Q1 If Λ is even then $\Theta_\Lambda(z) = \sum_{x\in\Lambda} q^{\frac{1}{2}\|x\|^2}$ is a power series in q

so $\Theta_\Lambda(z) = \Theta_\Lambda(z+1)$.

Let $A = (e_1 | \ldots | e_k) \in GL_n(\mathbb{R})$, columns $e_1 \ldots e_k$ basis for Λ.

As $\Lambda = \Lambda^*$, ${}^TA A = B \in GL_k(\mathbb{Z})$. So WLOG $\det A = 1$. (reorder basis if nec.)

$$\therefore \Theta_\Lambda(it) = \sum_{\underline{m}\in\mathbb{Z}^k} \exp -\pi t \|A\underline{m}\|^2 = \sum_{m\in\mathbb{Z}^k} f(\underline{m})$$

$$f(\underline{x}) = \exp(-\pi t\|A\underline{x}\|^2)$$

$$\hat{f}(\underline{y}) = \int_{\mathbb{R}^n} e^{-2\pi i \underline{x}\cdot\underline{y}} e^{-\pi t\|A\underline{x}\|^2} d\underline{x} \qquad \underline{x}' = t^{1/2}A\underline{x}, \quad d\underline{x}' = t^{k/2}\, d\underline{x}$$

$$= t^{-\frac{k}{2}}\int_{\mathbb{R}^n} e^{-2\pi i\, t^{-1/2}(A^{-1}\underline{x}')\cdot\underline{y}} e^{-\pi\|\underline{x}'\|^2} dx'$$

$$= t^{-k/2}\,\hat{g}(t^{-1/2}\, {}^TA^{-1}\underline{y}), \qquad g(\underline{x}) = e^{-\pi\|\underline{x}\|^2} = \prod_j e^{-\pi x_j^2} \quad (\text{so } \hat{g} = g)$$

$$= t^{-k/2}\exp(-\pi t^{-1}\|AB^{-1}\underline{y}\|^2) \qquad {}^TAA = B \Rightarrow AB^{-1} = {}^TA^{-1}$$

<u>Poisson summation:</u> $\Theta_\Lambda(it) = \sum_{\underline{m}\in\mathbb{Z}^k} f(\underline{m}) = \sum_{\underline{n}\in\mathbb{Z}^k}\hat{f}(\underline{n}) = t^{-k/2}\sum_{\underline{n}\in\mathbb{Z}^k} e^{-\pi t^{-1}\|AB^{-1}\underline{n}\|^2}$

$$= t^{-\frac{k}{2}}\sum_{\underline{n}'\in\mathbb{Z}^k} e^{-\pi t^{-1}\|A\underline{n}\|^2} = t^{-k/2}\Theta_\Lambda(i/t) \quad \text{as } B\in GL_k(\mathbb{Z}).$$

So $\Theta_\Lambda(-1/z) = (z/i)^{k/2}\,\Theta_\Lambda(z)$ by uniqueness of analytic continuation.

It's enough to show that $k \equiv 0 \pmod 8$; then $\Theta_\Lambda(-1/z) = z^{k/2}\Theta_\Lambda(z)$ $\Rightarrow \Theta \in M_{k/2}$. Suppose $k/8 = l/2^r$ with $r\geq 1$, l odd. Let $f = \Theta_\Lambda^{2^{r-1}}$.

Then $f(-\frac{1}{z}) = -z^{2l}f(z)$, ie. $f|_{2l}\begin{pmatrix} & 1\\ -1 & \end{pmatrix} = -f$, $f|_{2l}\begin{pmatrix}1 & 1\\ & 1\end{pmatrix} = f$.

Let $\gamma = \begin{pmatrix}0 & 1\\ -1 & -1\end{pmatrix} = \begin{pmatrix} & 1\\ -1 & \end{pmatrix}\begin{pmatrix}1 & 1\\ & 1\end{pmatrix}$; then $f|_{2l}\gamma = -f$

But $\gamma^3 = 1 \Rightarrow f = f|_{2l}\gamma^3 = -f \Rightarrow f = 0$.

[It also follows from classification for self-dual lattices that $k \equiv 0$ (8)]

Q2 (i) $\Lambda = \{\underline{x}\in\mathbb{R}^k \mid x_i \in \frac{1}{2}\mathbb{Z},\ x_i - x_j \in \mathbb{Z},\ \sum x_i \in 2\mathbb{Z}\}$.

$\underline{x}\in\Lambda$. Then either $\underline{x}\in\mathbb{Z}^k$, $\sum x_i \in 2\mathbb{Z}$
or $\underline{x} = (\frac{1}{2},\ldots,\frac{1}{2}) + \underline{x}'$, $\underline{x}'\in\mathbb{Z}^k$, $\sum x_i' \in 2\mathbb{Z}$.

So $\Lambda = \Lambda' + \mathbb{Z}.(\frac{1}{2},\ldots,\frac{1}{2})$. $\Lambda' = \{\underline{x}\in\mathbb{Z}^k \mid \sum x_i \in 2\mathbb{Z}\}$.

Obviously $\forall\, \underline{x},\underline{y}\in\Lambda'$, $\underline{x}\cdot\underline{y}\in\mathbb{Z}$; $\underline{x}.(\frac{1}{2},\ldots\frac{1}{2}) = \frac{1}{2}\sum x_i \in\mathbb{Z}$

and $(\frac{1}{2}\ldots\frac{1}{2}).(\frac{1}{2}\ldots\frac{1}{2}) = k/4 \in \mathbb{Z}$. So $\Lambda\subset\Lambda^*$

Suppose $y \in \Lambda^*$. Then $x = e_i - e_j = (0 \dots 1, \dots -1, \dots 0) \in \Lambda' \subset \Lambda$

$x \cdot y \in \mathbb{Z} \Rightarrow y_i - y_j \in \mathbb{Z}$

$x = 2e_i \in \Lambda' \Rightarrow y_i \in \frac{1}{2}\mathbb{Z}$. And $y \cdot (1/2 \dots 1/2) \in \mathbb{Z} \Rightarrow \sum y_j \in 2\mathbb{Z} \Rightarrow y \in \Lambda^*$.

(ii) $x \in \Lambda' \Rightarrow \sum x_i \equiv 0 \ (2) \Rightarrow \sum x_i^2 = (\sum x_i)^2 - 2 \sum_{i<j} x_i x_j \equiv 0 \ (2)$

$x = x' + (1/2, \dots, 1/2)$, $x' \in \Lambda \Rightarrow \|x\|^2 = \underbrace{\|x'\|^2}_{0 \ (2)} + \underbrace{\|(1/2 \dots 1/2)\|^2}_{k/4 \ \equiv 0 \ (\text{mod } 2)} + 2 \underbrace{x' \cdot (1/2 \dots 1/2)}_{\sum x_i' \equiv 0 \ (2)}$.

(iii) $k = 8 \Rightarrow \Theta_\Lambda \in M_4 = \mathbb{C} E_4$. Compare $a_0 \Rightarrow \Theta_\Lambda = E_4$.

And $a_1(\Theta_\Lambda) = \#\{x \in \Lambda \mid \|x\|^2 = 2\} = a_1(E_4) = 240$.

Q3 (i) $f\left(\frac{-1}{N^2 z} + \frac{a}{N}\right) = f(\gamma(z))$, $\gamma = \begin{pmatrix} aN & -1 \\ N^2 & 0 \end{pmatrix}$

$f\left(\tau - \frac{d}{N}\right) = f(\delta(z))$, $\delta = \begin{pmatrix} 1 & -d/N \\ & 1 \end{pmatrix}$

$\gamma \delta^{-1} = \begin{pmatrix} aN & -1 \\ N^2 & \end{pmatrix} \begin{pmatrix} 1 & -dN^{-1} \\ & 1 \end{pmatrix} = \begin{pmatrix} aN & 1-ad \\ N^2 & -dN \end{pmatrix} = \begin{pmatrix} N & \\ & N \end{pmatrix} \cdot \underbrace{\begin{pmatrix} a & \frac{1-ad}{N} \\ N & -d \end{pmatrix}}_{\in \Gamma(1)}$

$\Rightarrow f|_k \gamma \delta^{-1} = f \Rightarrow$ identity.

(ii) $M(g(iy), s) = \int_0^\infty \sum_{n \geq 1} c_n e^{2\pi i n (a/N + iy)} y^s \frac{dy}{y}$

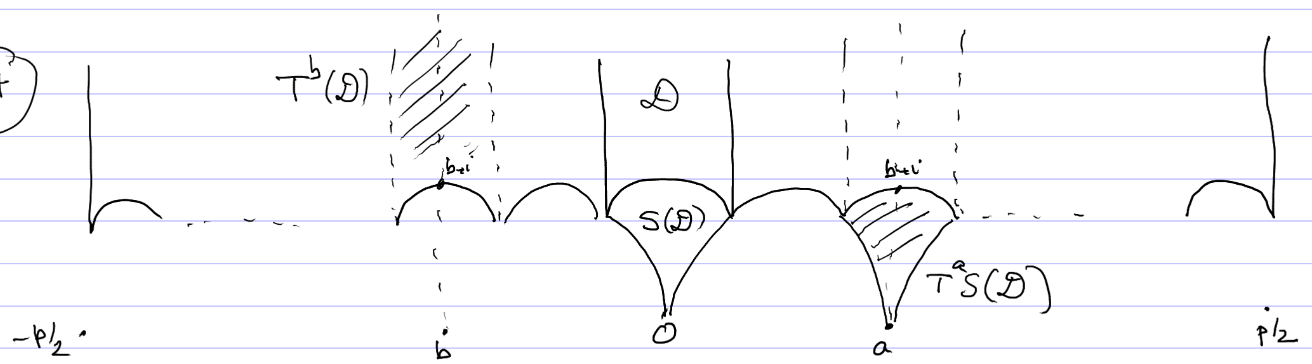
$= (2\pi)^{-s} \Gamma(s) \sum_{n \geq 1} e^{2\pi i an/N} c_n n^{-s}$.

$M(f, a/N, s) = \int_0^\infty N^s f\left(\frac{a}{N} + iy\right) y^s \frac{dy}{y} = \int_{1/N}^\infty + \int_0^{1/N}$

and (i) $\Rightarrow \int_0^{1/N} = \int_{1/N}^\infty (-1)^{k/2} f(-d/N + iy) (Ny)^{k-s} \frac{dy}{y}$.

from which rest follows.

Q4



$\begin{pmatrix} 1 & p \\ & 1 \end{pmatrix} : z \longmapsto z + p$.

Suppose $ab = -1 - pc \equiv -1 \pmod p$. Let $\gamma = \begin{pmatrix} a & pc \\ 1 & -b \end{pmatrix} \in \Gamma^0(p)$

Claim $\gamma : T^b(\mathcal{D}^{cl}) \longrightarrow T^a S(\mathcal{D}^{cl})$: since

$\gamma T^b = \begin{pmatrix} a & pc \\ 1 & -b \end{pmatrix} \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a+b & ab+pc \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} a & -1 \\ 1 & 0 \end{pmatrix} = T^a S$.

Q5 ells of finite order in $\Gamma(1) = SL_2(\mathbb{Z})$ are $\{\pm 1\}$ and elements conjugate to $\pm S$, $\pm (TS)^{\pm 1}$. $= \pm\begin{pmatrix} & -1 \\ 1 & \end{pmatrix}, \pm\begin{pmatrix} 1 & -1 \\ 1 & \end{pmatrix}, \pm\begin{pmatrix} & -1 \\ 1 & -1 \end{pmatrix}$.

(Proof: if $\gamma \in \Gamma(1) \setminus \{\pm 1\}$ has finite order n, then it is diagonalisable in $GL_2(\mathbb{C})$
ie. $\gamma = A \begin{pmatrix} \alpha & \\ & \alpha^{-1} \end{pmatrix} A^{-1}$, α primitive n-th root of 1, $n > 2$.
so if $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ then $\gamma\begin{pmatrix} a \\ c \end{pmatrix} = \alpha\begin{pmatrix} a \\ c \end{pmatrix}$, and $a/c \notin \mathbb{R}$ as eigenvalue $\alpha \notin \mathbb{R}$; $\gamma(a/c) = a/c$.)

i) $\Gamma(N) \triangleleft \Gamma(1)$. ; so just have to check $\pm S$ etc.

ii) $N \geq 4$. $\Gamma_1(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| \begin{array}{l} a \equiv d \equiv 1 \pmod N \\ c \equiv 0 \pmod N \end{array} \right\}$

If γ is conjugate to $\begin{cases} S \\ (TS)^{\pm 1} \end{cases}$ then $Tr(\gamma) = \begin{cases} 0 \\ \pm 1 \end{cases}$

If $\gamma \in \Gamma_1(N)$ then $Tr(\gamma) \equiv 2 \pmod N$ so for $N \geq 4$, $Tr(\gamma) \notin \{0, \pm 1\}$.

Q6 $\Gamma^* = \begin{pmatrix} -1 & \\ & 1 \end{pmatrix} \Gamma \begin{pmatrix} -1 & \\ & 1 \end{pmatrix} = \left\{ \gamma^* = \begin{pmatrix} a & -b \\ -c & d \end{pmatrix} \middle| \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma \right\}$.

(i) $(f^* |_k \gamma^*)(z) = (-cz+d)^{-k} f^*(\gamma^*(z)) = (-cz+d)^{-k} \overline{f(-\overline{\gamma^*(z)})} = (-cz+d)^{-k} \overline{f(\gamma(-\bar z))}$

$-\gamma^*(\bar z) = -\dfrac{a\bar z - b}{-c\bar z + d} = \gamma(-\bar z)$ $\quad = (-cz+d)^{-k} \overline{(c(-\bar z)+d)}\, \overline{f(-\bar z)} = f^*(z)$

$f(z) = \sum a_n q^n \Rightarrow f^*(z) = \sum \overline{a_n(f)}\, \overline{e^{-2\pi i n \bar z}} = \sum \overline{a_n(f)} \cdot q^n$.

(ii) Let $\{f_i\}$ be a basis for $M_k(\Gamma)$. Let $\begin{cases} g_i = \dfrac{f_i + f_i^*}{2} = \sum \mathrm{Re}\, a_n(f_i)\, q^n \\ h_i = \dfrac{f_i - f_i^*}{2i} = \sum \mathrm{Im}\, a_n(f_i)\, q^n \end{cases}$

Obviously $\{g_i, h_i\}$ span $M_k(\Gamma)$, so some subset is a basis with $\mathbb{R}$ Fourier coefficients

Q7. i) Consider $\bar\Gamma \subset PSL_2(\mathbb{Z})$. It contains $\pm\begin{pmatrix} 1 & 1 \\ & 1 \end{pmatrix}$ as ∞ has width 1.

It contains $\pm\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$ as 0 has width 1.

$$\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ & 1 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix} = TS. \Rightarrow \bar\Gamma \ni S \Rightarrow \bar\Gamma = PSL_2(\mathbb{Z})$$

So then enough to show that $\bar\Gamma = PSL_2(\mathbb{Z}) \Rightarrow \Gamma = SL_2(\mathbb{Z})$.

Well, $(z \mapsto -1/z) \in \bar\Gamma$, so one of $\pm\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \in \Gamma$

$\Rightarrow \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}^2 = -I \in \Gamma \Rightarrow \Gamma = SL_2(\mathbb{Z})$

ii) (**) $-1 \in \Gamma$ congruence subgroup, every cusp has width dividing N.
Need to show $\Gamma \supset \Gamma(N)$.

As $-1 \in \Gamma$, this means that $\gamma\begin{pmatrix} 1 & N \\ & 1 \end{pmatrix}\gamma^{-1} \in \Gamma \quad \forall \gamma \in SL_2(\mathbb{Z})$.

Enough to show: $N \geq 1$, p prime. Then

$$\left\langle \Gamma(Np), \left\{ \gamma\begin{pmatrix} 1 & N \\ & 1 \end{pmatrix}\gamma^{-1} \middle| \gamma \in \Gamma(1) \right\} \right\rangle = \Gamma(N).$$

– equivalently, the subgroup of $G_{Np} := SL_2(\mathbb{Z}/Np\mathbb{Z})$ generated by $\{\gamma\begin{pmatrix}1&N\\&1\end{pmatrix}\gamma^{-1} \mid \gamma \in G\}$ is $H = \ker\left(G_{Np} \xrightarrow[\text{mod } N]{} G_N\right)$.

2 cases: (A) $p \nmid N$. In this case $G_{Np} \simeq G_N \times G_p$ (by Chinese remainder thm)

So $H \simeq G_p$. Then use:–

Fact – for any field F, $SL_2(F)$ is generated by $\left\{\begin{pmatrix}1&a\\&1\end{pmatrix}, \begin{pmatrix}1&\\a&1\end{pmatrix} \,\middle|\, a \in F\right\}$.

[Proof :– use identity $\begin{pmatrix}1&\\\frac{1-x}{x}&1\end{pmatrix}\begin{pmatrix}1&1\\&1\end{pmatrix}\begin{pmatrix}1&\\x-1&1\end{pmatrix}\begin{pmatrix}1&-1/x\\&1\end{pmatrix} = \begin{pmatrix}x&\\&1/x\end{pmatrix}$.]

So as $F = \mathbb{Z}/p\mathbb{Z}$ and $H \ni \begin{pmatrix}1&1\\0&1\end{pmatrix}$ and $\begin{pmatrix}1&0\\1&1\end{pmatrix} = \begin{pmatrix}&-1\\1&\end{pmatrix}^{-1}\begin{pmatrix}1&-1\\&1\end{pmatrix}\begin{pmatrix}&-1\\1&\end{pmatrix}$, $H = SL_2(\mathbb{Z}/p\mathbb{Z})$.

(B) $p \mid N = pM$. Then $H = \left\{ g = \begin{pmatrix}1+aN & bN\\ cN & 1+dN\end{pmatrix} \in G_{Np} \right\}$

and $\det g = 1 \iff 1 \equiv (1+aN)(1+dN) - bcN^2 \pmod{Np} \iff a+d \equiv 0 \pmod N$. (since $N^2 \equiv 0 \bmod Np$)

Consider the map $g \xmapsto{\varphi} \begin{pmatrix}a&b\\c&d\end{pmatrix} \pmod p$.

Claim: φ is an isomorphism $H \xrightarrow{\sim} sl_2(\mathbb{Z}/p\mathbb{Z}) = \left\{\begin{pmatrix}a&b\\c&d\end{pmatrix} \in M_2(\mathbb{Z}/p\mathbb{Z}) \,\middle|\, a+d=0\right\}$ (under addition)

(Proof: $g = 1 + N\varphi(g)$ so $gh = (1+N\varphi(g))(1+N\varphi(h)) = 1 + N(\varphi(g)+\varphi(h))$.)

Now $\varphi\begin{pmatrix}1&N\\&1\end{pmatrix} = \begin{pmatrix}0&1\\&0\end{pmatrix}$ $\varphi\begin{pmatrix}1&\\N&1\end{pmatrix} = \begin{pmatrix}0&\\1&0\end{pmatrix}$ and

$\varphi : \begin{pmatrix}1&-1\\&1\end{pmatrix}\begin{pmatrix}1&\\N&1\end{pmatrix}\begin{pmatrix}1&1\\&1\end{pmatrix} \mapsto \begin{pmatrix}1&-1\\1&-1\end{pmatrix}$; and these 3 matrices generate $sl_2(\mathbb{Z}/p\mathbb{Z})$. □

Remarks ① with a little more work one can remove the hypothesis $-1 \in \Gamma$. (exercise)

② For $N \geq 6$, the subgroup $K_N = \langle \gamma\begin{pmatrix}1&N\\&1\end{pmatrix}\gamma^{-1} \mid \gamma \in \Gamma(1)\rangle \subset \Gamma(1)$ has infinite index.

[Proof requires things we haven't covered, but for the curious, here it is:–
let $X = \Gamma(N)\backslash\mathfrak{H} \cup \{\text{cusps}\}$ be compactified quotient surface. For $N \geq 6$, X has genus ≥ 1. So there are unramified coverings $X' \to X$ of arbitrary degree $d > 1$. Then X' corresponds to a subgroup $\Gamma_d \subset \Gamma$ containing -1 of index d, and since $X' \to X$ is unramified, the cusps of Γ_d also have width N, so $\Gamma_d \supset K_N$ $\forall d$, and $(\Gamma(1) : \Gamma_d) \to \infty$.]

Q8. $E_2(z) = 1 - 24\sum_{n\geq 1}\sigma_1(n)q^n = E_2(z+1)$.

(probably deserves a (*))

Recall $E_2(-1/z) = z^2E_2(z) + \lambda z$, $\lambda = 12/2\pi i$.

Define $\tilde{E}_2(z) = E_2(z) + \dfrac{\lambda}{z-\bar{z}} = \tilde{E}_2(z+1)$

$$\tilde{E}_2(-1/z) = E_2(-1/z) + \frac{\lambda}{-1/z + 1/\bar{z}} = z^2E_2(z) + \lambda z + \frac{\lambda z\bar{z}}{z-\bar{z}} = z^2\tilde{E}_2(z)$$

$\Rightarrow$ $\tilde{E}_2|_2\gamma = \tilde{E}_2$ $\forall \gamma \in SL_2(\mathbb{Z})$, but $\tilde{E}_2$ not holomorphic.

$\Rightarrow$ $\tilde{E}_2(Mz)|_2\gamma = \tilde{E}_2(Mz)$ $\forall \gamma \in \Gamma_0(M)$.

$$\underbrace{\sum_M c(M)\tilde{E}_2(Mz)}_{\tilde{f}} = \underbrace{\sum_M c(M) E_2(Mz)}_{f} + \lambda \underbrace{\sum \frac{c(M)}{M}}_{g} \cdot \frac{1}{z-\bar{z}}$$

$\tilde{f}|_2\gamma = \tilde{f}$ $\forall\ \gamma \in \Gamma_0(N)$. So same true for $f \iff$ true for g.

But $\frac{1}{z-\bar{z}}\Big|_2 \underbrace{\begin{pmatrix} a & b \\ c & d\end{pmatrix}}_{\gamma} = (cz+d)^{-2} \cdot \frac{1}{2i \operatorname{Im}\gamma(z)} = (cz+d)^{-2} \frac{|cz+d|^2}{2i \operatorname{Im} z} \neq \frac{1}{z-\bar{z}}$

if $c \neq 0$ eg. $\gamma = \begin{pmatrix} 1 & 0 \\ N & 1\end{pmatrix}$

So $f|_2\gamma = f$ $\forall \gamma \in \Gamma_0(N) \iff \sum \frac{c(M)}{M} = 0$

Remains to show: if $\sum \frac{c(M)}{M} = 0$ then f is holo. at the cusps.

Let's first do case $N = p$ prime, $f = E_2(z) - pE_2(pz)$. Two cusps ∞, $0 = S(\infty)$,

so need to check:
- f holo. at ∞ (obvious)
- $f|_2 S$ holo. at ∞: but $f|_2 S = E_2(z) - \frac{1}{p}E_2\left(\frac{z}{p}\right)$ (compute) is holo. at ∞.

General: $f = \sum_{M|N} c(M) E_2(Mz)$ with $\sum \frac{c(M)}{M} = 0$.

Then can write $f = \sum_{p|N} \sum_{M|N/p} a(M) \cdot \underbrace{\left(E_2(Mz) - pE_2(Mpz)\right)}_{\in M_2(\Gamma_0(Mp))}$ $\square$

[Alternatively: $E_2|_2\gamma = E_2 + \lambda \frac{c}{cz+d}$ $\forall \gamma \in \Gamma(1)$.]

(write γ as word in S, T and use induction on length of word.)

Q9 $\quad G_{\underline{r},k}(z) = \sum_{\underline{m}\in\mathbb{Z}^2}' \frac{1}{[(m_1+r_1)z + m_2 + r_2]^k}$ $\quad k \geq 3$, $\underline{r} = (r_1, r_2) \in \mathbb{Q}^2$

(i) The series converges absolutely as $k \geq 3$, so can be rearranged.

If $g = \begin{pmatrix} a & b \\ c & d\end{pmatrix}$, $\underline{u} \in \mathbb{R}^2$

$$\frac{1}{(u_1 z + u_2)^k}\Big|_k g = \frac{(\det g)^{k/2}}{((au_1 + cu_2)z + (bu_1 + du_2))^k} = \frac{(\det g)^{k/2}}{(v_1 z + v_2)^k}, \quad \underline{v} = \underline{u} g$$

So $G_{\underline{r},k}\big|_k \gamma = \sum_{\underline{m}}' \frac{1}{[((\underline{m}+\underline{r})g)_1 z + ((\underline{m}+\underline{r})g)_2]^k} = \sum_{\underline{m}'}' \frac{1}{((m_1'+r_1')z + m_2' + r_2')^k} = G_{\underline{r}',k}$

$\underline{r}' = \underline{r}\gamma$, $\underline{m}' = \underline{m}\gamma$.

(ii) If $\gamma \in \Gamma(N)$ and $N\underline{r} \in \mathbb{Z}^2$, $N\underline{r}\gamma \equiv N\underline{r} \pmod N \Rightarrow \underline{r}\gamma \equiv \underline{r} \bmod \mathbb{Z}^2$

So by (i), $G_{\underline{r},k}|\gamma = G_{\underline{r}\gamma,k} = G_{\underline{r},k}$. So STP $G_{\underline{r},k}$ holomorphic at cusps.

By (i), enough to show that $G_{\underline{r},k}$ is holo. at ∞ $\forall \underline{r}$.

Let's assume WLOG $0 \le r_1 < 1$.

Write $$G_{\underline{r},k} = \sum_{\substack{m_1+r_1>0 \\ m_2}} + \sum_{\substack{m_1+r_1=0 \\ m_2}} + \sum_{\substack{m_1+r_1<0 \\ m_2}}$$

$$\text{1st sum} = \sum_{m_1>-r_1} \sum_{m_2\in\mathbb{Z}} \frac{1}{[(m_1+r_1)z+r_2+m_2]^k} = \frac{(-2\pi i)^k}{(k-1)!} \sum_{m>-r_1} \sum_{d\ge 1} d^{k-1} e^{2\pi i d r_2} q^{(m+r_1)d}$$

$$\text{2nd sum} = \sum_{m_2\in\mathbb{Z}}' \frac{1}{(m_2+r_2)^k} = \zeta(k;r_2) + \zeta(k;-r_2), \qquad \zeta(s;\alpha) = \sum_{\substack{m>0 \\ m-\alpha\in\mathbb{Z}}} m^{-s}$$

(only exists if $r_1 = 0$)

$$\text{3rd} = \sum_{m_1<-r_1} \sum_{m_2\in\mathbb{Z}} \frac{(-1)^k}{((-m_1-r_1)z-r_2-m_2)^k} = \frac{(2\pi i)^k}{(k-1)!} \sum_{m>r_1} \sum_{d\ge 1} d^{k-1} e^{-2\pi i d r_2} \cdot q^{(m-r_1)d}$$

Let $\underline{r} = (s_1/N, s_2/N)$ where $(s_1,s_2,N)=1$. So $(m\pm r_1)d = (Nm\pm s_1)d$

$$\sum_{m>-r_1} \sum_{d\ge 1} d^{k-1} e^{2\pi i d r_2} q^{(m+r_1)d} = \sum_{m>-r_1} \sum_{d\ge1} d^{k-1} \zeta_N^{s_2 d} q_N^{\overbrace{(Nm+s_1)d}^{=n}}$$
$$= \sum_{n\ge1} \Bigg[\sum_{\substack{n=dd' \\ d'\equiv s_1 \ (\mathrm{mod}\ N)}} d^{k-1} \zeta_N^{s_2 d} \Bigg] q_N^n.$$

$$\sum_{m>r_1} \sum_{d\ge1} d^{k-1} e^{-2\pi i d r_2} q^{(m-r_1)d} = \sum_{m-r_1>0} \sum_{d\ge1} d^{k-1} \zeta_N^{-s_2 d} q_N^{(Nm-s_1)d}$$
$$= \sum_{n\ge1} \Bigg[\sum_{\substack{dd'=n \\ d'\equiv -s_1 \ (\mathrm{mod}\ N)}} d^{k-1} \zeta_N^{-s_2 d} \Bigg] q_N^n$$

So $$G_{\underline{r},k}(z) = A_0 + \frac{(-2\pi i)^k}{(k-1)!} \sum_{n\ge1} A_n q^{n/N}$$ where

$$A_0 = \begin{cases} \zeta(k;r_2) + \zeta(k;-r_2) & \text{if } r_1 \in \mathbb{Z} \\ 0 & \text{if } r_1 \notin \mathbb{Z} \end{cases}$$

$$A_n = \sum_{\substack{dd'=n \\ d'\equiv s_1 \ (\mathrm{mod}\ N)}} d^{k-1} \zeta_N^{s_2 d} + (-1)^k \sum_{\substack{dd'=n \\ d'\equiv -s_1 \ (\mathrm{mod}\ N)}} d^{k-1} \zeta_N^{-s_2 d} \qquad (n\ge 1).$$

$d' \equiv s_1 \ (N)$

$\Rightarrow d \equiv n/s_1 \ (N).$

Q10 (*) $$E_{\Gamma,k}(z) = \frac{1}{2} \sum_{\gamma\in\Gamma_\infty\backslash\Gamma} \frac{1}{(cz+d)^k} \qquad \gamma = \begin{pmatrix} * & * \\ c & d \end{pmatrix}$$
$$= \frac{1}{2} \sum_{\gamma\in\Gamma_\infty\backslash\Gamma} j(\gamma,z)^{-k}$$

$\Gamma \subset SL_2(\mathbb{Z})$, $-1\in\Gamma$.

$\Gamma_\infty = \begin{pmatrix} 1 & * \\ & 1 \end{pmatrix} \cap \Gamma = \left\langle \begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix} \right\rangle$ $m \ge 1$

$$E_{\Gamma,k}\Big|_k \delta\,(z) = \tfrac{1}{2}\sum_\gamma j(\gamma,\delta(z))^{-k}\cdot j(\delta,z)^{-k}$$
$$= \tfrac{1}{2}\sum_\gamma j(\gamma\delta,z)^{-k} = E_{\Gamma,k}.$$

As for E_k, converges (uniformly on compact subsets of $\mathfrak{H}$) $\Rightarrow$ holomorphic on $\mathfrak{H}$.

q-expansion: $$E_{\Gamma,k}(z) = 1 + \tfrac{1}{2}\sum_{\substack{\gamma\in\Gamma_\infty\backslash\Gamma \\ \gamma\notin\Gamma_\infty}} \frac{1}{(cz+d)^k}$$

$$= 1 + \sum_{c\geq 1}\sum_{\substack{d\in\mathbb{Z} \\ \text{s.t. } \left(\begin{smallmatrix}* & * \\ c & d\end{smallmatrix}\right)\in\Gamma}} \frac{1}{(cz+d)^k} = 1 + \sum_{c\geq1}\sum_{\substack{0\leq d<mc \\ \left(\begin{smallmatrix}* & * \\ c & d\end{smallmatrix}\right)\in\Gamma}}\sum_{r\in\mathbb{Z}}\frac{1}{(cz+d+rmc)^k}$$

$$\begin{pmatrix} a & b \\ c & d\end{pmatrix} = \gamma\in\Gamma \Rightarrow \gamma\begin{pmatrix}1 & rm \\ & 1\end{pmatrix} = \begin{pmatrix} * & * \\ c & d+rmc\end{pmatrix}$$

$$= 1 + \sum_{\substack{c\geq1 \\ 0\leq d<mc \\ \left(\begin{smallmatrix}* & * \\ c & d\end{smallmatrix}\right)\in\Gamma}}\sum_{r\in\mathbb{Z}}\frac{1}{(mc)^k}\cdot\frac{1}{(z/m + d/mc + r)^k}$$

$$= 1 + \sum_{\substack{c\geq1 \\ d}}\frac{1}{(mc)^k}\cdot\sum_{n\geq1} n^{k-1} e^{2\pi i n(z/m + d/mc)}$$

$$= 1 + \sum_{n\geq1}\left(\sum_{c,d}\frac{1}{(mc)^k}\cdot e^{2\pi i n d/mc}\right) n^{k-1} q^{n/m}.$$

Another cusp: $\alpha = \delta(\infty)$, $\delta\in\Gamma(1)\setminus\Gamma$.

$$E_{k,\Gamma}\big|\delta = \tfrac{1}{2}\sum_{\gamma\in\Gamma_\infty\backslash\Gamma} j(\gamma\delta,z)^{-k} = \tfrac{1}{2}\sum_{\gamma\in\Gamma_\infty\backslash\Gamma\delta} j(\gamma,z)^{-k}.$$

If $\delta\notin\Gamma$ then $\begin{pmatrix} * & * \\ c & d\end{pmatrix} = \gamma\in\Gamma\delta \Rightarrow \gamma\neq\pm I \Rightarrow c\neq 0$.

So $E_{k,\Gamma}|\delta \to 0$ as $\operatorname{Im} z\to\infty$, by comparison w/ $E_k \sim \frac{1}{2}\sum_{c\neq0}\frac{1}{c^k}$.

Now fix Γ, let $\alpha_1 = \infty, \alpha_2, \ldots, \alpha_\nu$ be cusps of Γ.

$\alpha_r = g_r(\infty)$ $\quad \alpha_1 = 1$ say.

Consider $F_r = E_{k,g_r^{-1}\Gamma g_r}\big|_k g_r^{-1}$. If $\gamma\in\Gamma$ then

$$F_r\big|_k\gamma = \underbrace{E_{k,g_r^{-1}\Gamma g_r}\big|g_r^{-1}\gamma g_r}_{=E_{k,g_r^{-1}\Gamma g_r}}\big|g_r^{-1} = F_r.$$

And by defn, F_r is non-vanishing at the cusp α_r & vanishes at all the others.

Have $M_k(\Gamma)\xrightarrow{\psi}\mathbb{C}^\nu$, $f\longmapsto\left(a_0(f|_k g_r)\right)_{1\leq r\leq\nu}$ whose kernel is $S_k(\Gamma)$.

and $\psi: \bigoplus_{r=1}^{\nu}\mathbb{C}F_r \xrightarrow{\sim}\mathbb{C}^\nu$, so $M_k(\Gamma) = S_k(\Gamma)\oplus\bigoplus_r\mathbb{C}F_r$. $\square$