Modular forms and L-functions (Michaelmas 2017) — example sheet #1

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1. Let G be a finite abelian group, and χ, χ' characters of G. Show that

$$\sum_{g\in G} \overline{\chi(g)}\chi'(g) = \begin{cases} 0 & \text{if } \chi \neq \chi' \\ \#G & \text{if } \chi = \chi' \end{cases}$$

2. Show that every continuous homomorphism $\chi\colon \mathbb{R} \to \mathbb{C}^\times$ is of the form $\chi(x) = z^x$, for some $z \in \mathbb{C}^\times$. Deduce that the character group $\widehat{\mathbb{R}}$ equals $\{\chi_y \mid y \in \mathbb{R}\}$, where $\chi_y(x) = e^{2\pi i x y}$.

   Show also that every continuous homomorphism $\chi\colon \mathbb{R}^\times_{>0} \to \mathbb{C}^\times$ is of the form $\chi(x) = x^s$ for some $s \in \mathbb{C}$, and identify the characters of $\mathbb{R}^\times_{>0}$.

   [Hint: first describe all continuous homomorphisms $\mathbb{R} \to \mathbb{C}$.]

3. *Dirichlet characters*. Let $\chi\colon (\mathbb{Z}/N\mathbb{Z})^\times \to \mathbb{C}^\times$ be a Dirichlet character mod N. We say that χ is *primitive* if there does not exist $M|N$, $1 \leq M < N$ and $\chi'\colon (\mathbb{Z}/M\mathbb{Z})^\times \to \mathbb{C}^\times$ such that $\chi = \chi' \circ \mathrm{red}_{N,M}$. (Here $\mathrm{red}_{N,M}\colon \mathbb{Z}/N\mathbb{Z} \to \mathbb{Z}/M\mathbb{Z}$ denotes the reduction mod M map.)

   We say that Dirichlet characters χ_i mod N_i ($i = 1, 2$) are *equivalent* if there exists a Dirichlet character χ mod N such that for $i = 1, 2$, $N_i|N$ and $\chi = \chi_i \circ \mathrm{red}_{N,N_i}$.

   (i) Show that every character is equivalent to a unique primitive character, and that if χ is a primitive character mod N, then the characters equivalent to it are precisely the characters $\chi \circ \mathrm{red}_{ND,N}$ for $D \geq 1$.

   (ii) Let $\chi_i\colon (\mathbb{Z}/N_i\mathbb{Z})^\times \to \mathbb{C}^\times$ ($i = 1, 2$) be equivalent. Show that

$$L(\chi_2, s) = \prod_{p|N_1,\ p\nmid N_2} \left(1 - \frac{\chi_2(p)}{p^s}\right)^{-1} \prod_{p\nmid N_1,\ p|N_2} \left(1 - \frac{\chi_1(p)}{p^s}\right) L(\chi_1, s).$$

4. Show that

$$-\frac{\zeta'(s)}{\zeta(s)} = \sum_{n\geq 1} \Lambda(n) n^{-s}$$

   where Λ is the *Von Mangoldt function*:

$$\Lambda(n) = \begin{cases} \log p & \text{if } n = p^k,\ k \geq 1,\ p \text{ prime} \\ 0 & \text{if } n = 1 \text{ or } n \text{ is not a prime power} \end{cases}$$

   Show also that $\sum_{d|n} \Lambda(d) = \log n$.

5. Evaluate $\zeta(2k)$ is terms of Bernoulli numbers. Deduce that $(-1)^{k-1}B_{2k} > 0$ for every $k \geq 1$. (It is not easy to prove this directly from the generating function definition!)

6. (i) The *Bernoulli polynomials* $B_n(X)$ are defined by the formula

$$\sum_{k=0}^{\infty} B_k(X)\frac{t^k}{k!} = \frac{te^{tX}}{e^t - 1}.$$

   Let $\psi\colon \mathbb{Z}/N\mathbb{Z} \to \mathbb{C}$ be any periodic function. Use the analytic continuation to write the values at negative integers of the L-series $L(\psi, s)$ in terms of the values $B_k(j/N)$.

   (ii) Consider the (inverse) Fourier transform

$$\widehat{B}_{n,N}(\zeta) = \sum_{j=0}^{N-1} \zeta^j B_n(j/N) \qquad (\zeta^N = 1)$$

   Show that if $\zeta \neq 1$, then $\widetilde{B}_n(\zeta) \overset{\text{def}}{=} N^{1-n} B_{n,N}(\zeta) = P_n(\zeta)/(\zeta - 1)^n$ for some polynomial P_n which does not depend on N. (The substitution $u = e^{-t/N}$ may be useful.)

   (iii) Obtain for $D > 1$ the *distribution relation*: if $\zeta^N = 1$ then

$$\sum_{\eta^D = \zeta} \widetilde{B}_n(\eta) = D^n \widetilde{B}_n(\zeta)\,.$$