Modular forms and L-functions (Michaelmas 2017) — example sheet #3

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1. (i) Let $\Lambda \subset \mathbb{R}^n$ be a lattice. The *dual lattice* is the set

$$\Lambda' = \{x \in \mathbb{R}^n \mid \text{for all } y \in \Lambda,\ (x,y) \in \mathbb{Z}\}$$

(where (x,y) is the usual inner product on $\mathbb{R}^n$). Show that Λ' is a lattice.

(ii) Consider the function $f(\underline{x}) = \exp\left(-\pi t \left\|A\underline{x}\right\|^2\right)$, where $\Lambda = A\,\mathbb{Z}^n$, $A \in GL_n(\mathbb{R})$. Compute its Fourier transform, and show by Poisson summation that the theta function

$$\vartheta_\Lambda(z) = \sum_{x\in\lambda} e^{\pi i \|x\|^2}$$

satisfies the transformation law

$$\vartheta_\Lambda(-1/z) = (z/i)^{n/2} v(\mathbb{R}^n/\Lambda)^{-1} \vartheta_{\Lambda'}(z)$$

where $v(\mathbb{R}^n/\Lambda)$ is the volume of the quotient.

(iii) We say Λ is *self-dual* if $\Lambda' = \Lambda$, and *even* if $\|x\|^2 \in 2\mathbb{Z}$ for every $x \in \Lambda$. Show that if Λ is self-dual and even, then ϑ_Λ is a modular form of weight $n/2$ and level 1, and that n is divisible by 8. (Hint: compute $\vartheta_\Lambda^r|ST$, for r a suitable power of 2.)

2. Let $\Lambda \subset \mathbb{R}^8$ be the set of all $x = (x_1 \ldots, x_8) \in \mathbb{R}^8$ satisfying

$$2x_i \in \mathbb{Z}, \qquad x_i - x_j \in \mathbb{Z}, \qquad \sum_{i=1}^{8} x_i \in 2\mathbb{Z}.$$

Show that Λ is an even self-dual lattice. [Λ is usually denoted E_8.] Show that ϑ_Λ is the normalised Eisenstein series E_4. Hence (or directly) show that there are exactly 240 elements $x \in \Lambda$ with $\|x\|^2 = 2$.

3. (i) Let $f \in S_k(\Gamma(1))$ and $N \geq 1$, a, d integers with $ad \equiv 1 \pmod{N}$. Show that

$$f\Big(\frac{-1}{N^2\tau} + \frac{a}{N}\Big) = (N\tau)^k f\Big(\tau - \frac{d}{N}\Big).$$

(ii) Suppose further that f has q-expansion $\sum_{n\geq 1} c_n q^n$. By considering the Mellin transform of $g(\tau) = f(a/N + \tau)$, show that the function

$$M(f, a/N, s) = \Big(\frac{N}{2\pi}\Big)^s \Gamma(s) \sum_{n=1}^{\infty} e^{2\pi i a n/N} c_n n^{-s}$$

has the integral representation

$$M(f, a/N, s) = \int_{1/N}^{\infty} \left(f\Big(\frac{a}{N} + iy\Big)(Ny)^s + (-1)^{k/2} f\Big(\frac{-d}{N} + iy\Big)(Ny)^{k-s} \right) \frac{dy}{y}$$

and deduce that $M(f, a/N, s)$ has an analytic continuation to $\mathbb{C}$ which satisfies the functional equation

$$M(f, a/N, k-s) = (-1)^{k/2} M(f, -d/N, s).$$

4. Let $p > 2$ be prime. Draw a fundamental domain for $\Gamma^0(p)$ as given in the lectures, and and show that the identifications of points along the boundary are given as follows:

   - the vertical lines $\mathrm{Im}(z) = \pm p/2$ are identified by the translation $z \mapsto z + p$.
   - the circular arcs $C_a = \{|z - a| = 1\}$, for integers a with $0 < |a| < p/2$, are identified as follows: C_a is identified with C_b iff $ab \equiv -1 \pmod{p}$.

5. (i) Show that every element of finite order of $SL_2(\mathbb{Z})$ is conjugate to one of S^a, $(ST)^a$ for some $a \in \mathbb{Z}$.

   (ii) Show that $\Gamma(N)$ is torsionfree if $N \geq 3$, and that the only elements of finite order of $\Gamma(2)$ are $\{\pm 1\}$.

   (iii) Show that if $N \geq 4$ then $\Gamma_1(N)$ is torsionfree.

6. Let
$$\Gamma^* = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \Gamma \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

   (i) Show that if $f \in M_k(\Gamma)$, then the function $f^*(z) = \overline{f(-\bar{z})}$ belongs to $M_k(\Gamma^*)$.

   (ii) Show that if $\Gamma = \Gamma^*$ (for example, any one of $\Gamma_0(N)$, $\Gamma_1(N)$, $\Gamma(N)$) then $M_k(\Gamma)$ has a basis all of whose elements have real Fourier coefficients.

7. (i) Show that if every cusp of Γ has width one then Γ must be $\Gamma(1)$.

   (ii)** Show that if Γ is a congruence subgroup containing -1, then $\Gamma \supset \Gamma(N)$ where N is the least common multiple of the widths of the cusps of Γ. (This gives a way to tell whether or not a given group is a congruence subgroup.)

8. Let $N > 1$, and let $c(M) \in \mathbb{C}$ be given for each $M|N$. Show that
$$\sum_{M|N} c(M) E_2(M\tau)$$
   is a modular form of weight 2 on $\Gamma_0(N)$ if and only if $\sum M^{-1} c(M) = 0$.

9. Let $k \geq 3$ and $\underline{r} = (r_1, r_2) \in \mathbb{Q}^2$ (row vectors). Define
$$G_{\underline{r},k}(z) = \sum_{\underline{m} \in \mathbb{Z}^2}{}' \frac{1}{\big((m_1 + r_1)z + m_2 + r_2\big)^k}$$
   where $'$ means that the term $\underline{m} + \underline{r} = \underline{0}$ (if it exists) is omitted.

   (i) Show that if $\gamma \in \Gamma(1)$, the $G_{\underline{r},k}|_k \gamma = G_{\underline{r}\gamma,k}$.

   (ii) Suppose $N \geq 1$ and $N\underline{r} \in \mathbb{Z}^2$. Show that $G_{\underline{r},k} \in M_k(\Gamma(N))$.

10. * Let $\Gamma \subset SL_2(\mathbb{Z})$ be a subgroup of finite index containing -1. Let $\Gamma_\infty \subset \Gamma$ be the stabiliser of the cusp ∞. Show that if $k > 2$ is even, then the series
$$E_{\Gamma,k}(\tau) = \frac{1}{2} \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} \frac{1}{(c\tau + d)^k} \qquad \text{where } \gamma = \begin{pmatrix} * & * \\ c & d \end{pmatrix}$$
    converges and defines an element of $M_k(\Gamma)$ which is not a cusp form, but which vanishes at every cusp of Γ other than ∞.

    By considering modular forms of the shape $E_{k,\Gamma'}|_k \gamma$, deduce that if Γ has ν cusps, then for even $k > 2$ one has
$$\dim M_k(\Gamma) - \dim S_k(\Gamma) = \nu.$$