Example sheet 2

(1) Identify $X = \begin{pmatrix} a & b \\ b & c \end{pmatrix}$ with quadratic form $aU^2 + 2bUV + cV^2 = (U \ V) X \begin{pmatrix} U \\ V \end{pmatrix}$.

Congruence of matrices $X \underset{g}{\longmapsto} gX{}^tg \iff$ equivalence of quadratic forms under change of variables $(U \ V) \underset{g}{\longmapsto} (U\ V)g$.

Under $SL_2(\mathbb{R})$, every $X$ with $\det X = D > 0$ is equivalent to one of $\pm\begin{pmatrix} \sqrt{D} & \\ & \sqrt{D} \end{pmatrix}$.

Stabiliser of $\begin{pmatrix} \sqrt{D} & \\ & \sqrt{D} \end{pmatrix}$ in $SL_2(\mathbb{R})$ is $\{g \in SL_2(\mathbb{R}) \mid g\cdot{}^tg = I\} = SO(2)$.

$\therefore$ $\quad \left(\text{orbit of } \begin{pmatrix} \sqrt{D} & \\ & \sqrt{D} \end{pmatrix}\right) \simeq SL_2(\mathbb{R})/SO(2) \simeq \mathfrak{H}$

$$g\begin{pmatrix} \sqrt{D} & \\ & \sqrt{D} \end{pmatrix} g^{-1} = \frac{\sqrt{D}}{y}\begin{pmatrix} x^2+y^2 & x \\ x & 1 \end{pmatrix} \longleftrightarrow g\,SO(2) \longleftrightarrow \tau = g(i), \quad g = \begin{pmatrix} y^{1/2} & xy^{-1/2} \\ 0 & y^{-1/2} \end{pmatrix}$$

Last part: Gauss's reduction for positive definite binary quadratic forms states that each $SL_2(\mathbb{Z})$-orbit of quadratic forms $aU^2 + 2bUV + cV^2$ of discriminant $-4D = 4(b^2 - ac) < 0$ which are positive definite contains a unique representative with $-a < 2b \le a < c$ or $0 \le 2b \le a = c$.

If $a = \frac{\sqrt{D}}{y}$, $b = x\frac{\sqrt{D}}{y}$, $c = (x^2+y^2)\frac{\sqrt{D}}{y}$

this is the same as the condition that $z = x + iy$ belongs to the fundamental set $\mathcal{D} \subset \mathfrak{H}$.

(2) (i) If $\Lambda = \bigoplus_{i=1}^{m} \mathbb{Z}x_i$, $(x_i)$ lin. indep. $/\mathbb{R}$ then after change of basis,
$\Lambda = \mathbb{Z}^m \oplus 0^{n-m} \subset \mathbb{R}^n$, obviously discrete.

(ii) Let $\Lambda \subset \mathbb{R}^n$ be discrete, $V = \mathbb{R}$-span of $\Lambda$. Replacing $\mathbb{R}^n$ by $V$, and choosing a basis for $V$ contained in $\Lambda$, we may assume $\mathbb{Z}^n \subset \Lambda \subset \mathbb{R}^n$.

Then every elt. $\underline{x} \in \Lambda$ can be (uniquely) written $\underline{x} = \underline{m} + \underline{y}$, $\underline{m} \in \mathbb{Z}^n$, $\underline{y} \in \Lambda \cap [0,1)^n$.

But as $\Lambda$ is discrete, $\Lambda \cap [0,1)^n$ is finite. So $(\Lambda : \mathbb{Z}^n) = d < \infty$, so $1/d\, \mathbb{Z}^n \supset \Lambda \supset \mathbb{Z}^n$, hence $\Lambda = \bigoplus_{i=1}^{n} \mathbb{Z}x_i$ with $x_i \in \frac{1}{d}\mathbb{Z}^n$ linearly independent $/\mathbb{Z}$ and $\therefore$ also lin. indep. $/\mathbb{R}$.

(3) Copy proof for $\tau(n) \equiv \sigma_{11}(n) \pmod{691}$.

(4) $\sum_{\tau_0 \ne i, \rho} \operatorname{ord}_{\tau_0} f + \operatorname{ord}_\infty f + \frac{1}{2}\operatorname{ord}_i f + \frac{1}{3}\operatorname{ord}_\rho f = k/12$, so

$$3\operatorname{ord}_i f + 2\operatorname{ord}_\rho f \equiv k/2 \pmod 6.$$

So $k \equiv 2 \pmod 4 \Rightarrow \operatorname{ord}_i f \equiv 1 \pmod 2 \Rightarrow f(i) = 0$

$k \equiv \pm 1 \pmod 3 \Rightarrow \operatorname{ord}_\rho f \equiv \pm 1 \pmod 3 \Rightarrow f(\rho) = 0$.

(5) Let $f \in M_k^!$, $\mathrm{ord}_\infty f \geq -r$ say. Then $\Delta^r f \in M_{k+12r}$.

Conversely, as $\Delta \neq 0$ on $\mathfrak{H}$, $\Delta^{-r} M_{k+12r} \subset M_k^!$.

So $$M_k^! = \bigcup_{r \geq 0} \Delta^{-r} M_{k+12r}.$$

$M_k$ has basis $\{E_4^a E_6^b \Delta^c \mid a, c \geq 0,\ b \in \{0,1\},\ 4a + 6b + 12c = k\}$

(and $b = \begin{cases} 0 & \text{if } k \equiv 0 \pmod 4 \\ 1 & \text{—"— } 2 \text{ —"—} \end{cases}$

$\Rightarrow$ $M_k^!$ has basis $\left\{ E_4^a E_6^b \Delta^c \;\middle|\; \begin{array}{l} a \geq 0,\ c \in \mathbb{Z},\ b \in \{0,1\} \\ 4a + 6b + 12c = k \end{array} \right\}$.

(6) $f\left(\frac{az+b}{cz+d}\right) = (cz+d)^k f(z) \Rightarrow \frac{1}{(cz+d)^2} f'\left(\frac{az+b}{cz+d}\right) = k(cz+d)^{k-1} f(z) + (cz+d)^k f'(z)$

i.e. $f'\left(\frac{az+b}{cz+d}\right) = k(cz+d)^{k+1} f(z) + (cz+d)^{k+2} f'(z)$

$\Rightarrow (f'g)\left(\frac{az+b}{cz+d}\right) = k(cz+d)^{k+l+1} (fg)(z) + (cz+d)^{k+l+2} (f'g)(z).$

$\Rightarrow l f'g - k g'f \in M_{k+l+2}$.

(7) $\mathrm{Im}\, \gamma(z) = \frac{\mathrm{Im}\, z}{|cz+d|^2}$, so $(\mathrm{Im}\, \gamma(z))^{k/2} |f(\gamma(z))| = (\mathrm{Im}\, z)^{k/2} |f(z)|$.

i.e. $y^{k/2}|f|$ is invariant under $\Gamma$ (= modular of weight 0)

So $y^{k/2}|f|$ bounded on $\mathfrak{H}$ $\Leftrightarrow$ $y^{k/2}|f|$ bounded on $\mathcal{D}$.

$f$ holomorphic, $k > 0$. Then $f \in S_k \Rightarrow f = O(|q|) = O(e^{-2\pi y})$ as $|q| \to 0$ i.e. $y \to \infty$

So $y^N |f|$ is bounded on $\mathcal{D}$ for <u>any</u> $N$.

Conversely, let $f$ have $q$-expansion $\tilde{f}(q) = \sum_{n \in \mathbb{Z}} a_n q^n$ (convergent for $0 < |q| < 1$)

Then $y^{k/2}|f|$ bounded on $\mathcal{D}$ $\Rightarrow$ $\tilde{f}(q) \to 0$ as $q \to 0$ (since $k > 0$)

$$\overset{\shortparallel}{\left(-\frac{\log|q|}{2\pi}\right)^{k/2}|f|}$$

$\Rightarrow \tilde{f}$ has removable singularity at $q = 0$

and $a_n = 0\ \forall n \leq 0$ i.e. $f \in S_k$.

(8) From lectures, $\sum_{n \in \mathbb{Z}} \frac{1}{(z+n)^2} = -4\pi^2 \sum_{d=1}^{\infty} d\, q^d$. So

$$G_2(z) = \sum_{0 \neq n \in \mathbb{Z}} \frac{1}{n^2} + 2 \sum_{m \geq 1} \left( \sum_{n \in \mathbb{Z}} \frac{1}{(mz+n)^2} \right) = 2\zeta(2) - 8\pi^2 \sum_{\substack{m \geq 1 \\ d \geq 1}} d\, q^{md} = \frac{\pi^2}{3}\left(1 - 24 \sum_{n=1}^{\infty} \sigma_1(n) q^n\right)$$

(as $\zeta(2) = \frac{\pi^2}{6}$).

$G_2 \notin M_2$ as $M_2 = \{0\}$. Reason the proof for $k \geq 4$ doesn't work

is that $G_2$ is <u>not</u> absolutely convergent, and

$$z^{-2} G_2(-1/z) = z^{-2} \sum_m \left( \sum_n \frac{1}{(m/z + n)^2} \right) = \sum_m \left( \sum_n \frac{1}{(nz+m)^2} \right) \neq G_2(z).$$