Modular forms and L-functions (Michaelmas 2017) — example sheet #2

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1. Fix a positive real number D. Let $\mathcal{X}_D$ be the set of real symmetric 2×2 matrices of determinant D. Let $SL_2(\mathbb{R})$ act on $\mathcal{X}_D$ by $g \colon X \to gXg^t$ ($g^t =$ transpose of g). Describe the orbits of this action, and identify one of them with the upper halfplane.

   If you know about binary quadratic forms, describe the connection between the fundamental domain of $SL_2(\mathbb{Z})\backslash\mathcal{H}$ and Gauss's reduction theory.

2. A subgroup $\Lambda \subset \mathbb{R}^n$ is *discrete* if is is discrete as a topological space (i.e. for any ball $B \subset \mathbb{R}^n$, $\Lambda \cap B$ is finite). Show that Λ is discrete iff $\Lambda = \sum \mathbb{Z}x_i$ where $\{x_i\} \subset \mathbb{R}^n$ is an $\mathbb{R}$-linearly independent set.

3. Write $E_6(z)\Delta(z) = \sum_{n=1}^{\infty} c_n q^n$. Show that $c_n \equiv \sigma_{17}(n) \pmod{43867}$. Obtain similar congruences for the coefficients of $E_4\Delta$, $E_8\Delta$, $E_{10}\Delta$ and $E_{14}\Delta$.

   [NB: $B_{16} = -3617/510$, $B_{18} = 43867/798$, $B_{20} = -174611/330$, $B_{22} = 854513/138$, $B_{26} = 8553103/6$.]

4. Let $f \in M_k$. Show that if $k \not\equiv 0 \pmod 4$ then $f(i) = 0$, and that if $k \not\equiv 0 \pmod 3$ then $f(\rho) = 0$, where $\rho = e^{\pi i/3}$.

5. Let $M_k^!$ denote the space of weakly modular (meromorphic at infinity) forms of weight $k \in 2\mathbb{Z}$. Find a basis for $M_k^!$ in terms of E_4, E_6 and Δ.

6. Let $f \in M_k$ and $g \in M_l$ be modular forms. Show that $lf'g - kfg' \in M_{k+l+2}$.

7. Let $f \colon \mathcal{H} \to \mathbb{C}$ satisfy $f|_k\gamma = f$ for all $\gamma \in \Gamma(1)$. Show that $y^{k/2}\,|f(x+iy)|$ is invariant under $z = x+iy \mapsto \gamma(z)$. Show that if moreover f is holomorphic on $\mathcal{H}$ and $k > 0$, then f is a cusp form if and only if $y^{k/2}\,|f|$ is bounded on $\mathcal{H}$ (or equivalently, is bounded on $\mathcal{D}$).

8. Define
$$G_2(z) = \sum_{m=-\infty}^{\infty}{}' \left(\sum_{n=-\infty}^{\infty} \frac{1}{(mz+n)^2} \right)$$
   where the inner sum is over all integers n, except where $m = 0$, in which case the term $n = 0$ is omitted. By rewriting the inner sum, show that the series converges to
$$\frac{\pi^2}{3}\left(1 - 24\sum_{n=1}^{\infty} \sigma_1(n) q^n\right).$$
   Explain why $G_2(z)$ is not a modular form of weight 2. (Later we will see that G_2 satisfies a somewhat more complicated transformation law for $z \mapsto -1/z$.)

9. Fix an even weight $k \geq 4$. Let $\mathbb{T} \subset \operatorname{End} S_k$ be the subalgebra of endomorphisms generated over $\mathbb{Z}$ by the Hecke operators T_n, $n \geq 1$. Let $S_k(\mathbb{Z}) = S_k \cap \mathbb{Z}[[q]]$ denote the submodule of cusp forms with integral Fourier coefficients. Show that $S_k(\mathbb{Z})$ is stable under $\mathbb{T}$, and that the map
$$\begin{aligned} S_k(\mathbb{Z}) \times \mathbb{T} &\to \mathbb{Z} \\ (f, T_n) &\mapsto a_1(T_n f) \end{aligned}$$
   gives an isomorphism between $S_k(\mathbb{Z})$ and $\operatorname{Hom}_{\mathbb{Z}}(\mathbb{T}, \mathbb{Z})$, which is an isomorphism of $\mathbb{T}$-modules.