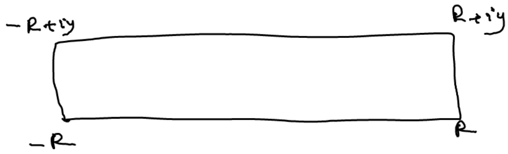
$\boxed{Q1}$ "Standard" proof : $f(x) = e^{-\pi x^2}$

$\therefore \hat{f}(y) = \int_{-\infty}^{\infty} e^{-\pi x^2 - 2\pi i x y}\, dx = e^{-\pi y^2}\int_{-\infty}^{\infty} e^{-\pi(x+iy)^2}\, dx = e^{-\pi y^2}. I$ where

$I = \lim_{R\to\infty} \int_{-R+iy}^{R+iy} e^{-\pi z^2}\, dz$. Consider rectangle



Then $\int_{\pm R}^{\pm R+iy} e^{-\pi z^2}\, dz \to 0$ as $R \to \infty$

so by Cauchy's thm $I = \lim_{R\to\infty}\int_{-R}^{R} e^{-\pi z^2}\, dz = \int_{-\infty}^{\infty} e^{-\pi x^2}\, dx = 1$, so $\hat{f}(y) = f(y)$

Another proof. $\left(\frac{d}{dx} + 2\pi x\right) f(x) = 0$. Now $g(x) = f'(x) \Rightarrow \hat{g}(y) = 2\pi i y \hat{f}(y)$ (integration by parts!)

$g(x) = x f(x) \Rightarrow \hat{g}(y) = \frac{-1}{2\pi i}(\hat{f})'(y)$ (apply FT to previous identity)

So FT gives $2\pi i y \hat{f}(y) + 2\pi\left(\frac{-1}{2\pi i}\right)(\hat{f})'(y) = 0$

i.e. $\left(\frac{d}{dy} + 2\pi y\right)\hat{f}(y) = 0 \Rightarrow \hat{f}(y) = c. e^{-\pi y^2}$.

and $\hat{f}(0) = \int_{-\infty}^{\infty} f(x)\, dx = \int_{-\infty}^{\infty} e^{-\pi x^2}\, dx = 1 \Rightarrow c = 1$.

$\boxed{Q2}$ $\chi : G \to U(1)$ so $\overline{\chi(g)} = \chi(g)^{-1}$ So if $\chi = \chi'$, sum $= \sum_{g\in G} 1 = \# G$.

$\chi \neq \chi' \Rightarrow \exists h \in G$ st. $\chi(h) \neq \chi'(h)$. Then

$$\sum_{g\in G} \overline{\chi(g)}\, \chi'(g) = \sum_{g \in h^{-1}G = G} \overline{\chi(hg)}\, \chi'(hg) = \overline{\chi(h)}\, \chi'(h) \sum_{g\in G} \overline{\chi(g)}\, \chi'(g)$$

$$\overline{\chi(h)}\, \chi'(h) \neq 1 \Rightarrow \sum_{g\in G} \overline{\chi(g)}\, \chi'(g) = 0 .$$

$\boxed{Q3}$ $\varphi : \mathbb{R} \to \mathbb{C}$ cts. hom$^{=}$. Let $a = \varphi(1) \in \mathbb{C}$. Then $\forall\, m/n \in \mathbb{Q}$,

$n\,\varphi(m/n) = \varphi(m) = m\,\varphi(1) = ma \Rightarrow \varphi(m/n) = m/n \cdot a$

So as $\mathbb{Q} \subset \mathbb{R}$ is dense and φ is cts, $\varphi(x) = ax$ $\forall x \in \mathbb{R}$.

i.e. $\mathrm{Hom}_{cts}(\mathbb{R}, \mathbb{C}) = \{\varphi_a : x \mapsto ax \mid a \in \mathbb{C}\}$.

$\hat{\mathbb{R}}$: enough to show that if $\chi \in \hat{\mathbb{R}}$ then $\chi = \chi_y = \exp(2\pi i y\, -)$, some $y \in \mathbb{R}$.

Method 1 :

$$\begin{tikzcd} \mathbb{R} \arrow[r, "\chi"] \arrow[d, dashed] & U(1) \\ \mathbb{R} \arrow[r] & \mathbb{R}/\mathbb{Z} \arrow[u, "\cong\ x \mapsto e^{2\pi i x}"'] \end{tikzcd}$$

Topology (homotopy lifting property) $\Rightarrow$ if $\chi : \mathbb{R} \to U(1)$ is a cts. map with $\chi(0) = 1$, $\exists!$ cts. $\tilde{\chi} : \mathbb{R} \to \mathbb{R}$ st. $\chi(x) = \exp 2\pi i \tilde{\chi}(x)$ and $\tilde{\chi}(0) = 0$

If χ is a hom$^{=}$, then $f(a,b) = \tilde{\chi}(a+b) - \tilde{\chi}(a) - \tilde{\chi}(b)$ satisfies $\exp 2\pi i f(a,b) = 1$, so $f : \mathbb{R}\times\mathbb{R} \to \mathbb{Z}$ cts, hence constant, so (as $\tilde{\chi}(0) = 0$) $f = 0$ ie. $\tilde{\chi}$ is a hom$^{=}$.

By previous part, $\tilde{\chi}(x) = \varphi_y$, some $y \in \mathbb{R} \Rightarrow \chi = \chi_y$

(3 ctd) Method 2 $\quad \chi$ cts. $\Rightarrow \exists n_0$ st. $\forall n \geq n_0$, $|\chi(\frac{1}{2^n}) - 1| < \sqrt{2}$ $\quad$ ie. $\operatorname{Re}\chi(\frac{1}{2^n}) > 0$

So $\chi(\frac{1}{2^n}) = e^{i\theta_n}$ for $-\frac{\pi}{2} < \theta_n < \frac{\pi}{2}$, unique.

By $n \geq n_0$, $e^{i\theta_n} = \chi(1/2^n) = \chi(1/2^{n+1})^2 = e^{i.2\theta_{n+1}} \Rightarrow \theta_n - 2\theta_{n+1} \in 2\pi\mathbb{Z}$

But $|\theta_n|, |\theta_n| < \frac{\pi}{2} \Rightarrow \theta_n = 2\theta_{n+1}$ $\forall n \geq n_0$.

Let $\theta = \theta_n/2^n$, any $n \geq n_0$. Then if $x = \frac{m}{2^n}$ for some $n \geq n_0$, $m \in \mathbb{Z}$,

$\chi(x) = e^{im\theta_n} = e^{i\theta x}$. As $\{\frac{m}{2^n} \mid {m \in \mathbb{Z} \atop n \geq n_0}\} \subset \mathbb{R}$ is dense, $\chi(x) = e^{i\theta x}$ $\forall x \in \mathbb{R}$.

Finally:
$$\begin{array}{ccccc} \chi : \mathbb{R}^\times_{>0} & \longrightarrow & \mathbb{C}^\times & = & U(1) \times \mathbb{R}^\times \\ \uparrow \scriptstyle{2\,\exp} & & & & \| \; (\mathrm{id}, \exp) \\ \mathbb{R} & & & & U(1) \times \mathbb{R} \end{array}$$

So $\quad \chi(e^x) = \psi(x) . \exp\varphi(x)$ for $\psi \in \widehat{\mathbb{R}}^\times$, $\varphi : \mathbb{R} \to \mathbb{R}$

$\quad = e^{2\pi i y x + a x}$ say, $a, y \in \mathbb{R}$

ie. $\chi(x) = x^{a + 2\pi i y} = x^s$ $(s \in \mathbb{C})$ say $\forall x \in \mathbb{R}^\times_{>0}$.

$\boxed{Q4}$ FE for ζ is $Z(s) = \pi^{-s/2}\Gamma(s/2)\zeta(s) = \pi^{(s-1)/2}\Gamma(\frac{1-s}{2})\zeta(1-s)$

ie. $\zeta(1-s) = \pi^{1/2 - s}\dfrac{\Gamma(s/2)}{\Gamma(\frac{1-s}{2})}\zeta(s) = 2(2\pi)^{-s}\cos\frac{\pi s}{2}\,\Gamma(s)\,\zeta(s)$

[Duplication formula $\Rightarrow \pi^{1/2}\Gamma(s) = 2^{s-1}\Gamma(s/2)\Gamma(\frac{s+1}{2})$

Reflection formula $\Rightarrow \Gamma(\frac{s+1}{2})\Gamma(\frac{1-s}{2}) = \dfrac{\pi}{\sin\frac{1}{2}\pi(s+1)} = \dfrac{\pi}{\cos\frac{\pi s}{2}}$]

$\boxed{Q6}$ From lectures, $\zeta(1-2k) = -\dfrac{B_{2k}}{2k}$ $\forall k \geq 1$.

So from F.E. (say in the form of Q.4) $\forall k \geq 1$

$0 < \zeta(2k) = \frac{1}{2}(2\pi)^{2k}(\cos\pi k)^{-1}\Gamma(2k)^{-1}\zeta(1-2k)$

$= \frac{1}{2}(2\pi)^{2k}(-1)^k\dfrac{1}{(2k-1)!}\cdot\dfrac{-B_{2k}}{2k} = -\dfrac{(2\pi i)^k B_{2k}}{2(2k)!}$; so $(-1)^{k-1}B_{2k} > 0$.

$\boxed{Q7}$ $f(x) = e^{-\pi(x+c)^2 t} = g((x+c)\sqrt{t})$, $g(x) = e^{-\pi x^2}$

$\Rightarrow \hat{f}(y) = e^{2\pi i c y}\, t^{-1/2}\,\hat{g}(y/\sqrt{t}) = e^{2\pi i c y}\, t^{-1/2}\, e^{-\pi y^2/t}$

Poisson summation $\Rightarrow$

$$\Theta(t;c) = \sum_{n\in\mathbb{Z}} f(n) = \sum_{n\in\mathbb{Z}} \hat{f}(n) = t^{-1/2}\,\Theta^*(1/t;c).$$

Q5

Recall: $\Gamma(s)\zeta(s) = \int_0^\infty \frac{y^{s-1}}{e^y-1}\,\frac{dy}{y} = \int_Y^\infty + \int_0^Y$

i) $\int_Y^\infty \frac{y^{s-1}}{e^y-1}\,dy$ converges uniformly on compact subsets of $\mathbb{C}$. Putting $s=1$:

$$\int_Y^\infty \frac{dy}{e^y-1} = \int_Y^\infty \frac{e^{-y}}{1-e^{-y}}\,dy = \left[\log(1-e^{-Y})\right]_Y^\infty = -\log(1-e^{-Y}).$$

ii) $\frac{1}{e^y-1} = \frac{1}{y} + g(y)$ for $g \in C^\infty(\mathbb{R})$. So for $\mathrm{Re}(s)>1$,

$$\int_0^Y \frac{y^{s-1}}{e^y-1}\,dy = \int_0^Y y^{s-2} + y^{s-1}g(y)\,dy = \left[\frac{y^{s-1}}{s-1}\right]_0^Y + \int_0^Y y^{s-1}g(y)\,dy$$

Now $\int_0^Y y^{s-1}g(y)\,dy = \int_0^Y g(y)\,dy + O(s-1)$ since

$$\left|\int_0^Y g(y)(y^{s-1}-1)\right| \le \sup_{[0,Y]}|g| \cdot \int_0^Y |y^{s-1}-1|\,dy \le \sup_{[0,Y]}|g| \cdot Y\,|Y^{s-1}-1| = O(s-1).$$

and $\left[\frac{y^{s-1}}{s-1}\right]_0^Y = \frac{Y^{s-1}}{s-1} = \frac{1}{s-1} + \log Y + O(s-1).$

iii) So $\Gamma(s)\zeta(s) = \frac{1}{s-1} + \log Y - \log(1-e^{-Y}) + \int_0^Y g(y)\,dy + O(s-1)$

and therefore $\left.\left(\Gamma(s)\zeta(s) - \frac{1}{s-1}\right)\right|_{s=1} = \log Y - \log(1-e^{-Y}) + \int_0^Y g(y)\,dy$

and letting $Y \to 0$, this becomes 0

iv) FE for ζ is $Z(s) = \pi^{-s/2}\Gamma(s/2)\zeta(s) = \pi^{(s-1)/2}\Gamma\left(\frac{1-s}{2}\right)\zeta(1-s)$

ie. $\zeta(1-s) = \pi^{1/2-s}\frac{\Gamma(s/2)}{\Gamma\left(\frac{1-s}{2}\right)}\zeta(s) = 2(2\pi)^{-s}\cos\frac{\pi s}{2}\,\Gamma(s)\,\zeta(s)$

$= h(s)\,\Gamma(s)\zeta(s)$ say

and $h(1) = 0$.

[Duplication formula $\Rightarrow \pi^{1/2}\Gamma(s) = 2^{s-1}\Gamma(s/2)\Gamma\left(\frac{s+1}{2}\right)$

Reflection formula $\Rightarrow \Gamma\left(\frac{s+1}{2}\right)\Gamma\left(\frac{1-s}{2}\right) = \frac{\pi}{\sin\frac{1}{2}\pi(s+1)} = \frac{\pi}{\cos\frac{\pi s}{2}}$]

Expanding RHS about $s=1$:

$$2\left[\frac{1}{2\pi} - \frac{\log 2\pi}{2\pi}(s-1) + O((s-1)^2)\right]\cdot\left[-\frac{\pi}{2}(s-1) + O((s-1)^3)\right]\left[\frac{1}{s-1} + O(s-1)\right]$$

$= -\frac{1}{2} + \frac{1}{2}\log 2\pi\,(s-1) + O((s-1)^2)$, hence $\boxed{\zeta(0) = -1/2,\ \zeta'(0) = -\frac{1}{2}\log 2\pi.}$