# The analytic continuation of $\zeta_K(s)$ and the class number formula

Let $V$ be a real vector space of dimension $n \geq 1$, $(-,-)\colon V \times V \to \mathbb{R}$ an inner product with corresponding norm $\|-\|$. Let $\{e_i\}$ be an ON basis for $V$, and $\mu$ (or $dv$) the associated measure (for which $\mu(V/\sum \mathbb{Z}e_i) = 1$) — it doesn't depend on the choice of ON basis. Write $m(\Lambda)$ for the covolume of $\Lambda$ (i.e., the volume of the quotient $V/\Lambda$).

Recall that the **Epstein zeta function** of the quadratic lattice $\Lambda$ is

$$G(\Lambda, s) = \sum_{0 \neq x \in \Lambda} \frac{1}{\|x\|^{2s}}.$$

It converges absolutely for $\mathrm{Re}(s) > n/2$, and:

**Theorem.** *$Z(\Lambda, s) = \pi^{-s}\Gamma(s)\, G(\Lambda, s)$ has a meromorphic continuation to $\mathbb{C}$, analytic apart from simple poles at $s = 0$, $n/2$ with residues $-1$, $m(\Lambda)^{-1}$ respectively. It satisfies the* ***functional equation***

$$Z(\Lambda, s) = m(\Lambda)^{-1} Z(\Lambda', \frac{n}{2} - s).$$

*In particular, $G(\Lambda, 0) = -1$.*

Let $K$ be a number field. For a prime ideal $P \subset \mathfrak{o}_K$, write $q_P = N(P)$ for its norm.

**Definition.** The *Dedekind zeta function* is the function

$$\zeta_K(s) = \sum_I \frac{1}{N(I)^s}$$

the sum taken over non-0 ideals $I \subset \mathfrak{o}_K$.

**Proposition.** *$\zeta_K(s) = \prod_P (1 - q_P^{-s})^{-1}$, and the product converges absolutely for $\mathrm{Re}(s) > 1$.*

*Proof.* As formal series, the product follows from unique factorisation of ideals: writing $I = \prod P^{n_P}$ gives $N(I) = \prod q_P^{n_P}$, hence

$$\zeta_K(s) = \prod_P (1 + q_P^{-s} + q_P^{-2s} + \cdots)$$

Now $\#\{P \mid p\} \leq n$ and $q_P \geq p$ if $P|p$, so product converges by comparison with

$$\prod_p (1 - p^{-s})^{-n} = (\sum_{N \geq 1} N^{-s})^n = \zeta(s)^n.$$

□

**Theorem.** *(i) $\zeta_K(s)$ has a meromorphic continuation to $\mathbb{C}$ whose only singularity is a simple pole at $s=1$. Moreover*

$$\lim_{s\to 0}\frac{\zeta_K(s)}{s^{r_1+r_2-1}} = -\frac{h_K R_K}{w_K}, \qquad \mathrm{Res}_{s=1}\,\zeta_K(s) = \frac{2^{r_1+r_2}\pi^{r_2}\,h_K R_K}{|d_K|^{1/2}\,w_K}.$$

*(ii) Write $Z_K(s) = \Gamma_{\mathbb{R}}(s)^{r_1}\Gamma_{\mathbb{C}}(s)^{r_2}\zeta_K(s)$, where $\Gamma_{\mathbb{R}}(s) = \pi^{-s/2}\Gamma(s/2)$, $\Gamma_{\mathbb{C}}(s) = 2(2\pi)^{-s}\Gamma(s)$. Then*

$$Z_K(s) = |d|^{1/2-s}\,Z_K(1-s).$$

Here:

- $h_K = \#Cl(K)$, the *class number* of $K$
- $w_K = \#\mu(K)$ the order of the group of roots of unity of $K$
- $d_K$ and $R_K$ are the *discriminant* and *regulator* of $K$.

We recall the definitions. Write $r = r_1 + r_2$, and let

$$\sigma_1,\dots,\sigma_{r_1}\colon K \hookrightarrow \mathbb{R}$$
$$\sigma_{r_1+1} = \overline{\sigma}_{r+1},\dots,\sigma_r = \overline{\sigma}_n\colon K \hookrightarrow \mathbb{C}$$

be the complex embeddings of $K$. Write $\sigma = (\sigma_i,\dots\sigma_{r_1+r_2})\colon K \hookrightarrow \mathbb{R}^{r_1}\times\mathbb{C}^{r_2}$. If $\{\theta_1,\dots,\theta_n\}$ is an integral basis for $K$, then by definition $d_K = \det(\sigma_j(\theta_k))^2$.

Write the group of units of $K$ as $\mathfrak{o}_K^\times = \mu_K \times \langle\varepsilon_1,\dots,\varepsilon_{r-1}\rangle$. Let $e_i = 1$ if $\sigma_i$ is real, 2 otherwise, so that $\left|N_{K/\mathbb{Q}}(x)\right| = \prod_i |\sigma_i(x)|^{e_i}$.

Write

$$\mathbb{R}^{r,1}_{>0} = \Big\{(u_j)\in\mathbb{R}^r \;\Big|\; u_j>0,\ \prod_j u_j = 1\Big\}.$$

It is convenient to define, for $\varepsilon\in\mathfrak{o}_K^\times$, $\iota(\varepsilon) = (|\sigma_j(\varepsilon)|^{e_j})_j$, so that $\iota\colon \mathfrak{o}_K^\times \to \mathbb{R}^{r,1}_{>0}$. The proof of the unit theorem shows that $\ker(\iota) = \mu(K)$, that $\iota(\mathfrak{o}_K^\times)$ is a discrete subgroup and that $\mathbb{R}^{r,1}_{>0}/\iota(\mathfrak{o}_K^\times)$ is compact. As measure on $\mathbb{R}^{r,1}_{>0}$ we will always take

$$d^\times u := \frac{du_1\cdots du_{r-1}}{u_1\cdots u_{r-1}}.$$

This measure is invariant under multiplication $u_j\mapsto b_j u_j$, for any $(b_j)\in\mathbb{R}^{r,1}_{>0}$.

Consider the $(r-1)\times r$ real matrix

$$(e_j\log|\sigma_k(\varepsilon_k)|) = (\log\iota(\varepsilon_k)_j) \qquad (1\le k\le r-1,\ 1\le j\le r).$$

The sum of the $k$-th row of this matrix is $\log\left|N_{K/\mathbb{Q}}(\varepsilon_k)\right| = 0$. So all of its $(r-1)\times(r-1)$ minors have the same absolute value, which is by definition $R_K$. The proof of the unit theorem shows that $R_K\neq 0$.

**Lemma.** *(i) The covolume of the lattice $\sigma(\mathfrak{o}_K)\subset\mathbb{R}^{r_1}\times\mathbb{C}^{r_2}$ with respect to Lebesgue measure on $\mathbb{R}^{r_1}\times\mathbb{C}^{r_2}$ equals $2^{-r_2}|d_K|^{1/2}$.*

*(ii) The volume of $\mathbb{R}^{r,1}_{>0}/\iota(\mathfrak{o}_K^\times)$ with respect to the measure $d^\times u$ equals $R_K$.*

*Proof.* (i) We have $\det(\sigma_j(\theta_k)) = (-2i)^{r_2} \det(\Delta_{kj})$, where the $k^{\text{th}}$ row of the matrix $(\Delta_{kj})$ is

$$(\sigma_1(\theta_k), \dots, \sigma_{r_1}(\theta_k), \operatorname{Re}\sigma_{r_1+1}(\theta_k), \operatorname{Im}\sigma_{r_1+1}(\theta_k), \dots, \operatorname{Im}\sigma_r(\theta_k))$$

and $|\det(\Delta_{kj})| = m(\sigma(\mathfrak{o}_K))$.

(ii) The isomorphism $\mathbb{R}^{r,1}_{>0} \xrightarrow{\sim} \mathbb{R}^{r-1}$, $(u_j) \mapsto (\log u_j)_{1\le j\le r-1}$ transforms the measure $d^\times u$ into Lebesgue measure $dv_1 \cdots dv_{r-1}$. So the result follows from the definition of $R_K$. □

*Proof of Theorem.* We begin by breaking the sum up into ideal classes:

$$\zeta_K(s) = \sum_{\mathcal{C}\in Cl(K)} \zeta_K(\mathcal{C}, s), \quad \text{where } \zeta_K(\mathcal{C}, s) = \sum_{I\subset \mathfrak{o}_K,\ I\in\mathcal{C}} N(I)^{-s}.$$

Fix $I_0 \in \mathcal{C}^{-1}$. Then $\mathcal{C} = \{xI_0^{-1} \mid x \in I_0\}$ and

$$\zeta_K(\mathcal{C}, s) = N(I_0)^s \sum_{x\in I_0}{}^{*} \left|N_{K/\mathbb{Q}}(x)\right|^{-s}$$

where $^*$ denotes the sum over a set of representatives of $I_0 \setminus \{0\}q$ modulo $\mathfrak{o}_K^\times$.

Recall that if $K = \mathbb{Q}$ then $\zeta(s)$ is the Epstein zeta function $(1/2)G(\mathbb{Z}, s/2)$. If $\mathbb{Q}(\sqrt{-D})$ is imaginary quadratic, then as $\mathfrak{o}_K$ is finite, the sum is just $1/w_K$ times the sum over all nonzero $x \in I_0$, and $N_{K/Q}(x) = x\bar{x}$ is a positive definite quadratic form. So $\zeta_K(\mathcal{C}, s)$ is also an Epstein zeta function.

We saw in the lectures that, when $K$ is real quadratic, $\zeta_K(\mathcal{C}, s)$ can be written as a 1-dimensional integral of Epstein zeta functions. For general $K$, we will express it as an $(r-1)$-dimensional integral of Epstein zeta functions. The proof is not very different, the main difficuly being notational. We begin with (a weighted version of) the Hecke transform in $r$ dimensions.

**Lemma** (The Hecke transform)**.** *Let $a_j > 0$, $z_j \in \mathbb{C}^\times$ $(1 \le j \le r)$. Write $A = \sum a_j$, $B = \prod a_j$. Then if $\operatorname{Re}(s) > 0$,*

$$\frac{B}{A} \prod_{j=1}^{r} \frac{\Gamma(a_j s)}{|z_j|^{2a_j s}} = \Gamma(As) \int_{\mathbb{R}^{r,1}_{>0}} \frac{1}{\left(\sum_{j=1}^r u_j^{1/a_j} |z_j|^2\right)^{As}} \, d^\times u$$

*Proof.*

$$\begin{aligned}
(\text{RHS}) &= \int_0^\infty \int_{\mathbb{R}^{r,1}_{>0}} \frac{e^{-t} t^{As}}{\left(\sum_{j=1}^r u_j^{1/a_j} |z_j|^2\right)^{As}} \, d^\times u \, \frac{dt}{t} \\
&= \int_0^\infty \int_{\mathbb{R}^{r,1}_{>0}} \exp\Big[-\Big(\sum_{j=1}^r u_j^{1/a_j} |z_j|^2\Big)t\Big] \, t^{As} \, d^\times u \, \frac{dt}{t}.
\end{aligned}$$

Put $y_j = u_j^{1/a_j} t$. Then $t^A = \prod y_j^{a_j}$ and

$$\frac{dy_j}{y_j} = \frac{dt}{t} + \frac{1}{a_j}\frac{du_j}{u_j} \qquad (1 \le j < r), \qquad \frac{dy_r}{y_r} = \frac{dt}{t} - \frac{1}{a_r}\Big(\frac{du_1}{u_1} + \dots + \frac{du_{r-1}}{u_{r-1}}\Big)$$

so that

$$\frac{dy_1}{y_1}\cdots\frac{dy_r}{y_r} = |J|\,\frac{dt}{t}\,d^\times u$$

with

$$|J| = \begin{Vmatrix} 1 & a_1^{-1} & 0 & \cdots & 0 \\ 1 & 0 & a_2^{-1} & \cdots & 0 \\ \vdots & & & \ddots & \vdots \\ 1 & -a_r^{-1} & \cdots & \cdots & -a_r^{-1} \end{Vmatrix} = \begin{Vmatrix} 1 & a_1^{-1} & 0 & \cdots & 0 \\ 1 & 0 & a_2^{-1} & \cdots & 0 \\ \vdots & & & \ddots & \vdots \\ A/a_r & 0 & \cdots & \cdots & 0 \end{Vmatrix} = \frac{A}{B}$$

So

$$\begin{aligned}(\mathrm{RHS}) &= \frac{B}{A}\int_{\mathbb{R}^n_{>0}} \exp\Bigl[-\sum_{j=1}^{r} |z_j|^2\, y_j\Bigr]\,\bigl(\prod y_j^{a_j}\bigr)^a\,\frac{dy_1}{y_1}\cdots\frac{dy_r}{y_r} \\ &= \prod_{j=1}^{r}\int_0^\infty e^{-|z_j|^2 y} y^{a_j s}\,\frac{dy}{y} = \prod_{j=1}^{r} |z_j|^{-2a_j s}\,\Gamma(a_j s). \qquad \square\end{aligned}$$

For each $u = (u_j) \in \mathbb{R}^{r,1}_{>0}$, let $V_u$ be the real vector space $\mathbb{R}^{r_1}\times\mathbb{C}^{r_2}$ together with the inner product

$$\begin{aligned}(\underline{x},\underline{y})_u &= \sum_{1\le j\le r_1} u_j^2 x_j y_j + 2\sum_{r_1<j\le r} u_j\bigl(\mathrm{Re}(x_j)\,\mathrm{Re}(y_j) + \mathrm{Im}(x_j)\,\mathrm{Im}(y_j)\bigr) \\ &= \sum_{1\le j\le r_1} u_j^2 x_j y_j + \sum_{r_1<j\le r} u_j(x_j\overline{y}_j + \overline{x}_j y_j)\end{aligned}$$

whose associated norm $\|-\|_u$ is

$$\|(x_j)\|_u^2 = \sum_{1\le j\le r_1} u_j^2\,|x_j|^2 + 2\sum_{r_1<j\le r} u_j\,|x_j|^2 = \sum_{j=1}^{r} e_j u_j^{1/e_j}\,|x_j|^2$$

If $\varepsilon \in \mathfrak{o}_K^\times$ then

$$\|\sigma(\varepsilon)(x_j)\|_u^2 = \sum_{j=1}^{r} e_j u_j^{2/e_j}\,|\sigma_j(\varepsilon)|^2\,|x_j|^2 = \|(x_j)\|^2_{\iota(\varepsilon)u}\,.$$

Because $\prod u_j = 1$ the measure on $V_u$ associated to the inner product is independent of $u$, and is

$$\prod_{j=1}^{r_1} dx_j \prod_{j=r_1+1}^{r} 2d\,\mathrm{Re}(x_j)\,d\,\mathrm{Im}(x_j) = 2^{r_2}\times(\text{Lebesgue measure on } \mathbb{R}^{r_1}\times\mathbb{C}^{r_2}).$$

Let $x \in K^\times$. We compute the Hecke transform for

$$z_j = \begin{cases}\sigma_j(x) \\ \sqrt{2}\sigma_j(x)\end{cases}, \qquad a_j = \begin{cases}1/2 & (1\le j\le r_1) \\ 1 & (r_1 < j \le r)\end{cases}$$

so that $A = r_1/2 + r_2 = n/2$, $B = 2^{-r_1}$, and $\prod |z_j|^{2a_j s} = 2^{r_2 s}\,\bigl|N_{K/\mathbb{Q}}(x)\bigr|^s$, to get:

**Corollary.** *If* $x \in K^\times$ *and* $\mathrm{Re}(s) > 0$ *then*

$$\Gamma(s/2)^{r_1}\,\Gamma(s)^{r_2}\left|N_{K/\mathbb{Q}}(x)\right|^{-s} = 2^{r_1+r_2 s-1} n\,\Gamma(ns/2)\int_{\mathbb{R}^{r,1}_{>0}} \frac{1}{\|\sigma(x)\|_u^{ns}}\,d^\times u. \quad \square$$

Now let $I_0 \subset \mathfrak{o}_K$ be an ideal, and let $\Lambda_u = \Lambda_u(I_0)$ be the lattice $\sigma(I_0)$ in the inner product space $V_u$. If $\mathrm{Re}(s) > n/2$ then the Epstein zeta function of $\Lambda_u$ is

$$G(\Lambda_u, s) = \sum_{0\neq x\in I_0} \frac{1}{\|\sigma(x)\|_u^{2s}} = \sum_{x\in I_0}{}^{*} \sum_{\varepsilon\in\mathfrak{o}_K^\times} \frac{1}{\|\sigma(\varepsilon x)\|_u^{2s}} = \sum_{x\in I_0}{}^{*} \sum_{\varepsilon\in\mathfrak{o}_K^\times} \frac{1}{\|\sigma(x)\|_{\iota(\varepsilon)u}^{2s}}.$$

In particular, for every $\varepsilon \in \mathfrak{o}_K^\times$, $G(\Lambda_u, s) = G(\Lambda_{\iota(\varepsilon)u}, s)$, and so the integral

$$\int_{\mathbb{R}^{n,1}_{>0}/\iota(\mathfrak{o}_K^\times)} G\Big(\Lambda_u, \frac{ns}{2}\Big)\,d^\times u$$

is well-defined and equals

$$\begin{aligned}\sum_{x\in I_0}{}^{*} \int_{\mathbb{R}^{n,1}_{>0}/\iota(\mathfrak{o}_K^\times)} \sum_{\varepsilon\in\mathfrak{o}_K^\times} \frac{1}{\|\sigma(x)\|_{\iota(\varepsilon)u}^{ns}}\,d^\times u &= w_K \sum_{x\in I_0}{}^{*} \int_{\mathbb{R}^{n,1}_{>0}/\iota(\mathfrak{o}_K^\times)} \sum_{\delta\in\iota(\mathfrak{o}_K^\times)} \frac{1}{\|\sigma(x)\|_{\delta u}^{ns}}\,d^\times u \\ &= w_K \sum_{x\in I_0}{}^{*} \int_{\mathbb{R}^{n,1}_{>0}} \frac{1}{\|\sigma(x)\|_u^{ns}}\,d^\times u\end{aligned}$$

using in the first equality the fact that $\ker(\iota) = \mu(K)$, and in the second the invariance of $d^\times u$ under multiplication $u \mapsto \delta u$. If apply the Corollary to this last expression we obtain the desired representation of $\zeta_K(\mathcal{C}, s)$ as an integral of Epstein zeta functions:

$$\begin{aligned}\zeta_k(\mathcal{C}, s) &= N(I_0)^s \frac{2^{r_1+r_2 s-1} n\,\Gamma(ns/2)}{w_K \Gamma(s/2)^{r_1}\,\Gamma(s)^{r_s}} \int_{\mathbb{R}^{r,1}_{>0}/\iota(\mathfrak{o}_K^\times)} G\Big(\Lambda_u, \frac{ns}{2}\Big)\,d^\times u. \\ &= \frac{2^{r-1} n N(I_0)^s}{w_K \Gamma_{\mathbb{R}}(s)^{r_1} \Gamma_{\mathbb{C}}(s)^{r_2}} \int_{\mathbb{R}^{r,1}_{>0}/\iota(\mathfrak{o}_K^\times)} Z\Big(\Lambda_u, \frac{ns}{2}\Big)\,d^\times u.\end{aligned}$$

As $Z(\Lambda_u, s)$ has an analytic continuation to $\mathbb{C} \smallsetminus \{n/2, 0\}$, and $\Gamma(s)$ has no zeroes, this gives the desired analytic continuation of $\zeta_K(\mathcal{C}, s)$ away from $s = 1$, $0$.

At $s = 0$, $G(\Lambda_u, s)$ is holomorphic and $G(\Lambda_u, 0) = -1$. So

$$\int_{\mathbb{R}^{r,1}_{>0}/\iota(\mathfrak{o}_K^\times)} G\Big(\Lambda_u, \frac{ns}{2}\Big)\,d^\times u\Big|_{s=0} = -\int_{\mathbb{R}^{r,1}_{>0}/\iota(\mathfrak{o}_K^\times)} d^\times u = -R_K.$$

Therefore

$$\zeta_K(\mathcal{C}, s) \underset{s=0}{\sim} 1 \times \frac{2^{r_1-1} n (ns/2)^{-1}}{w_K\,(s/2)^{-r_1} s^{-r_2}} (-R_K) = -\frac{R_K}{w_K} s^{r-1}$$

as required.

Finally compute at $s = 1$. Recalling that the measure on $V_u$ is $2^{r_2}$ times Lebesgue measure, we have $m(\Lambda_u) = N(I_0)\,|d_K|^{1/2}$, and $Z(\Lambda_u, ns/2) = m(\Lambda_u)^{-1}(ns/2 - n/2)^{-1} + (\text{analytic})$. Therefore

$$\int_{\mathbb{R}^{r,1}_{>0}/\iota(\mathfrak{o}_K^\times)} Z\Big(\Lambda_u, \frac{ns}{2}\Big)\,d^\times u = \frac{2R_K}{N(I_0)\,|d_K|^{1/2}\,n(s-1)} + (\text{analytic at } s = 1).$$

Using $\Gamma_{\mathbb{R}}(1) = \pi^{-1/2}\Gamma(1/2) = 1$ and $\Gamma_{\mathbb{C}}(1) = 2(2\pi)^{-1}\Gamma(1) = \pi^{-1}$, this becomes

$$\zeta_K(\mathcal{C}, s) \underset{s=1}{\sim} \frac{2^r \pi^{r_2}}{w_k \left|d_K\right|^{1/2}(s-1)}.$$

Summing over ideal classes then gives the result.

To prove the functional equation, we have

$$Z_K(\mathcal{C}, s) := \Gamma_{\mathbb{R}}(s)^{r_1}\Gamma_{\mathbb{C}}(s)^{r_2}\zeta_K(\mathcal{C}, s) = \frac{2^{r-1}n}{w_K}\int_{\mathbb{R}^{r_1}_{>0}/\iota(\mathfrak{o}_K^\times)} N(I_0)^s Z\Big(\Lambda_u(I_0), \frac{ns}{2}\Big)\, d^\times u.$$

Recall the definition of the *different* ideal $\mathcal{D}_K$ of $K$: it is the ideal whose inverse is the dual of $\mathfrak{o}_K$ under the trace form:

$$\mathcal{D}_K^{-1} := \{y \in K \mid \forall x \in K,\ \mathrm{tr}_{K/\mathbb{Q}}(xy) \in \mathbb{Z}\}.$$

The norm of $\mathcal{D}_K$ is $|d_K|$.

**Proposition 0.1.** *$Z(\Lambda_u(I_0), s) = N(I_0)^{-1}\left|d_K\right|^{-1/2} Z(\Lambda_{u^{-1}}(I_1), n/2 - s)$, where $I_1 = I_0^{-1}\mathcal{D}_K^{-1}$.*

*Proof.* We need to identify the dual lattice to $\Lambda_u(I_0)$. Define a map $\phi_u \colon V_u \xrightarrow{\sim} V_{u^{-1}}$ by $\phi_u(\underline{x})_j = u_j^{-1/e_j}\overline{x}_j$. From the definition of the inner product on $V_u$, this map is an isometry between $V_u$ and $V_{u^{-1}}$, and for $\underline{x} \in V_u$, $\underline{y} \in V_{u^{-1}}$,

$$\big(\phi_u(\underline{x}), \underline{y}\big)_{u^{-1}} = \sum_{j=1}^{r_1} x_j y_j + \sum_{j=r_1+1}^{r} x_j y_j + \overline{x_j y_j}.$$

**Claim:** $\phi_u \colon \Lambda_u(I_0) \xrightarrow{\sim} \Lambda_{u^{-1}}(I_1)'$.

Granted this, we have (since $\phi_u$ is an isometry)

$$\begin{aligned} Z(\Lambda_u(I_0), s) = Z(\Lambda_{u^{-1}}(I_1)', s) &= m(\Lambda_{u^{-1}}(I_1)')^{-1} Z(\Lambda_{u^{-1}}(I_1), n/2 - s) \\ &= m(\Lambda_u(I_0))^{-1} Z(\Lambda_{u^{-1}}(I_1), n/2 - s) \end{aligned}$$

so the Proposition follows from the formula above for $m(\Lambda_u)$. To prove the claim, let $\theta_i$ $(1 \le i \le n)$ be a $\mathbb{Z}$-basis for $I_0$, and let $(\theta_i')_i$ be the dual basis for $K$ with respect to the trace form — so it is a $\mathbb{Z}$-basis for $I_0^{-1}\mathcal{D}_k^{-1} = I_1$. We then have, for $i, k \in \{1, \dots, n\}$,

$$\begin{aligned} \big(\phi_u(\sigma(\theta_i)), \sigma(\theta_k')\big)_{u^{-1}} &= \sum_{1 \le j \le r_1} \sigma_j(\theta_i\theta_k') + \sum_{r_1 < j \le r} \sigma_j(\theta_i\theta_k') + \overline{\sigma_j}(\theta_i\theta_k') \\ &= \mathrm{tr}_{K/Q}(\theta_i\theta_k') = \delta_{ik} \end{aligned}$$

so $\phi_u(\sigma(I_0))$ and $\sigma(I_1)$ are dual lattices in $V_{u^{-1}}$. □

We then have

$$\begin{aligned} N(I_0)^s Z(\Lambda_u(I_0), ns/2) &= N(I_0)^{s-1}\left|d_K\right|^{-1/2} Z(\Lambda_{u^{-1}}(I_1), n(1-s)/2) \\ &= \left|d_K\right|^{1/2-s} N(I_1)^{1-s} Z(\Lambda_{u^{-1}}(I_1) \end{aligned}$$

since $N(I_1) = \left|d_K\right|^{-1} N(I_0)^{-1}$. As the measure $d^\times u$ is invariant under $u \mapsto u^{-1}$, we obtain $Z_K(\mathcal{C}, s) = \left|d_K\right|^{1/2-s} Z_K(\mathcal{C}', 1-s)$ where $\mathcal{C}'$ is the class of $I_1^{-1}$. Summing over ideal classes gives the functional equation for $Z_K(s)$. □