$\boxed{\text{Ex sheet 2}}$

① Identify $X = \begin{pmatrix} a & b \\ b & c \end{pmatrix}$ with quadratic form $aU^2 + 2bUV + cV^2 = (U\ V) X \begin{pmatrix} U \\ V \end{pmatrix}$

Congruence $g: X \longmapsto gXg^t$ of matrices $\iff$ equivalence of quadratic forms under change of variables $(U\ V) \longmapsto (U\ V)g$

$\therefore$ Under $SL_2(\mathbb{R})$, each X with $\det X = D$ is equiv. to one of $\pm\begin{pmatrix} \sqrt{D} & \\ & \sqrt{D} \end{pmatrix}$.

Stabilizer of $\begin{pmatrix} \sqrt{D} & \\ & \sqrt{D} \end{pmatrix}$ is $SO(2) = \{ g \in SL_2(\mathbb{R}) \mid gg^t = I \}$ so

orbit $\simeq SL_2(\mathbb{R})/SO(2) \simeq \mathfrak{H}$.

$$g \cdot \begin{pmatrix} \sqrt{D} & \\ & \sqrt{D} \end{pmatrix} g^t = \frac{\sqrt{D}}{y} \begin{pmatrix} x^2+y^2 & x \\ x & 1 \end{pmatrix} \longleftrightarrow \tau = g(i), \quad g = \begin{pmatrix} y^{1/2} & xy^{-1/2} \\ 0 & y^{-1/2} \end{pmatrix}$$

[Gauss's reduction for positive definite binary quadratic forms states that each $SL_2(\mathbb{Z})$-orbit of quadratic forms $aU^2 + 2bUV + cV^2$ of discriminant $-4D = 4(b^2 - ac) < 0$ which are positive definite contains a unique representative with $-a < 2b \le a < c$ or $0 \le 2b \le a = c$.
If $a = \sqrt{D}/y$, $b = x\sqrt{D}/y$, $c = (x^2+y^2)\frac{\sqrt{D}}{y}$, this is the condition that $\tau = x + iy$ belongs to the fundamental domain $\mathcal{D}$ for $SL_2(\mathbb{Z})$ acting on $\mathfrak{H}$.]

② (i) If $\Lambda = \bigoplus \mathbb{Z}x_i$, x_i lin. independent, then after change of basis $\Lambda = \mathbb{Z}^m \subset \mathbb{R}^n$, obviously discrete.

(ii) Let $V \subset \mathbb{R}$ be the $\mathbb{R}$-span of Λ. Replacing $\mathbb{R}^n$ by V, and choosing a basis for V consisting of elts. of Λ, reduce to the case when $\Lambda \subset \mathbb{Z}^n$.
Let $\epsilon \in (0, 1/2)$ s.t. $\Lambda \cap B(\underline{0}, \epsilon) = \{0\}$. Then if $\underline{m} \in \mathbb{Z}^n$, $\Lambda \cap B(\underline{m}, \epsilon) = \{\underline{m}\}$ as Λ is a subgroup. Consider $\pi: \mathbb{R}^n \to \mathbb{R}^n/\mathbb{Z}^n$.
The $\pi(\Lambda) \subset \mathbb{R}^n/\mathbb{Z}^n$ is discrete: indeed if $B(\underline{0}, \epsilon) \subset \mathbb{R}^n/\mathbb{Z}^n$ is open ball (we chose $\epsilon < 1/2$) then $\pi^{-1}(B(\underline{0}, \epsilon)) = \coprod_{\underline{m} \in \mathbb{Z}^n} B(\underline{m}, \epsilon)$
and $\therefore \pi^{-1}(B(\underline{0}, \epsilon) \cap \pi(\Lambda)) = \coprod_{\underline{m} \in \mathbb{Z}^n} B(\underline{m}, \epsilon) \cap \Lambda = \mathbb{Z}^n$,
hence $B(\underline{0}, \epsilon) \cap \pi(\Lambda) = \{0\}$. But $\mathbb{R}^n/\mathbb{Z}^n$ compact $\Rightarrow \pi(\Lambda)$ is finite; so $\Lambda \supset \mathbb{Z}^n$ with finite index d say $\Rightarrow \Lambda = \bigoplus \mathbb{Z}x_i$, $x_i \in \frac{1}{d}\mathbb{Z}^n$ lin indep. over $\mathbb{Z}$ $\Rightarrow$ x_i lin. indep. over $\mathbb{R}$.

③ Same proof as for $\tau(n) \equiv \sigma_{11}(n) \pmod{691}$.

④. For both parts just use $\sum_{\tau_0 \neq i, \rho} \mathrm{ord}_{\tau_0} f + \frac{1}{2}\mathrm{ord}_i f + \frac{1}{3}\mathrm{ord}_\rho f = k/12$.

(5) Let $f \in M_k^!$, $\mathrm{ord}_\infty f \geq -r$ say $\Rightarrow \Delta^r f \in M_{k+12r}$

$\Delta \neq 0$ on upper ½-plane $\Rightarrow$ $\Delta^{-r} M_{k+12r} \subset M_k^!$.

So $\{ f \in M_k^! \mid \mathrm{ord}_\infty f \geq -r \} \xrightarrow[\Delta^r]{\sim} M_{k+12r}$

i.e. $M_k^! = \bigcup_{r \geq 0} \Delta^{-r} M_{k+12r}$

M_k has basis $\Delta^a E_4^b E_6^{(k-12a-4b)/6}$ $\quad 0 \leq a \leq \frac{k-4b}{12}$.

$b = 0$ or 1 depending on k

$\therefore$ $M_k^! = \text{span of } \left\{ \Delta^a E_4^b E_6^{(k-12a-4b)/6} \;\middle|\; \begin{array}{l} b = 0 \text{ or } 1 \text{ (depending)} \\ \mathbb{Z} \ni a \leq k \leq \frac{k-4b}{12} \end{array} \right\}$.

(6) Differentiate the formula $f\left(\frac{az+b}{cz+d}\right) = (cz+d)^k f(z)$.

(7) $|f(\gamma(z))|^2 = f(\gamma(z)) \cdot \overline{f(\gamma(z))} = |cz+d|^{2k} \cdot |f(z)|^2$

and $\mathrm{Im}\,\gamma(z) = \frac{\mathrm{Im}\,z}{|cz+d|^2}$; so $(\mathrm{Im}\,z)^{k/2} |f| = g$ is Γ-invariant.

So g bounded on $\mathcal{H}$ $\Leftrightarrow$ g bounded on $\mathcal{D}$.

- suppose $f \in S_k(\Gamma(1))$. Then $f = O(e^{-y})$ as $y = \mathrm{Im}\,z \longrightarrow \infty$

$\Rightarrow$ $g = O(y^{k/2} e^{-y})$ so g bounded on $\mathcal{D}$.

- conversely, suppose f holomorphic on $\mathfrak{H}$. So f has q-expansion

$\tilde{f}(q) = \sum_{n \in \mathbb{Z}} a_n(f) q^n \qquad (|q| < 1)$.

g bounded $\Rightarrow$ $\tilde{f}(q) \longrightarrow 0$ as $q \longrightarrow 0$ $\Rightarrow$ $\tilde{f}$ has a removable singularity at $q = 0$,
(assuming $k > 0$) with value $= 0$

$\Rightarrow a_n(f) = 0 \;\forall n \leq 0 \;\Rightarrow\; f \in S_k$.

(8) $\displaystyle\sum_{n \in \mathbb{Z}} \frac{1}{(z+n)^2} = -4\pi^2 \sum_{d=1}^{\infty}{}' d\, e^{2\pi i d z}$ (proved in lectures), so $(m \neq 0)$-sum equals

$\displaystyle\sum_{m=1}^{\infty} 2 \sum_{n \in \mathbb{Z}} \frac{1}{(mz+n)^2} = -8\pi^2 \sum_{m \geq 1} \sum_{d \geq 1} d\, q^{md}$. This series is absolutely convergent,

so can be rearranged to $\displaystyle -8\pi^2 \sum_{n \geq 1} \sigma_1(n) q^n$. The $(m=0)$-sum

is $2\zeta(2) = \frac{\pi^2}{3}$, so $\displaystyle G_2 = \frac{\pi^2}{3}\left(1 - 24 \sum_{n \geq 1} \sigma_1(n) q^n\right)$

$G_2 \notin M_2$ since $M_2 = 0$; the proof $G_k \in M_k$ fails for $k = 2$ as the double series is only conditionally convergent, so

$$z^{-2} G_2(-1/z) = z^{-2} \sum_m{}' \left(\sum_n \frac{1}{(-m/z+n)^2} \right) = \sum_m{}' \sum_n \frac{1}{(nz+m)^2}$$

but this isn't the same as the series for $G_2(z)$.

(9) In the lectures we proved $S_k(\mathbb{Z}) \xrightarrow{\sim} \mathbb{Z}^{m=\dim S_k}$, $f \mapsto (a_i(f))_{1\le i\le m}$. So $\exists$ $\mathbb{Z}$-basis $g_1,\dots,g_m$ of S_k with

$$a_i(g_j) = \delta_{i,j} \quad \text{for} \quad 1\le i\le m$$

and if $f\in S_k(\mathbb{Z})$ then $f=\sum_{j=1}^{m} a_j(f).g_j$.

Let $\mathbb{T}\subset \operatorname{End} S_k$ be the $\mathbb{Z}$-subalgebra generated by the Hecke operators $(T_n)_{n\ge 1}$. Recall $S_k(\mathbb{Z})$ invariant under $\{T_n\}$.

Consider the pairing

$$\langle\ ,\ \rangle : S_k(\mathbb{Z}) \times \mathbb{T} \longrightarrow \mathbb{Z}$$
$$(f, T) \longmapsto a_1(Tf)$$

So $\langle f, T_n\rangle = a_1(T_n f) = a_n(f)$.

If $1\le j\le m$, $1\le n\le m$ then $\langle g_j, T_n\rangle = a_n(g_j) = \delta_{n,j}$.

So, if $\mathbb{T}'\subset\mathbb{T}$ is the $\mathbb{Z}$-submodule generated by $T_1,\dots,T_m$,

$\langle\ ,\ \rangle : S_k(\mathbb{Z})\times\mathbb{T}'\to\mathbb{Z}$ is a <u>perfect pairing</u>, i.e. induces isomorphisms

$$S_k(\mathbb{Z}) \xrightarrow{\sim} \operatorname{Hom}_{\mathbb{Z}}(\mathbb{T}', \mathbb{Z}), \qquad \mathbb{T}' \xrightarrow{\sim} \operatorname{Hom}_{\mathbb{Z}}(S_k(\mathbb{Z}), \mathbb{Z}).$$
$$f \longmapsto (T\mapsto \langle f,T\rangle) \qquad T\longmapsto (f\mapsto\langle f,T\rangle)$$

(as (g_j), $(T_n)_{n\le m}$ are dual bases for $S_k(\mathbb{Z})$ and $\mathbb{T}'$).

Now as the $\{T_n\}$ commute and are simultaneously diagonalisable, $\mathbb{T}$ has rank at most m ($*$ below). As $\mathbb{T}'$ has rank m and $\mathbb{T}$ is torsion-free, this means that $\mathbb{T}'\subset\mathbb{T}\subset\mathbb{T}'\otimes\mathbb{Q} = \bigoplus_{n=1}^{m}\mathbb{Q}.T_n \subset \operatorname{End} S_k$

Suppose $T = \sum_{n\ge 1}^{m} c_n T_n \in \mathbb{T}$, so $c_n\in\mathbb{Q}$. Then

$\mathbb{Z}\ni\langle g_j, T\rangle = \sum c_n\langle g_j, T_n\rangle = c_n$. So $T\in\mathbb{T}'$ i.e. $\mathbb{T}=\mathbb{T}'$.

Finally we show that $\langle\ ,\ \rangle$ induces an isomorphism of <u>$\mathbb{T}$-modules</u>

$$S_k(\mathbb{Z}) \xrightarrow{\sim} \operatorname{Hom}_{\mathbb{Z}}(\mathbb{T}, \mathbb{Z})$$

The $\mathbb{T}$-module structure on $\operatorname{Hom}(\mathbb{T},\mathbb{Z})$ is given by

$T\in\mathbb{T}$, $\varphi\in\operatorname{Hom}(\mathbb{T},\mathbb{Z})$: the $(T\varphi)(Y) = \varphi(TY)$ $\forall Y\in\mathbb{T}$.

(As $\mathbb{T}$ is commutative there is no difference between left and right modules).

So taking $\varphi = \langle f, -\rangle$ we need to check that $\forall T, Y\in\mathbb{T}$,

$$\langle f, TY\rangle = \langle Tf, Y\rangle$$

But $\langle f, TY\rangle = a_1((TY)f) = a_1(Y(Tf)) = \langle Tf, Y\rangle$.

(∗) Choosing a basis of $S_k(\mathbb{Q})$ [e.g. the basis (g_j)] identifies $\mathbb{T}$ with a subalgebra of $\mathrm{Mat}_{m\times m}(\mathbb{Q})$. To show $\mathbb{T}$ has rank $\leq m$ it's enough to show that $\mathbb{T} \otimes_{\mathbb{Z}} \mathbb{C} \subset \mathrm{Mat}_{m\times m}(\mathbb{C})$ has dimension $\leq m$. But simultaneously diagonalising the T_n's shows that $\mathbb{T} \otimes_{\mathbb{Z}} \mathbb{C}$ is conjugate to an algebra of diagonal matrices, hence has dimension $\leq m$.