SHEET 1

(1) If $\chi \neq \chi'$ $\exists h \in G$ st. $\chi(h) \neq \chi'(h)$ ie. $\bar\chi(h)\chi'(h) \neq 1$; then

$$\sum_g \bar\chi(g)\chi'(g) = \bar\chi\chi'(h)\sum_g \bar\chi(gh^{-1})\chi'(gh^{-1}) = \bar\chi\chi'(h)\sum_g \bar\chi(g)\chi'(g) \quad (g \mapsto gh^{-1})$$

$$\Rightarrow \sum_g \bar\chi(g)\chi'(g) = 0 \quad \text{If } \bar\chi\chi' = 1 \text{ obviously } \sum = \sum 1 = \#G.$$

(2) $\chi : \mathbb{R} \to \mathbb{C}^\times$ continuous hom. Then (as $\mathbb{R}$ simply-connected) $\chi = \exp \circ \psi$ for some cts. hom $\psi : \mathbb{R} \to \mathbb{C}$. Let $\psi(1) = s \in \mathbb{C}$. Then $\psi(1/n) = s/n \Rightarrow$ $\psi(x) = sx$ $\forall x \in \mathbb{Q} \Rightarrow \psi(x) = sx$ $\forall x \in \mathbb{R}$ by continuity. So $\chi(x) = e^{sx}$.
χ unitary $\iff$ $s = 2\pi i y \in i\mathbb{R}$ ie. $\chi = \chi_y$

So if $\chi : \mathbb{R}^\times_{>0} \to \mathbb{C}^\times$, using $\mathbb{R} \xrightarrow{\sim} \mathbb{R}^\times_{>0}$, $x \mapsto e^x$, see that $\chi(x) = x^s$, and χ unitary $\iff$ $s \in i\mathbb{R}$.

(3) (i) Trivial to see that this is an equivalence relation.

Lemma. Suppose $\chi_1 \pmod{N_1}$ and $\chi_2 \pmod{N_2}$ are equivalent. Let $N_3 = \gcd(N_1, N_2)$. Then $\exists!\ \chi_3 \pmod{N_3}$ equivalent to χ_1 & χ_2.

Proof By hypothesis, $\exists N$ with $N_1 | N$, $N_2 | N$ such that $\chi_1 \circ \mathrm{red}_{N,N_1} = \chi_2 \circ \mathrm{red}_{N,N_2}$. We'll show $\exists!\ \chi_3 \pmod{N_3}$ st. $\chi_i = \chi_3 \circ \mathrm{red}_{N_i,N}$. As reduction maps are surjective, χ_3 is unique if it exists.

(The proof I gave in class was a bit confusing, I confess. Here is a different one.)

Proof By hypothesis, $\chi_1(x) = \chi_2(x)$ for every x with $(x, N) = 1$.
The map $(\mathbb{Z}/N\mathbb{Z})^\times \to (\mathbb{Z}/(N_1,N_2)\mathbb{Z})^\times$ is surjective, so $\forall x$ with $(x, (N_1,N_2)) = 1$, we can find $x' \in \mathbb{Z}$ with $(x, N) = 1$ and $x \equiv x' \pmod{(N_1, N_2)}$.
Define $\chi_3(x) = \chi_1(x') = \chi_2(x')$.
Claim this is well-defined and depends only on x mod (N_1, N_2).
Suppose x'' also is prime to N and $\equiv x$ mod (N_1, N_2).
So $x'' \equiv y x' \pmod N$ for some y with $(y, N) = 1$, $y \equiv 1$ mod (N_1, N_2)
Then $\chi_i(x'')/\chi_i(x') = \chi_i(y)$ $(i = 1, 2)$. So enough to show: $(y, N) = 1$, $y \equiv 1 \pmod{(N_1, N_2)} \Rightarrow \chi_1(y) (= \chi_2(y)) = 1$.

For this: $\exists y_1$ with $y_1 \equiv y \pmod{N_1}$, $y_1 \equiv 1 \pmod{N_2}$ and $(y_1, N) = 1$
$\Rightarrow \chi_1(y) = \chi_1(y_1) = \chi_2(y_1) = 1$. □ (of Lemma).

Now let $\chi : (\mathbb{Z}/N\mathbb{Z})^\times \to \mathbb{C}^\times$ be a Dirichlet character mod N.
As $(\mathbb{Z}/ND\mathbb{Z})^\times \twoheadrightarrow (\mathbb{Z}/N\mathbb{Z})^\times$, if $\chi' : (\mathbb{Z}/N\mathbb{Z})^\times \to \mathbb{C}^\times$ is equivalent to χ then $\chi = \chi'$.

Let $S = S_\chi = \{ M \mid \exists \chi' : (\mathbb{Z}/M\mathbb{Z})^\times \to \mathbb{C}^\times$ equivalent to $\chi \}$.

Then $N \in S_\chi$, and $S_\chi = S_{\chi'}$ if χ, χ' are equivalent. Also:

- If $M \in S_\chi$, $D \geq 1$ then $MD \in S_\chi$ (as $\chi' \circ \mathrm{red}_{MD,M}$ is equivalent to χ).
- By the lemma, if $N_1, N_2 \in S_\chi$ then $(N_1, N_2) \in S_\chi$.

These 2 properties $\Rightarrow$ $S_\chi = \{ MD \mid D \geq 1\}$ where M = least element of S_χ. For this M, the corresponding character χ^* mod M is then primitive, and the characters equivalent to it are $\{ \chi^* \circ \mathrm{red}_{MD,M} \mid D \geq 1\}$. So χ^* is the unique primitive character equivalent to χ. $\square$

<u>Remark</u>: This question (and everything concerning Dirichlet characters) becomes clearer if one takes the following point of view:–

Consider $\hat{\mathbb{Z}} = \varprojlim_{N \geq 1} (\mathbb{Z}/N\mathbb{Z})$ (inverse limit over integers N ordered by divisibility)

$$\cong \prod_p \mathbb{Z}_p \quad ; \text{ it is a profinite ring.}$$

$$\hat{\mathbb{Z}}^\times = \varprojlim_N (\mathbb{Z}/N\mathbb{Z})^\times \cong \prod_p \mathbb{Z}_p^\times \qquad (1)$$

A Dirichlet character χ determines a homomorphism $\tilde{\chi} : \hat{\mathbb{Z}}^\times \to \mathbb{C}^\times$ with open kernel (ie continuous for the discrete(*) topology on $\mathbb{C}^\times$). If $\chi_i : (\mathbb{Z}/N_i\mathbb{Z})^\times \to \mathbb{C}^\times$ are Dirichlet characters, then they are equivalent $\iff$ $\tilde{\chi}_1 = \tilde{\chi}_2$. So

$$\{ \text{cts. hom's } \hat{\mathbb{Z}}^\times \to \mathbb{C}^\times \} \simeq \{ \text{equivalence classes of Dirichlet characters} \}.$$

Now, the subgroups $U_N = (1 + N\hat{\mathbb{Z}}) \cap \hat{\mathbb{Z}}^\times$ form a basis of open neighbourhoods of 1 in $\hat{\mathbb{Z}}^\times$, and under the isomorphism (1),

$$U_N \xrightarrow{\sim} \prod_{p \nmid N} \mathbb{Z}_p^\times \times \prod_{p \mid N} (1 + p^{v_p(N)} \mathbb{Z}_p).$$

Therefore

$$U_{N_1} U_{N_2} = \prod_{p \nmid (N_1, N_2)} \mathbb{Z}_p^\times \times \prod_{p \mid (N_1, N_2)} 1 + p^{\min(v_p(N_1), v_p(N_2))} \mathbb{Z}_p = U_{(N_1, N_2)} \qquad (2)$$

Now suppose that $\psi : \hat{\mathbb{Z}}^\times \to \mathbb{C}^\times$ is a homomorphism with open kernel. Let M be the least integer for which $U_M \subset \ker \psi$. Then (2) implies that $U_N \subset \ker \psi \iff M \mid N$. So there is a (unique) Dirichlet character χ mod M such that $\tilde{\chi} = \psi$, and the characters equivalent to it are the characters $\chi \circ \mathrm{red}_{N,M}$ for $M \mid N$. In particular, χ is the unique primitive character in its equivalence class.

(*) in fact, this is equivalent to being continuous for the usual topology on $\mathbb{C}^\times$. This is a special case of the following fact:

> if G is a profinite group and $\rho : G \to GL_n(\mathbb{C})$ is a continuous homomorphism (for usual topology on $GL_n(\mathbb{C})$) then $\ker \rho$ is an open subgroup of G.

(ii). We may assume $\chi_i = \chi \circ \mathrm{red}_{N_i, N}$ where $\chi : (\mathbb{Z}/M\mathbb{Z})^\times \to \mathbb{C}^\times$, $M = (N_1, N_2)$.

Then $L(\chi, s) = \prod_{p \nmid M} (1 - \chi(p) p^{-s})^{-1} = \prod_{p \nmid N_1} (1 - \chi(p) p^{-s})^{-1} \cdot \prod_{p | N_1, p \nmid M} (1 - \chi(p) p^{-s})^{-1}$

$= L(\chi_1, s) \prod_{p | N_1, p \nmid N_2} (1 - \chi(p) p^{-s})$

as if $p | N_1$ then $p \nmid M = (N_1, N_2) \iff p \nmid N_2$. Same for $\chi_2 \Rightarrow$ result.

(4) $\zeta(s) = \prod_p (1 - p^{-s})^{-1} \Rightarrow -\frac{\zeta'(s)}{\zeta(s)} = \sum_p \frac{d}{ds} \log(1 - p^{-s})$

$= \sum_p \frac{\log p \cdot p^{-s}}{1 - p^{-s}} = \sum_p \log p \left(p^{-s} + p^{-2s} + \cdots \right) = \sum_n \Lambda(n) n^{-s}$.

[Note: for this to make analytic sense: the Euler product converges uniformly on compact subsets of $\mathrm{Re}(s) > 1$, hence (general facts about ∞ products, see Ahlfors) so does $\sum_p \log(1 - p^{-s})$ (principal branch of logarithm: each $1 - p^{-s}$ has $\mathrm{Re} > 0$) – and it converges to some logarithm of $\zeta(s)$. Then can differentiate term-by-term.]

Let $n = \prod_i p_i^{e_i}$ $(e_i \geq 1)$. Then $\sum_{d | n} \Lambda(d) = \sum_i \sum_{r=1}^{e_i} \Lambda(p_i^r) = \sum_i (\log p_i) e_i = \log n$.

(5) From lectures, $\zeta(1 - 2k) = -\frac{B_{2k}}{2k}$. By FE, $Z(2k) = Z(1 - 2k)$

ie. $\pi^{-k} \Gamma(k) \cdot \zeta(2k) = \pi^{k - 1/2} \Gamma(\tfrac{1}{2} - k) \cdot \zeta(1 - 2k)$

ie. $\zeta(2k) = \pi^{2k - 1/2} \cdot (k-1)! \cdot \frac{\Gamma(1/2)}{-1/2 \cdot -3/2 \cdot \ldots \cdot -\frac{(2k-1)}{2}} \cdot \frac{-B_{2k}}{2k}$

and $1.3. \ldots (2k-1) = \frac{(2k-1)!}{2.4 \ldots (2k-2)} = \frac{(2k-1)!}{2^{k-1} (k-1)!}$; $\Gamma(1/2) = \sqrt{\pi}$

$\Rightarrow \zeta(2k) = \pi^{2k} \times (-2)^k \cdot \frac{2^{k-1}}{(2k-1)!} \cdot \frac{-B_{2k}}{2k} = (-1)^{k-1} \frac{(2\pi)^{2k}}{2(2k)!} \cdot B_{2k}$.

As $\zeta(s) > 0$ for $s \in \mathbb{R}_{>1}$, $(-1)^{k-1} B_{2k} > 0$.

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6 (i) $\psi : \mathbb{Z}/N\mathbb{Z} \to \mathbb{C}$. From lectures,

$$(2\pi)^{-s}\Gamma(s)L(\psi,s) = \sum_{n=1}^{N}\psi(n).\left[\frac{1}{2\pi N(s-1)} + \frac{c_{0,n}}{s} + \dots + \frac{c_{L-1,n}}{s+L-1}\right]$$

$+$ (analytic for $Re(s) > -L$)

where $$\frac{e^{2\pi(N-n)y}}{e^{2\pi Ny}-1} = \frac{1}{2\pi Ny} + \sum_{r=0}^{L-1} c_{r,n}y^r + O(y^L). \qquad y \in (0,1].$$

$$= \frac{1}{2\pi Ny}\cdot\sum_{k\geq 0}\frac{B_k\left(\frac{N-n}{N}\right)}{k!}.(2\pi Ny)^k$$

$\Rightarrow \quad c_{k-1,n} = (2\pi N)^{k-1}.B_k(1-\frac{n}{N})/k!$

$Res_{s=1-k}\ (2\pi)^{-s}\Gamma(s)L(\psi,s) = (2\pi)^{k-1}(-1)^{k-1}/(k-1)!\ .L(\psi,1-k)$

$$\Rightarrow \quad L(\psi,1-k) = \frac{(-N)^{k-1}}{k}.\sum_{j=0}^{N-1}\psi(-j)B_k(j/N)$$

(ii) $$\hat{B}_{n,N}(\zeta) = \sum_{j=0}^{N-1}\zeta^j B_n(j/N) = \sum_j \zeta^j.\ n! \times \text{coeff}\left(t^{n-1}\,\Big|\,\frac{e^{jt/N}}{e^t-1}\right)$$

$$= n! \times \text{coeff}\left(t^{n-1}\,\Big|\,\sum_{j=0}^{N-1}e^{jt/N}.\zeta^j\Big/e^t-1\right)$$

$$= n! \times \text{coeff}\left(t^{n-1}\,\Big|\,\frac{1}{\zeta e^{t/N}-1}\right)$$

$\boxed{\text{if } \zeta\neq 1}$ $$= n\left(\frac{d}{dt}\right)^{n-1}\frac{1}{\zeta e^{t/N}-1}\Bigg|_{t=0} = n.\left(\frac{-1}{N}u\frac{d}{du}\right)^{n-1}\frac{u}{\zeta-u}\Bigg|_{u=1} \qquad (u = e^{-t/N})$$

$$\therefore \quad \tilde{B}_n(\zeta) = (-1)^n.n.\left(u\frac{d}{du}\right)^{n-1}\left(\frac{u}{u-\zeta}\right)\Bigg|_{u=1} = \frac{P_n(\zeta)}{(\zeta-1)^n}$$

(NB. there was a typo in the original question — it should read
$\tilde{B}_n(\zeta) = N^{n-1}\hat{B}_{n,N}(\zeta)$.)

(iii) $$\sum_{\eta^D=\zeta}\frac{1}{u-\eta} = \frac{Du^{D-1}}{u^D-\zeta} \Rightarrow \sum_{\eta^D=\zeta}\tilde{B}_n(\eta) = \sum_{\eta^D=\zeta}(-1)^n.n.\left(u\frac{d}{du}\right)^{n-1}\left(\frac{u}{u-\eta}\right)\Bigg|_{u=1}$$

$$= (-1)^n n\left(u\frac{d}{du}\right)^{n-1}\left(\frac{Du^D}{u^D-\zeta}\right) \qquad v = u^D \quad \frac{dv}{du} = D.\frac{v}{u}$$

$$= D^n.(-1)^n n\left(v\frac{d}{dv}\right)^{n-1}\left(\frac{v}{v-\zeta}\right). \qquad \therefore\ u\frac{d}{du} = D.v\frac{d}{dv}$$

$$= D^n\tilde{B}_n(\zeta).$$