\}.$$

(ii) $G_k(z) = \sum_{(m,n)\neq(0,0)} \frac{1}{(mz+n)^k}$, and every pair (m,n) can be written

uniquely as $m = ce$, $n = de$ with $e = (m,n) \geq 1$, $(c,d) = 1$.

$$\therefore\ G_k(z) = \sum_{e\geq 1} \sum_{\substack{c,d\in\mathbb{Z}\\(c,d)=1}} \frac{1}{e^k(cz+d)^k} = \zeta(k) \sum_{\substack{c,d\in\mathbb{Z}\\(c,d)=1}} \frac{1}{(cz+d)^k}$$

$$\Rightarrow E_k(z) = \frac{1}{2} \sum_{\substack{c,d\in\mathbb{Z}\\(c,d)=1}} \frac{1}{(cz+d)^k} = \sum_{\gamma\in\Gamma_\infty\backslash\Gamma} \gamma'(z)^{k/2} \quad \text{by (i),}$$

since if $\gamma = \pm\begin{pmatrix} a & b \\ c & d\end{pmatrix}$, $\gamma'(z) = \frac{1}{(cz+d)^2}$.

Likewise, $$G(z,s) = \sum_{e\geq 1} \sum_{\substack{c,d\in\mathbb{Z}\\(c,d)=1}} \frac{y^s}{e^{2s}|cz+d|^{2s}} = \zeta(2s) \sum_{\substack{c,d\in\mathbb{Z}\\(c,d)=1}} \frac{y^s}{|cz+d|^{2s}}$$

$$= 2\zeta(2s) \sum_{\gamma\in\Gamma_\infty\backslash\Gamma} (\operatorname{Im}\gamma(z))^s \quad \text{as } \operatorname{Im}\begin{pmatrix} a & b \\ c & d\end{pmatrix}(z) = \frac{\operatorname{Im} z}{|cz+d|^2}.$$

Q4 (i) Proved in lectures that if $\Theta_\Lambda(t) = \sum_{x\in\Lambda} e^{-\pi\|x\|^2 t}$ then

$$\Theta_\Lambda(1/t) = t^{N/2} m(\Lambda)^{-1} \Theta_{\Lambda'}(t).$$

Now $\Theta_\Lambda(t) = \vartheta_\Lambda(it)$, so transformation law holds for $z \in i\mathbb{R}_{>0}$

Both sides holomorphic $\Rightarrow$ identity holds $\forall z \in \mathfrak{H}$.

(ii) If Λ is self-dual, then since $m(\Lambda') = m(\Lambda)^{-1}$, have $m(\Lambda) = 1$, and

so $\vartheta_\Lambda(-1/z) = (z/i)^{N/2}\vartheta_\Lambda(z)$

and Λ even $\Rightarrow$ $\vartheta_\Lambda(z) = \sum_{x\in\Lambda} q^{\|x\|^2/2}$ is a power series in q,

so $\vartheta_\Lambda(z+1) = \vartheta_\Lambda(z)$.

To show $N \equiv 0 \pmod 8$: suppose not. Then $N/8 = \ell/2^{r+1}$ with ℓ odd, $r\geq 0$.

Let $f = \vartheta_\Lambda^{2^r}$. Then $f(z+1) = f(z)$ and

$$f(-1/z) = \left(\sqrt{z/i}\right)^{2^r N} f(z) = \left(\sqrt{z/i}\right)^{4\ell} = -z^{2\ell} f(z).$$

So $f|_{2\ell}T = f$, $f|_{2\ell}S = -f$. Therefore $f|_{2\ell}(ST) = -f$, and so

$f|_{2\ell}(ST)^3 = -f$. But $(ST)^3 = \pm I$, so $f = -f$ i.e. $f = 0$, contradiction.

Therefore $N \equiv 0 \pmod 8$, so $\vartheta_\Lambda(-1/z) = z^{N/2}\vartheta_\Lambda(z)$, and ϑ is modular of wt $\frac{N}{2}$.

Holomorphic on $\mathfrak{H}$ and at ∞ $\Rightarrow$ $\vartheta_\Lambda \in M_{N/2}$.

(iii) $\Lambda = \{(x_1, \dots x_8) \in \mathbb{R}^8 \mid 2x_i \in \mathbb{Z},\ x_i - x_j \in \mathbb{Z},\ \sum_1^\infty x_i \in 2\mathbb{Z}\}$.

As $\frac{1}{2}\mathbb{Z}^8 \supset \Lambda \supset 2\mathbb{Z}^8$, Λ is a lattice.

$\underline{x} \in \mathbb{R}^8$. Then $\underline{x} \in \Lambda$ iff either $\underline{x} \in \mathbb{Z}^8$, $\sum x_i \in 2\mathbb{Z}$

or $\underline{x} = \underline{e} + \underline{x}'$, $\underline{x}' \in \mathbb{Z}^8$, $\sum x_i' \in 2\mathbb{Z}$, where $\underline{e} = (1/2, \dots, 1/2)$.

ie $\Lambda = \Lambda_1 \cup (\underline{e} + \Lambda_1)$, $\Lambda_1 = \{\underline{x} \in \mathbb{Z}^8 \mid \sum x_i \in 2\mathbb{Z}\}$
$= \{\underline{x} \in \mathbb{Z}^8 \mid \underline{e} \cdot \underline{x} \in \mathbb{Z}\}$

<u>Claim</u> i) $\forall \underline{x}, \underline{y} \in \Lambda$, $\underline{x} \cdot \underline{y} \in \mathbb{Z}$
ii) $\forall \underline{x} \in \Lambda$, $\underline{x} \cdot \underline{x} \in 2\mathbb{Z}$.

1) $\underline{x} = \underline{x}' + a\underline{e}$, $\underline{y} = \underline{y}' + b\underline{e}$, $a, b \in \{0, 1\}$, $\underline{x}', \underline{y}' \in \Lambda_1$

$\Rightarrow \underline{x} \cdot \underline{y} = \underline{x}' \cdot \underline{y}' + \underline{e} \cdot (b\underline{x}' + a\underline{y}') + \underline{e} \cdot \underline{e} \in \mathbb{Z}$ as $\underline{e} \cdot \underline{e} = 2$

2) $\underline{x} = \underline{x}' + a\underline{e} \Rightarrow \underline{x} \cdot \underline{x} = \underline{x}' \cdot \underline{x}' + 2a\, \underline{e} \cdot \underline{x}' + 2a^2$

and $\underline{x}' \cdot \underline{x}' = \sum (x_i')^2 \equiv \sum x_i' \pmod 2$
$\equiv 0 \pmod 2$.

So $\Lambda \subset \Lambda'$. Now $(\mathbb{Z}^8 : \Lambda_1) = 2$ and $(\Lambda : \Lambda_1) = 2$, so

$m(\Lambda) = 1 = m(\Lambda')^{-1}$. As $m(\Lambda') = (\Lambda : \Lambda')\, m(\Lambda)$, $\Lambda = \Lambda'$.

So Λ is even & self dual. Therefore $\vartheta_\Lambda \in M_4 = \mathbb{C} E_4$. Comparing

constant terms $\Rightarrow \vartheta_\Lambda = E_4$. Now coeff. of q in ϑ_Λ

$= \#\{\underline{x} \in \Lambda \mid \|x\|^2 = 2\} = (\text{coeff. of } q \text{ in } E_4) = 240$

<u>Q5.</u> (i) Put $z' = z - d/N$. Then

$$\text{LHS} = f\left(\frac{-1}{N^2(z' + d/N)} + \frac{a}{N}\right) = f\left(\frac{-1 + a(Nz' + d)}{N(Nz' + d)}\right) = f\left(\frac{az' + (ad-1)/N}{Nz' + d}\right)$$

$$= (Nz' + d)^k f(z') = \text{RHS} \quad \text{since} \begin{pmatrix} a & (ad-1)/N \\ N & d \end{pmatrix} \in SL_2(\mathbb{Z})$$

(ii) $g(z) = f(a/N + z)$. Then

$$M(g(iy), s) = \sum_n e^{2\pi i a n/N} c_n M(e^{-2\pi n y}, s)$$

$$= (2\pi)^{-s}\Gamma(s) \sum_n e^{2\pi i a n/N} c_n n^{-s}$$

$$\therefore M(f, a/N, s) = N^s \int_0^\infty g(iy)\, y^s \frac{dy}{y}$$

$$= \int_0^\infty f\left(\frac{a}{N} + iy\right) (Ny)^s \frac{dy}{y} \qquad = \int_{1/N}^\infty + \int_0^{1/N}$$

Now $f\left(iy + \frac{a}{N}\right) = \left(\frac{i}{Ny}\right)^k f\left(\frac{i}{N^2y} - \frac{d}{N}\right)$ (putting $iy = \frac{-1}{N^2 z}$, $z = \frac{i}{N^2 y}$ in (i))

so

$$\int_0^{1/N} f\left(\frac{a}{N} + iy\right) (Ny)^s \frac{dy}{y} = \int_0^{1/N} \left(\frac{i}{Ny}\right)^k f\left(\frac{i}{N^2y} - \frac{d}{N}\right) (Ny)^s \frac{dy}{y}$$

$$\underset{(y \mapsto 1/N^2y)}{=} \int_{1/N}^\infty (iNy)^k f(iy - d/N) (1/Ny)^s \frac{dy}{y}$$

$$= (-1)^{k/2} \int_{1/N}^\infty f(iy - d/N) (Ny)^{k-s} \frac{dy}{y}$$

i.e. $$M(f, a/N, s) = \int_{1/N}^\infty f\left(\frac{a}{N} + iy\right) (Ny)^s + (-1)^{k/2} f(iy - d/N)(Ny)^{k-s} \frac{dy}{y}$$

which is an entire function of s, and

$$M(f, a/N, k-s) = \int_{1/N}^\infty f\left(\frac{a}{N} + iy\right) (Ny)^{k-s} + (-1)^{k/2} f(iy - d/N)(Ny)^s \frac{dy}{y}$$

$$= (-1)^{k/2} M(f, -d/N, s),$$

since $(a', d') = (-d, -a) \Rightarrow a'd' \equiv 1 \pmod N$.